Yirong Liu, Jibin Li, Wentao Huang

Planar Dynamical Systems

Yirong Liu, Jibin Li, Wentao Huang

Planar Dynamical Systems

Selected Classical Problems

DE GRUYTER
Science Press
Beijing

Mathematics Subject Classification 2010
34C05, 34C07, 34C14, 34C23, 34C25, 34C37, 37C80, 37G05, 37G10, 37G15, 37G20

Responsible Editor
Yanchao Zhao

ISBN 978-3-11-029829-1
e-ISBN 978-3-11-029836-9
Set-ISBN 978-3-11-029837-6

Library of Congress Cataloging-in-Publication Data
A CIP catalog record for this book has been applied for at the Library of Congress.

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

Printing and binding: CPI books GmbH, Leck
♾Printed on acid-free paper
Printed in Germany

www.degruyter.com

Preface

In 2008, November 23-28, the workshop of "Classical Problems on Planar Polynomial Vector Fields " were held in Banff International Research Station of Canada. So called "classical problems", it concerns with the following: (1) Problems on integrability of planar polynomial vector fields. (2) The problem of the center stated by Poincaré for real polynomial differential systems, asks us to recognize when a planar vector field defined by polynomials of degree at most n possesses a singurality which is a center. (3) Global geometry of specific classes of planar polynomial vector fields. (4) Hilbert's 16th problem.

These problems had been posed more than 110 years. Therefore, they are called "classical problem" in the studies of the theory of dynamical systems.

The qualitative theory and stability theory of differential equations, created by Poincaré and Lyapunov at the end of the 19th century had major developments as two branches of the theory of dynamical systems during the 20th century. As a part of the basic theory of nonlinear science, it is one of the very active areas in the new millennium.

This book presents in an elementary way the recent significant developments in the qualitative theory of planar dynamical systems. The subjects are covered as follows: The studies of center and isochronous center problems, multiple Hopf bifurcations and local and global bifurcations of the equivariant planar vector fields which concern with Hilbert's 16th problem.

We are interested in the study of planar vector fields, because they occur very often in applications. Indeed, such equations appear in modelling chemical reactions, population dynamics, travelling wave systems of nonlinear evolution equations in mathematical physics and in many other areas of applied mathematics and mechanics. In the other hand, the study of planar vector fields has itself theoretical signification. We would like to cite Canada's mathematician Dana Schlomiuk's words to explain this fact: "Planar polynomial vector fields and more generally, algebraic differential equations over the projective space are interesting objects of study for their own sake. Indeed, due to their analytic, algebraic and geometric nature they form a fertile soil for intertwining diverse methods, and success in finding solutions to problems in this area depends very much on the capacity we have to blend the diverse aspects into a unified whole."

We emphasize that for the problems of the planar vector fields, many sophisticated tools and theories have been built and still being developed, whose field of application goes far beyond the initial areas. In this book, we only state some important progress in the above directions which have attracted our study interest. The materials of this book are taken mainly from our published results.

This book is divided ten chapters. In Chapter 1 we provide some basic results in the theory of complex analytic autonomous systems. We discuss the normal forms, integrability and linearized problem in a neighborhood of an elementary singular point.

In order to clearly understand the content in Chapter 2∼Chapter 10 for young readers, and to save space in the following chapters, we shall describe in more detail the subjects which are written in this book and give brief survey of the historic literature.

I. Center-focus problem

We consider planar vector fields and their associated differential equations:

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{E}$$

where $X(x, y), Y(x, y)$ are analytic functions or polynomials with real coefficients. If X, Y are polynomials, we call degree of a system (E), the number $n = \max(\deg(X), \deg(Y))$. Without loss of generality, we assume that $X(0, 0) = Y(0, 0) = 0$, i.e., the origin $O(0, 0)$ is a singular point of (E) and the linearization at the origin of (E) has purely imaginary eigenvalues.

The origin $O(0, 0)$ is a *center* of (E) if there exists a neighborhood U of the origin such that every point in U other than $O(0, 0)$ is nonsingular and the orbit passing through the point is closed. In 1885, Poincaré posed the following problem.

The problem of the center Find necessary and sufficient conditions for a planar polynomial differential system (E) of degree m to possess a center.

This problem was solved in the case of quadratic systems by Dulac who proved that all quadratic systems with a center are integrable in finite terms. Actually they could be shown to be Darboux integrable by the method of Darboux by using invariant algebraic curves. Similar results were obtained for some classes of cubic differential systems with a center. Darboux integrability is an important tool, although not the only one. The problem of the center is open for general cubic systems and for higher degrees.

Poincaré considered the above problem. He gave an infinite set of necessary and sufficient conditions for such system to have a center at the origin. In his memoir on the stability of motion, Lyapunov studies systems of differential equations in n variables. When applied to the case $n = 2$, his results also gave an infinite set of necessary and sufficient conditions for system (E) with X, Y polynomials to have a center. (Actually, Lyapunov's result is more general since it is for the case where X and Y are analytic functions). In searching for sufficient conditions for a center, both Poincaré and Lyapunov's work involve the idea of trying to find a constant of the motion $F(x, y)$ for (E) in a neighborhood U of the origin, where

$$F(x, y) = \sum_{k=2}^{\infty} F_k(x, y), \tag{1}$$

F_k is a homogeneous polynomial of order k and F_2 is a positive definite quadratic form. If F is constant on all solution curve $(x(t), y(t))$ in U, we say that F is a first integral on U of system (E). If there exists such an F which is nonconstant on any open subset of U, we say that system (E) is integrable on U.

Poincaré and Lyapunov proved the following theorem.

Poincaré-Lyapunov Theorem The origin of the polynomial (or analytic) system (E) is a center if and only if in an open neighborhood U of the origin, (E) has a nonconstant first integral which is analytic.

Thus, we can construct a power series (1) such that

$$\left.\frac{dF}{dt}\right|_{(E)} = V_3(x^2+y^2)^2 + V_5(x^2+y^2)^3 + \cdots + V_{2k+1}(x^2+y^2)^{k+1} + \cdots \qquad (2)$$

with $V_3, V_5, \cdots, V_{2k+1}, \cdots$ constants. The first non-zero V_{2n+1} give the asymptotic stability or instability of the origin according to its negative or positive sign. Indeed, stopping the series at F_k, we obtain a polynomial which is a Lyapunov function for the system (E). The $V_{2k+1}'s$ are called the *Lyapunov constants.* Some people also use the term *focal values* for them. In fact, Andronov et al defined the focal values by the formula $\nu_i = \Delta^{(i)}(0)/i!$, where $\Delta^{(i)}(\rho_0)$ is the ith-derivative of the function $\Delta(\rho_0) = P(\rho_0) - \rho_0$, P is the Poincaré return map. The first non-zero focal value of Andronov corresponding to an odd number $i = 2n+1$. It had been proved that the first non-zero Lyapunov constant V_{2n+1} differs only by a positive constant factor from the first non-zero focal value, which is $\Delta^{(2n+1)}(0)$. Hence, the identification in the terminology is natural.

In terms of the $V_{2i+1}'s$, the conditions for a center of the origin become $V_{2k+1} = 0$, for all $k = 1, 2, 3, \cdots$. Now $V_3, V_5, \cdots, V_{2k+1}, \cdots$ are polynomial with rational coefficients in the coefficients of $X(x,y)$ and $Y(x,y)$. Theoretically, by using Hilbert's basis theorem, the ideal generated by these polynomials has a finite basis $B_1, B_2, \cdots, B_m$. Hence, we have a finite set of necessary and sufficient conditions for a center, i.e., $B_i = 0$ for $i = 1, 2, \cdots, M$. To calculate this basis, we reduce each V_{2k+1} modulo $\ll V_3, V_5, \cdots, V_{2k-1} \gg$, the ideal generated by $V_3, V_5, \cdots, V_{2k-1}$. The elements of the basis thus obtained are called the Lyapunov quantities or the focal quantities. The origin is said to be an k-order fine focus (or a focus of multiplicity k) of (E) if the fist $k-1$ Lyapunov quantities are 0 but the k-order one is not.

The above statement tell us that the solution of the center-focus for a particular system, the procedure is as follows: Compute several Lyapunov constants and when we get one significant constant that is zero, try to prove that the system obtained indeed has a center. Unfortunately, the described method has the following questions.

(1) How can we be sure that you have computed enough Lyapunov constants?

(2) How do we prove that some system candidate to have a center actually has a center?

(3) Do you know the general construction of Lyapunov constants in order to get general shortened expressions for Lyapunov constants $V_3, V_5, \cdots$.

In Chapter 2 we devote to give possible answer for these questions.

In addition, we shall consider the following two problems.

Problem of center-focus at infinite singular point

A real planar polynomial vector field V can be compactified on the sphere as follows: Consider the x, y plane as being the plane $Z = 1$ in the space $\mathcal{R}^3$ with coordinates X, Y, Z. The center projection of the vector field V on the sphere of radius one yields a diffeomorphic vector field on the upper hemisphere and also another vector field on the lower hemisphere. There exists an analytic vector filed $p(V)$ on the whole sphere such that its restriction on the upper hemisphere has the same phase curves as the one constructed above from the polynomial vector field. The projection of the closed northern hemisphere H^+ of $\mathcal{S}^2$ on $Z = 0$ under $(X, Y, Z) \to (X, Y)$ is called the *Poincaré disc*. A singular point q of $p(V)$ is called an *infinite* (or finite) singular point if $q \in \mathcal{S}^1$ (or $q \in \mathcal{S}^2/\mathcal{S}^1$). The vector field $p(V)$ restricted to the upper hemisphere completed with the equator is called *Poincaré compactification of a polynomial vector field*.

If a real polynomial vector field has no real singular point in the equator Γ_∞ of the *Poincaré disc* and Γ_∞ can be seen a orbit. All orbits in a inner neighborhood of Γ_∞ are spirals or closed orbits, then Γ_∞ is called the equator cycle of the vector field. Γ_∞ can be become a point by using the Bendixson reciprocal radius transformation. This point is called infinity of the system. For infinity, there exists the problem of the characterization of center for concrete families of planar polynomial (or analytic) systems. In Chapter 5, we introduce corresponding research results.

Problem of center-focus at a multiple singular point

The center-focus problem for a multiple (degenerate) singular point is essentially difficult problems. There is only a few results on this direction before 2000. This book shall give some basic results in Chapter 6.

II. Small-amplitude limit cycles created by multiple Hopf bifurcations

So called *Hopf bifurcation*, it means that a differential system exhibits the phenomenon that the appearance of periodic solution (or limit cycle in plane) branching off from an equilibrium point of the system when certain changes of the parameters occur. Hopf's original work on this subject appeared in 1942, in which the author considered higher dimensional (greater than two) systems. Before 1940s, Andronov and his co-workers had done the pioneering work for planar dynamical systems. Bautin showed that for planar quadratic systems at most three small-amplitude limit cycles can bifurcate out of one equilibrium point. By the work of Andronov et al, it is well known that the bifurcation of several limit cycles from a fine focus is directly related with the stability of the focus. The sign of the first nonvanishing Lyapunov constant determines the stability of the focus. Furthermore, the number of the leading $V_{2i+1}{}'s(i = 1, 2, \cdots)$ which vanish simultaneously is the number of limit cycles which may bifurcate from the focus. This is the reason why the investigation of the bifurcation of limit cycles deal with the computation of Lyaponov constants.

The appearance of more than one limit cycles from one equilibrium point is called *multiple Hopf bifurcation.* How these small-amplitude limit cycles can be generated?

The idea is to start with a system (E) for which the origin is a k-th weak focus, then to make a sequence of perturbations of the coefficients of $X(x,y)$ and $Y(x,y)$ each of which reverses the stability of the origin, thereby causing a limit cycle to bifurcate.

In Chapter 3 and Chapter 9 the readers shall see a lot of examples of systems having multiple Hopf bifurcation.

III. Local and non-local bifurcations of Z_q-equivariant perturbed planar Hamiltonian vector fields

The second part of Hilbert's 16th problem deals with the maximum number $H(n)$ and relative positions of limit cycles of a polynomial system

$$\frac{dx}{dt} = P_n(x,y), \qquad \frac{dy}{dt} = Q_n(x,y), \tag{E_n}$$

of degree n, i.e., $\max(\deg P, \deg Q) = n$. Hilbert conjectured that the number of limit cycles of (E_n) is bounded by a number depending only on the degree n of the vector fields.

Without any doubt, the most famous one of the classical problems on planar polynomial vector fields is the second part of Hilbert's 16th problem. This a doubly global problem: It involves the behavior of systems in the whole plane, even at infinity, and this for the whole class of systems defined by polynomials of a fixed degree n. Not only is this problem unsolved even in the case of quadratic systems, i.e. for $n = 2$, but it is still unproved that the uniform upper bound of the numbers of limit cycles occurring in quadratic systems is finite. This in spite of the fact that no one was ever able to construct an example of a quadratic system for which more than four limit cycles can be proven to exist.

Let χ_N be the space of planar vector fields $X = (P_n = \sum_{i+j=0}^{n} a_{ij}x^iy^j, Q_n = \sum_{i+j=0}^{n} b_{ij}x^iy^j)$ with the coefficients $(a_{ij}, b_{ij}) \in B \subset R^N$, for $0 \leqslant i+j \leqslant n, N = (n+1)(n+2)$. The standard procedure in the study of polynomial vector fields is to consider their behavior at infinity by extension to the Poincaré sphere. Thus, we can see (E_n) as an analytic N-parameter family of differential equations on S^2 with the compact base B. Then, the second part of Hilbert's 16th problem may be splited into three parts:

Problem A Prove the finiteness of the number of limit cycles for any concrete system $X \in \chi_N$ (given a particular choice for coefficients of (E_n) i.e,

$$\sharp\{L.C. \quad \text{of} \quad (E_n)\} < \infty.$$

Problem B Prove for every n the existence of an uniformly bounded upper bound for the number of limit cycles on the set B as the function of the parameters, i.e.,

$$\forall n, \forall (a_{ij}, b_{ij}) \in B, \exists H(n) \quad \text{such that} \quad \sharp\{L.C.\ of\ (E_n)\} \leqslant H(n),$$

and find an upper estimate for $H(n)$.

Problem C For every n and known $K = H(n)$, find all possible configurations (or schemes) of limit cycles for every number $K, K-i, i = 1, 2, \cdots, K-1$ respectively.

Hence, the second part of Hilbert's 16th problem consists of **Problems A∼Problem C**.

The Problem A for polynomial and analytic differential equations are already solved by J.Ecalle [1992] and Yu.Ilyashenko [1991] independently. Of course, as S. Small stated that "These two papers have yet to be thoroughly digested by mathematical community".

Up to now, there is no approach to the solution of the Problem B, even for $n = 2$, which seem to be very complicated. But there exists a similar problem, which seems to be a little bit easier. It is the weakened Hilbert's 16th problem proposed by Arnold [1977]:

"Let H be a real polynomial of degree n and let P be a real polynomial of degree m in the variables (x, y). How many real zeroes can the function

$$I(h) = \iint_{H \leqslant h} P dx dy$$

have ? "

The question is why zeroes of the Abelian integrals $I(h)$ is concerned with the second part of Hilbert's 16th problem ?

Let $H(x, y)$ be a real polynomial of degree n, and let $P(x, y)$ and $Q(x, y)$ be real polynomials of degree m. We consider a perturbed Hamiltonian system in the form

$$\frac{dx}{dt} = \frac{\partial H}{\partial y} + \varepsilon P(x, y, \lambda), \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x} + \varepsilon Q(x, y, \lambda), \qquad (E_H)$$

in which we assume that $0 < \varepsilon \ll 1$ and the level curves

$$H(x, y) = h$$

of the Hamiltonian system $(E_H)_{\varepsilon=0}$ contain at least a family Γ_h of closed orbits for $h \in (h_1, h_2)$.

Consider the Abelian integrals

$$I(h) = \int_{\Gamma_h} P dy - Q dx = \iint_{H \leqslant h} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy.$$

Poincaré-Pontrjagin-Andronov Theorem on the global center bifurcation

The following statements hold.

(i) If $I(h^*) = 0$ and $I'(h^*) \neq 0$, then there exists a hyperbolic limit cycle L_{h^*} of system (6.1) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$; and conversely, if there exists a hyperbolic limit cycle L_{h^*} of system (E_H) such that $L_{h^*} \to \Gamma_{h^*}$ as $\varepsilon \to 0$, then $I(h^*) = 0$, where $h^* \in (h_1, h_2)$.

(ii) If $I(h^*) = I'(h^*) = I''(h^*) = \cdots = I^{(k-1)}(h^*) = 0$, and $I^{(k)}(h^*) \neq 0$, then (E_H) has at most k limit cycles for ε sufficiently small in the vicinity of Γ_{h^*}.

(iii) The total number of isolated zeroes of the Abelian integral (taking into account their multiplicity) is an upper bound for the number of limit cycles of system

(E_H) that bifurcate from the periodic orbits of a period annulus of Hamiltonian system $(E_H)_{\varepsilon=0}$.

This theorem tells us that the weakened Hilbert's 16th problem posed by Arnold [1977] is closely related to the problem of determining an upper bound $N(n, m) = N(n, m, H, P, Q)$ for the number of limit cycles in a period annulus for the Hamiltonian system of degree $n-1$ under the perturbations of degree m, i.e., of determining the cyclicity on a period annulus. Since the problem is concerned with the number of limit cycles that occur in systems which are close to integrable ones (only a class of subsystems of all polynomial systems). So that it is called the weakened Hilbert's 16th problem.

A closed orbit Γ_{h^*} satisfying the above theorem (i) is called a generating cycle.

To obtain Poincaré-Pontrjagin-Andronov Theorem, the problem for investigating the bifurcated limit cycles is based on the Poincaré return mapping. It is reduced to counting the number of zeroes of the displacement function

$$d(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots + \varepsilon^k M_k(h) + \cdots ,$$

where $d(h, \varepsilon)$ is defined on a section to the flow, which is parameterized by the Hamiltonian value h. $I(h)$ just is equal to $M_1(h)$. The function $M_k(h)$ is called k-order Melnikov function. If $I(h) = M_1(h) \equiv 0$, we need to estimate the number of zeroes of higher order Melnikov functions. The zeroes of the first nonvanishing Melnikov function $M_k(h)$ determine the limit cycles in (E_H) emerging from periodic orbits of the Hamiltonian system $(E_H)_\varepsilon$.

In Chapter 8, we discuss a class of particular polynomial vector fields: Z_q-equivariant perturbed planar Hamiltonian vector fields, by using Poincaré-Pontrjagin-Andronov Theorem and Melnikov's result. The aim is to get some information for the studies of the second part of Hilbert's 16th problem.

IV. Isochronous center problem and periodic map

Suppose that system (E) has a center in the origin $(0, 0)$. Then, there is a family of periodic orbits of (E) enclosing the origin. The largest neighborhood of the center entirely covered by periodic orbits is called a *period annulus* of the center. If the period of the orbits is constant for all periodic orbits lying in the period annulus of the origin, then the center $(0, 0)$ is called an *isochroous center*. It has been proved that the isochronous center can exist if the period annulus of the center is unbounded.

If the origin is not an isochronous center, for a point $(\xi, 0)$ in a small neighborhood of the origin $(0, 0)$, we define $P(\xi)$ to be the minimum period of the periodic orbit passing through $(\xi, 0)$. The study for the period function $\xi \to P(\xi)$ is also very interesting problem, since monotonicity of the period function is a non-degeneracy condition for the bifurcation of subharmonic solutions of periodically forced integrable systems.

The history of the work on period functions goes back at least to 1673 when C. Huygens observed that the pendulum clock has a monotone period function and therefore oscillates with a shorter period when the energy is decreased, i.e., as the clock spring unwinds. He hope to design a clock with isochronous oscillations in order to have a more accurate clock to be used in the navigation of ships. His solution, the cycloidal pendulum, is perhaps the first example of nonlinear isochronous center.

In the last three decades of the 20th century, a considerable number of papers of the study for isochronous centers and period maps has been published. But, for a given polynomial vector field of the degree is more than two, the characterization of isochronous center is still a very difficult, challenging and unsolved problem.

In Chapter 4, we introduce some new method to treat these problems.

Except the mentioned seven chapters, we add three chapters to introduce our more recent study results.

In Chapter 7, we consider a class of nonanalytic systems which is called "quasi-analytic systems". We completely solve its center and isochronous center problems as well as the bifurcation of limit cycles.

In Chapter 8, as an example, for a class of Z_2-symmetric cubic systems, we give the complete answer for the center problem and the bifurcations of limit cycles. We prove that this class of cubic systems has at least 13 limit cycles.

In the final chapter (Chapter 10), we study the center-focus problem and bifurcations of limit cycles for three-multiple nilpotent singular points. The materials are taken by our recent new papers.

We would like to cite the following words written by Anna Schlomiuk in 2004 as the finale of this preface: "Planar polynomial vector fields are dynamical systems but to perceive them uniquely from this angle is limiting, missing part of their essence and hampering development of their theory. Indeed, as dynamical systems they are very special systems and the prevalent generic viewpoint pushes them on the side. This may explain in part why Hilbert's 16th problem as well as other problems are still unsolved even in their simple case, the quadratic one. But, Poincaré's work shows that he regarded these systems as interesting object of study from several viewpoints, and his appreciation of the work of Darboux which he qualifies as 'admirable' emphasizes this point. This area is rich with problems, very hard, it is true, but exactly for this reason an open mind and a free flow of ideas is necessary. It is to be hoped that in the future there will be a better understanding of this area which lies at a crossroads of dynamical systems, algebra, geometry and where algebraic and geometric problems go hand in hand with those of dynamical systems."

The book is intended for graduate students, post-doctors and researchers in dynamical systems. For all engineers who are interested the theory of dynamical systems, it is also a reasonable reference. It requires a minimum background of an one-year course on nonlinear differential equations.

The publication of this book is supported by the research foundation of the Center for Dynamical Systems and Nonlinear Scienu Science Studies given by Zhejiang Normal University. The work described in this book is supported by the grants from the National Natural Science Foundation of China (11371373,11071222 and 11261013). The third author would like to thank the support by Guangxi Key Laboratory of Trusted Software in Guilin University of Electronic Technology and Guangxi Education Department Key Laboratory of Symbolic Computation and Engineering Data Processing in Hezhou University.

Finally, we would like to acknowledge all members of the Center for Dynamical Systems and Nonlinear Studies of Zhejiang Normal University for their encouragement and help. We thank Mr.Yanchao Zhao of the Science Press of Chinese Academy of Sciences for their help in the publication of this book. We also thank the staff at De Gruyter for their interesting to publish this book and for their useful suggestions to improve the content of this book.

Jibin Li
The Center for Dynamical Systems and
Nonlinear Science Studies, Zhejiang Normal
University, Jinhua, Zhejiang, 321004
June 2013

Contents

Chapter 1

Basic Concept and Linearized Problem of Systems

In this chapter, we discuss the normal forms, integrability and linearized problem for the analytic autonomous differential system of two variables in a neighborhood of an elementary singular point. In addition, we give the definition of the multiplicity for a multiple singular point and study the quasi-algebraic integrals for some polynomial systems.

1.1 Basic Concept and Variable Transformation

Consider the following two-order differential equations

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y). \tag{1.1.1}$$

When x, y, t are real variables, $X(x, y), Y(x, y)$ are real functions of x and y, we say that system (1.1.1) is a real autonomous planar differential system, or (X, Y) is a real planar vector field. When x, y, t are complex variables and $X(x, y), Y(x, y)$ are complex functions of x and y, we say that system (1.1.1) is a two-order complex autonomous differential system.

When the functions $X(x, y), Y(x, y)$ are two polynomials of x and y of degree n, system (1.1.1) is called a polynomial system of degree n. It is often represented by (E_n).

If the functions $X(x, y), Y(x, y)$ can be expanded as a power series of $x - x_0$ and $y - y_0$ in a neighborhood of the point (x_0, y_0) with non-zero convergent radius, then system(1.1.1) is called an analytic system in a neighborhood of (x_0, y_0).

If system (1.1.1) is real and analytic in a neighborhood of (x_0, y_0), then, we can see x, y, t as complex variables in this small neighborhood to extend system (1.1.1) to complex field.

We next assume that $X(x, y), Y(x, y), F(x, y)$ is continuously differentiable in a region $\mathcal{D}$ of the (x, y) real plane (or (x, y) complex space).

Along the orbits of system (1.1.1), the total derivative of F in $\mathcal{D}$ is given by

$$\left.\frac{dF}{dt}\right|_{(1.1.1)} = \frac{\partial F}{\partial x}X + \frac{\partial F}{\partial y}Y. \tag{1.1.2}$$

If $F(x, y)$ is not constant function and $\left.\frac{dF}{dt}\right|_{(1.1.1)} \equiv 0$ in $\mathcal{D}$, then F is called a first integral of (1.1.1) in $\mathcal{D}$.

If $M(x, y)$ is a non-zero function in $\mathcal{D}$ and

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MY)}{\partial y} \equiv 0 \tag{1.1.3}$$

in $\mathcal{D}$, we say that M is an integral factor of (1.1.1) in $\mathcal{D}$. On the other hand, if M^{-1} is an integral factor of (1.1.1) in $\mathcal{D}$, we say that M is an inverse integral factor of (1.1.1) in $\mathcal{D}$.

If $X(x_0, y_0) = Y(x_0, y_0) = 0$, the point (x_0, y_0) is called a singular point or equilibrium point of (1.1.1). Otherwise, (x_0, y_0) is called an ordinary point of (1.1.1).

When (x_0, y_0) is a singular point of (1.1.1) and x_0, y_0 are real, we say that (x_0, y_0) is a real singular point. Otherwise, (x_0, y_0) is a complex singular point.

If (x_0, y_0) is a unique singular point in a neighborhood of the singular point (x_0, y_0) of (1.1.1), we say that (x_0, y_0) is an isolated singular point of (1.1.1). In this case, if

$$\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial x}\frac{\partial X}{\partial y}\right)_{x=x_0, y=y_0} \neq 0, \tag{1.1.4}$$

then, (x_0, y_0) is called an elementary singular point. Otherwise, it called a multiple singular point.

Suppose that the functions

$$u = \varphi(x, y), \quad v = \psi(x, y) \tag{1.1.5}$$

are continuously differentiable in $\mathcal{D}$ and when $(x, y) \in \mathcal{D}$, $(u, v) \in \mathcal{D}'$. Write that

$$J_1 = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}, \quad J_2 = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}. \tag{1.1.6}$$

If for all $(x, y) \in \mathcal{D}$, J_1 is bounded and $J_1 \neq 0$, we say that (1.1.5) is a non-singular transformation in $\mathcal{D}$.

Clearly, if (1.1.5) is non-singular in $\mathcal{D}$, then, for all $(x, y) \in \mathcal{D}$, $J_1 J_2 \equiv 1$.

It is easy to prove the following two conclusions by using the chain rule.

Proposition 1.1.1. *Suppose that the functions* $X(x,y), Y(x,y), M(x,y)$ *are continuously differentiable in* $\mathcal{D}$ *and* (1.1.5) *is a non-singular transformation. Under* (1.1.5)*, system* (1.1.1) *becomes*

$$\frac{du}{dt} = U(u,v), \quad \frac{dv}{dt} = V(u,v). \tag{1.1.7}$$

Then, for all $(x,y) \in \mathcal{D}$ *and any continuously differentiable function* $F(x,y)$*,*

$$\left.\frac{dF}{dt}\right|_{(1.1.1)} = \left.\frac{dF}{dt}\right|_{(1.1.7)}. \tag{1.1.8}$$

Proposition 1.1.2. *Under the conditions of Proposition* 1.1.1*, in addition, if the functions* $\varphi(x,y), \psi(x,y)$ *are two-order differentiable in* $\mathcal{D}$*, then, for all* $(x,y) \in \mathcal{D}$*,*

$$\frac{\partial(J_1X)}{\partial x} + \frac{\partial(J_1Y)}{\partial y} = J_1\left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right), \tag{1.1.9}$$

$$\frac{\partial(J_2U)}{\partial u} + \frac{\partial(J_2V)}{\partial v} = J_2\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right). \tag{1.1.10}$$

For system

$$\frac{dx}{dt} = M(x,y)X(x,y), \quad \frac{dy}{dt} = M(x,y)Y(x,y) \tag{1.1.11}$$

by Proposition 1.1.2, we have

Proposition 1.1.3. *Under the conditions of Propositions* 1.1.1 *and* 1.1.2*, for all continuously differentiable function* $M(x,y)$ *in* $(x,y) \in \mathcal{D}$*,*

$$\frac{\partial(J_1MX)}{\partial x} + \frac{\partial(J_1MY)}{\partial y} = J_1\left[\frac{\partial(MU)}{\partial u} + \frac{\partial(MV)}{\partial v}\right], \tag{1.1.12}$$

$$\frac{\partial(J_2MU)}{\partial u} + \frac{\partial(J_2MV)}{\partial v} = J_2\left[\frac{\partial(MX)}{\partial x} + \frac{\partial(MX)}{\partial y}\right]. \tag{1.1.13}$$

By the above proposition, we obtain

Proposition 1.1.4. *Under the conditions of Propositions* 1.1.1 *and* 1.1.2*, If* $M(x,y)$ *is an integral factor of* (1.1.1) *in* $\mathcal{D}$ *and* $M(x,y)$ *is continuously differentiable, then,* J_2M *is an integral factor of* (1.1.7) *in* $\mathcal{D}'$*.*

1.2 Resultant of the Weierstrass Polynomial and Multiplicity of a Singular Point

In this section, we first study the resultant of Weierstrass polynomials. By using their properties, we give the definition of multiplicity of singular points. For a multiple singular point, we investigate its division and composition from some simple singular points.

Suppose that

$$\begin{aligned} P(x,y) &= \varphi_0(x)y^m + \varphi_1(x)y^{m-1} + \cdots + \varphi_m(x), \\ Q(x,y) &= \psi_0(x)y^n + \psi_1(x)y^{n-1} + \cdots + \psi_n(x), \end{aligned} \tag{1.2.1}$$

are two polynomials of y, where m and n are two positive integers, $\varphi_k(x), \psi_k(x)$ are power series of x with non-zero convergent radius, x and y are complex variables. In addition, $\varphi_0(x)\psi_0(x)$ is not identically vanishing,

Definition 1.2.1. *The following $(n+m)$-order determinant*

$$\mathrm{Res}(P,Q,y) = \begin{vmatrix} \varphi_0 & \varphi_1 & \cdots & \cdots & \cdots & \varphi_m & & & & \\ & \varphi_0 & \varphi_1 & \cdots & \cdots & \cdots & \varphi_m & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \varphi_0 & \varphi_1 & \cdots & \cdots & \cdots & \varphi_m \\ \psi_0 & \psi_1 & \cdots & \cdots & \psi_n & & & & \\ & \psi_0 & \psi_1 & \cdots & \cdots & \psi_n & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & & \psi_0 & \psi_1 & \cdots & \cdots & \psi_n \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \end{matrix} \tag{1.2.2}$$

is called the resultant of $P(x,y)$ and $Q(x,y)$ with respect to y.

Definition 1.2.2. *Write that*

$$H(x,y) = y^m + h_1(x)y^{m-1} + h_2(x)y^{m-2} + \cdots + h_m(x), \tag{1.2.3}$$

where m is a positive integer, $h_k(x), k = 1, \cdots, m$, are power series of x with non-zero convergent radius. If

$$h_1(0) = h_2(0) = \cdots = h_m(0) = 0, \tag{1.2.4}$$

we say that $H(x,y)$ is a Weierstrass polynomial of degree m of y.

Definition 1.2.3. *Let $U(x,y)$ be a power series of x,y with a non-zero convergent radius and $U(0,0) = 1$. We say that $U(x,y)$ is an unitary power series of x,y.*

Definition 1.2.4. *Let $f(z)$ be a power series of z with a non-zero convergent radius, q be a positive integer. If $f(0) = 0$, we say that $x = 0$ is an algebraic zero of the function $f(x^{\frac{1}{q}})$. If there is a positive integer p, such that $f(x) = c_p x^p + o(x^p)$ and $c_p \neq 0$. Then, $c_p x^{\frac{p}{q}}$ is called the first term of $f(x^{\frac{1}{q}})$.*

By the theory of the algebraic curves, we know that

Theorem 1.2.1. *Let*

$$P(x,y)=\varphi_0(x)\prod_{k=1}^{m}(y-f_k(x)),\quad Q(x,y)=\psi_0(x)\prod_{j=1}^{n}(y-g_j(x)). \tag{1.2.5}$$

Then,

$$\begin{aligned}\mathrm{Res}(P,Q,y)&=\varphi_0^n(x)\psi_n^m(x)\prod_{k=1}^{m}\prod_{j=1}^{n}(f_k(x)-g_j(x))\\&=\varphi_0^n(x)\prod_{k=1}^{m}Q(x,f_k(x))=(-1)^{mn}\psi_0^m(x)\prod_{j=1}^{n}P(x,g_j(x)).\end{aligned} \tag{1.2.6}$$

Theorem 1.2.2. *Let $A_k(x)$ be the algebraic cofactor of* (1.2.2) *with the k-row and the $(m+n)$-column, $B_j(x)$ be the algebraic cofactor with $(n+j)$-row and $(m+n)$-column, $k=1,2,\cdots,n$, $j=1,2,\cdots,m$. Then*

$$\mathrm{Res}(P,Q,y)=A(x,y)P(x,y)+B(x,y)Q(x,y), \tag{1.2.7}$$

where

$$\begin{aligned}A(x,y)&=A_1(x)y^{n-1}+A_2(x)y^{n-2}+\cdots+A_n(x),\\B(x,y)&=B_1(x)y^{m-1}+B_2(x)y^{m-2}+\cdots+B_m(x).\end{aligned} \tag{1.2.8}$$

Theorem 1.2.3 (Weierstrass preparatory theorem). *Let $F(x,y)$ be a power series of x,y with a non-zero convergent radius. If there exists a positive integer m, such that*

$$F(0,y)=c_my^m+h.o.t.,\quad c_m\neq 0, \tag{1.2.9}$$

where h.o.t. stand for the high order terms. Then, there is a unique Weierstrass polynomial $H(x,y)$ of y with the degree m and an unitary power series $U(x,y)$, such that in a small neighborhood of the origin

$$F(x,y)=c_mH(x,y)U(x,y). \tag{1.2.10}$$

Theorem 1.2.4. *Under the conditions of Theorem* 1.2.3, *there exist two positive number σ_1 and σ_2, such that when $|x|<\sigma_1$, $F(x,y)$ as a function of y, it has exactly m complex zeros $y=f_k(x)$ inside the disk $|y|<\sigma_2$, where $f_k(0)=0$ and $x=0$ is an algebraic zero of $f_k(x)$, $k=1,2,\cdots,m$.*

Corollary 1.2.1. *If $H(x,y)$ is a Weierstrass polynomial of y with the degree m, then, there exist m functions $f_1(x),f_2(x),\cdots,f_m(x)$, for which $x=0$ is their algebraic zero, such that in a small neighborhood of the origin,*

$$H(x,y)\equiv\prod_{k=1}^{m}(y-f_k(x)). \tag{1.2.11}$$

Corollary 1.2.2. *Let $F(x,y)$ be a power series of x,y with a non-zero convergent radius. If there exist an integer m, such that*

$$F(0,0)=\frac{\partial F(0,0)}{\partial y}=\cdots=\frac{\partial F^{m-1}(0,0)}{\partial y^{m-1}}=0,\quad \frac{\partial F^{m}(0,0)}{\partial y^{m}}\neq 0. \tag{1.2.12}$$

Then, the implicit function equation

$$F(x,y)=0,\quad y|_{x=0}=0 \tag{1.2.13}$$

has exactly m complex solutions $y=f_k(x)$ in a small neighborhood of the origin and $x=0$ is an algebraic zero of $f_k(x), k=1,2,\cdots,m$.

We know consider the multiplicity of singular point for the complex autonomous differential system:

$$\frac{dx}{dt}=F_1(x,y),\quad \frac{dy}{dt}=F_2(x,y), \tag{1.2.14}$$

where $F_1(x,y)$ and $F_2(x,y)$ are power series of x,y with a non-zero convergent radius, $F_1(0,0)=F_2(0,0)=0$. Suppose that $O(0,0)$ is an isolated singular point of (1.2.14).

Without loss of generality, we assume that $F_1(0,y)\neq 0$. Hence, there is an integer m, such that

$$F_1(0,y)=a_m y^m+h.o.t.,\quad a_m\neq 0. \tag{1.2.15}$$

By Theorem 1.2.3 and Corollary 1.2.1, in a small neighborhood of the origin, $F_1(x,y)$ has the form as follows:

$$F_1(x,y)=a_m H_1(x,y)U_1(x,y)=a_m U_1(x,y)\prod_{k=1}^{m}(y-f_k(x)), \tag{1.2.16}$$

where

$$H_1(x,y)=\prod_{k=1}^{m}(y-f_k(x)) \tag{1.2.17}$$

is a Weierstrass polynomial of y of degree m. $x=0$ is an algebraic zero of $f_k(x)$. $U_1(x,y)$ is a unitary power series of x,y.

Consider the function

$$R(x)=\prod_{k=1}^{m}F_2(x,f_k(x)). \tag{1.2.18}$$

Lemma 1.2.1. *Let the origin be an isolated singular point of* (1.2.14). *Then there is an integer $N>0$, such that*

$$R(x)=Ax^N+o(x^N),\quad A\neq 0. \tag{1.2.19}$$

Proof. We can write that

$$F_2(x,y) = x^s \tilde{F}_2(x,y), \tag{1.2.20}$$

where $\tilde{F}_2(x,y)$ is a power series of x, y with a non-zero convergent radius and $\tilde{F}_2(0,y)$ is not identically vanishing. Consider the following two cases.

1. Suppose that $\tilde{F}_2(0,0) \neq 0$. Since $F_2(0,0) = 0$, so that, s is a positive integer. (1.2.18) and (1.2.20) follow that

$$R(x) = \tilde{F}_2^m(0,0)x^{sm} + o(x^{sm}). \tag{1.2.21}$$

By (1.2.21), when $\tilde{F}_2(0,0) \neq 0$, the conclusion of Lemma 1.2.1 holds.

2. Suppose that $\tilde{F}_2(0,0) = 0$. Since $\tilde{F}_2(0,y)$ is a non-zero function, thus, there is an integer $n > 0$, such that

$$\tilde{F}_2(0,y) = b_n x^n + h.o.t., \quad b_n \neq 0. \tag{1.2.22}$$

By Theorem 1.2.3 and Corollary 1.2.1, in a small neighborhood of the origin, $\tilde{F}_2(x,y)$ can be written as

$$\tilde{F}_2(x,y) = b_n H_2(x,y) U_2(x,y). \tag{1.2.23}$$

where $H_2(x,y)$ is a Weierstrass polynomial of y with the degree n and $U_2(x,y)$ is a unitary power series of x, y. By (1.2.18), (1.2.20) and (1.2.23), we know that

$$R(x) = b_n^m x^{sm} M(x) \prod_{k=1}^{m} H_2(x, f_k(x)), \tag{1.2.24}$$

where

$$M(x) = \prod_{k=1}^{m} U_2(x, f_k(x)) = 1 + o(1). \tag{1.2.25}$$

By Theorem 1.2.1,

$$\prod_{k=1}^{m} H_2(x, f_k(x)) = \text{Res}(H_1, H_2, y) \tag{1.2.26}$$

is a power series of x with a non-zero convergent radius. Because the origin is an isolated singular point of (1.2.14), it follows that $\text{Res}(H_1, H_2, y)$ is not zero function. By (1.2.24), (1.2.25) and (1.2.26), when $\tilde{F}_2(0,0) = 0$, the conclusion of Lemma 1.2.1 is also holds. □

Definition 1.2.5. *Suppose that the origin is an isolated singular point of* (1.2.14). $F_1(x,y)$ *is given by* (1.2.16), *where for all* $k, x = 0$ *is an algebraic zero of* $f_k(x)$ *and* $U_1(x,y)$ *is a unitary power series of* x, y, $a_m \neq 0$. *If there is a positive integer* N, *such that*

$$R(x) = \prod_{k=1}^{m} F_2(x, f_k(x)) = Ax^N + o(x^N), \quad A \neq 0, \tag{1.2.27}$$

then the origin is called a N*-multiple singular point of* (1.2.14). N *is called the multiplicity of the origin.*

In the theory of algebraic curves, there is the definition of the crossing number of two curves. We see from Definition 1.2.5 that if the origin is an isolated singular point of (1.2.14) and $F_1(x, y), F_2(x, y)$ are polynomial of x, y, then, the multiplicity of the origin is the same as the crossing number of the two curves of $F_1(x, y) = 0$ and $F_2(x, y) = 0$.

How to determine the multiplicity of a singular point? The following examples give some results.

Consider the following autonomous analytic system in a neighborhood of the origin:

$$\frac{dx}{dt} = \sum_{k=m}^{\infty} X_k(x, y) = X(x, y),$$
$$\frac{dy}{dt} = \sum_{k=n}^{\infty} Y_k(x, y) = Y(x, y), \tag{1.2.28}$$

where m, n are integers, for all k, $X_k(x, y), Y_k(x, y)$ are homogeneous polynomials of degree k of x, y. In addition, $X_m(x, y)Y_n(x, y)$ is not identically vanishing.

Theorem 1.2.5. *If $X_m(x, y)$ and $Y_n(x, y)$ are irreducible, then the origin is a mn-multiple singular point of* (1.2.28).

Proof. Since $Y_n(s, 1)$ is a polynomial of degree n, so that, there is a complex number s such that $Y_n(s, 1) \neq 0$. By the transformation

$$\xi = x - sy, \quad \eta = y \tag{1.2.29}$$

(1.2.28) becomes

$$\frac{d\xi}{dt} = X(\xi + s\eta, \eta) - sY(\xi + s\eta, \eta) = \tilde{X}(\xi, \eta),$$
$$\frac{d\xi}{dt} = Y(\xi + s\eta, \eta) = \sum_{k=n}^{\infty} \tilde{Y}_k(\xi, \eta) = \tilde{Y}(\xi, \eta). \tag{1.2.30}$$

Notice that $Y(0, \eta) = Y_n(s, 1)\eta^n + o(\eta^n)$. We can write

$$\tilde{Y}_n(\xi, \eta) = Y_n(s, 1) \prod_{k=1}^{n} (\eta - \lambda_k \xi). \tag{1.2.31}$$

By Corollary 1.2.2 and (1.2.30), in a neighborhood of the origin, the implicit function equation

$$\tilde{Y}(\xi, \eta) = 0, \quad \eta|_{\xi=0} = 0 \tag{1.2.32}$$

has exact n solutions $\eta = f_k(\xi) = \lambda_k \xi + o(\xi), \quad k = 1, 2, \cdots, n$. Denote $\tilde{X}_m(\xi, \eta) = X_m(\xi + s\eta, \eta)$, then

$$\begin{aligned}\prod_{k=1}^{n} \tilde{X}(\xi, f_k(\xi)) &= \prod_{k=1}^{n} X(\xi + s f_k(\xi), f_k(\xi)) \\ &= \prod_{k=1}^{n} X_m(\xi + s\lambda_k \xi, \lambda_k \xi) + o(\xi^{mn}) \\ &= \prod_{k=1}^{n} \tilde{X}_m(\xi, \lambda_k \xi) + o(\xi^{mn}) \\ &= \prod_{k=1}^{n} \tilde{X}_m(1, \lambda_k)\xi^{mn} + o(\xi^{mn}). \end{aligned} \tag{1.2.33}$$

By the irreducibility of $X_m(x, y)$ and $Y_n(x, y)$, we see from (1.2.31) that

$$\prod_{k=1}^{n} \tilde{X}_m(1, \lambda_k) \neq 0. \tag{1.2.34}$$

Thus, (1.2.33) follows the conclusion of Theorem 1.2.5. □

Obviously, by this theorem, the multiplicity of an elementary singular point is 1. In addition, by Definition 1.2.5, we have

Theorem 1.2.6. *Suppose that*

$$F_1(x, y) = ax + by + h.o.t., \quad F_2(x, y) = cx + dy + h.o.t., \quad b \neq 0. \tag{1.2.35}$$

If $y = f(x)$ *is the unique solution of the equation*

$$F_1(x, y) = 0, \quad y|_{x=0} = 0, \tag{1.2.36}$$

then, when $F_2(x, f(x)) \equiv 0$*, the origin is not a isolated singular point of* (1.2.14). *When* $F_2(x, f(x)) = Ax^N + o(x^N)$*, where* N *is a positive integer,* $A \neq 0$*, the origin is a* N*-multiple singular point of* (1.2.14).

Consider the polynomial system

$$\begin{aligned}\frac{dx}{dt} &= \sum_{k=1}^{m} X_k(x, y) = \mathcal{X}_m(x, y), \\ \frac{dy}{dt} &= \sum_{k=1}^{n} Y_k(x, y) = \mathcal{Y}_n(x, y), \end{aligned} \tag{1.2.37}$$

where m, n are positive integers, $X_k(x, y), Y_k(x, y)$ are homogeneous polynomials of degree k. $\mathcal{X}_m(x, y)$ and $\mathcal{Y}_n(x, y)$ are irreducible.

By Definition 1.2.5 and Bezout theorem in the theory of algebraic curves, we obtain

Theorem 1.2.7. *The sum of multiplicities of all finite singular points of* (1.2.37) *is less than mn or equals to mn.*

In addition, we have

Theorem 1.2.8. *If*

$$X_m(o,y) = ay^m, \quad Y_n(0,y) = by^n, \quad ab \neq 0 \tag{1.2.38}$$

and $\mathrm{Res}(\mathcal{X}_m, \mathcal{Y}_m, y) = Ax^N + o(x^N)$, $A \neq 0$, *then, the origin of* (1.2.37) *is a N-multiple singular point.*

Finally, we investigate the division and composition of the singular points. Take $m = n$. We consider the perturbed system of (1.2.37):

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k=1}^{n} X_k(x,y) + \Phi_n(x,y,\varepsilon), \\ \frac{dy}{dt} &= \sum_{k=1}^{n} Y_k(x,y) + \Psi_n(x,y,\varepsilon). \end{aligned} \tag{1.2.39}$$

where

$$\Phi_n(x,y,\varepsilon) = \sum_{k+j=0}^{n} \varepsilon_{kj} x^k y^j, \quad \Psi_n(x,y,\varepsilon) = \sum_{k+j=0}^{n} \varepsilon'_{kj} x^k y^j, \tag{1.2.40}$$

for all k, j, $\varepsilon_{kj}, \varepsilon'_{kj}$ are small parameters. ε stands for a vector consisting of all $\varepsilon_{kj}, \varepsilon'_{kj}$.

We have the following conclusions.

Theorem 1.2.9. *Suppose that when $\varepsilon = 0$, two functions of the right hands of* (1.2.39) *are irreducible and the origin is a N-multiple singular point of* $(1.2.39)_{\varepsilon=0}$. *Then, there exist two positive numbers r_0 and ε_0, such that when $|\varepsilon| < \varepsilon_0$, the sum of multiplicities of all singular points of* (1.2.39) *in the region $|x| < r_0, |y| < r_0$ is exact N. In addition, the coordinates of these singular points are continuous functions of ε. When $\varepsilon \to 0$, these singular points attend to the origin.*

By choosing the parameters of $\Phi_n(x,y,\varepsilon)$ and $\Psi_n(x,y,\varepsilon)$, system (1.2.39) *can have exactly N complex elementary singular points.*

1.3 Quasi-Algebraic Integrals of Polynomial Systems

In this section, we consider the polynomial system of degree n as follows:

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k=0}^{n} X_k(x,y) = \mathcal{X}_n(x,y), \\ \frac{dy}{dt} &= \sum_{k=0}^{n} Y_k(x,y) = \mathcal{Y}_n(x,y). \end{aligned} \tag{1.3.1}$$

In [Darboux, 1878], the author first studied systematically the invariant algebraic curve solutions of (1.3.1) and gave a method to construct first integrals and integral factors of (1.3.1), by using finitely many invariant algebraic curve solutions.

Definition 1.3.1. *Let $f(x,y)$ be a nonconstant polynomial of degree m. If there is a bounded function $h(x,y)$, such that*

$$\left.\frac{df}{dt}\right|_{(1.3.1)} = \frac{\partial f}{\partial x}\mathcal{X}_n(x,y) + \frac{\partial f}{\partial y}\mathcal{Y}_n(x,y) = h(x,y)f(x,y), \tag{1.3.2}$$

then, $f = 0$ is called an algebraic curve solution of (1.3.1). *The polynomial f is called an algebraic integral of* (1.3.1). *The function h is called a cofactor of f.*

Clearly, we see from (1.3.2) that the following conclusion holds.

Proposition 1.3.1. *If f is an algebraic integral of* (1.3.1), *then the cofactor h of f is a polynomial of degree at most $n-1$.*

In [Liu Y.R. etc, 1995], the authors developed Darboux's results to that f is not polynomial. They defined a quasi-algebraic integral of (1.3.1). We next introduce their main conclusions.

Definition 1.3.2. *Suppose that $f(x,y)$ is a continuously differentiable nonconstant function in a region $\mathcal{D}$. If there is a polynomial $h(x,y)$ of degree at most $n-1$, such that* (1.3.2) *holds in $\mathcal{D}$. We say that $f(x,y)$ is a quasi-algebraic integral of* (1.3.1) *in $\mathcal{D}$.*

It is easy to see that an algebraic integral must be a quasi-algebraic integral.

Example 1.3.1. *Obviously, the quintic system*

$$\begin{aligned}\frac{dx}{dt} &= -y + x(x^2+y^2-1)^2,\\ \frac{dy}{dt} &= x + y(x^2+y^2-1)^2\end{aligned} \tag{1.3.3}$$

has the following quasi-algebraic integrals:

$$\begin{aligned} &f_1 = x^2+y^2-1, && f_2 = e^{\frac{1}{x^2+y^2-1}},\\ &f_3 = x^2+y^2, && f_4 = e^{\arctan\frac{y}{x}},\\ &f_5 = x+iy, && f_6 = x-iy. \end{aligned} \tag{1.3.4}$$

They satisfy

$$\left.\frac{df_k}{dt}\right|_{(1.3.3)} = h_k(x,y)f_k(x,y), \quad k = 1,2,3,4,5,6, \tag{1.3.5}$$

where for all $k = 1-6$, $h_k(x,y)$ are as follows:

$$\begin{aligned} &h_1 = 2(x^2+y^2)(x^2+y^2-1), && h_2 = -2(x^2+y^2),\\ &h_3 = 2(x^2+y^2-1)^2, && h_4 = 1,\\ &h_5 = (x^2+y^2-1)^2+i, && h_6 = (x^2+y^2-1)^2-i. \end{aligned} \tag{1.3.6}$$

Because

$$h_1 + h_2 - h_3 + 2h_4 = 0, \tag{1.3.7}$$

by (1.3.5) *and* (1.3.7), (1.3.3) *has the first integral*

$$F = f_1 f_2 f_3^{-1} f_4^2 = constant, \tag{1.3.8}$$

which satisfies

$$\left.\frac{dF}{dt}\right|_{(1.3.3)} = (h_1 + h_2 - h_3 + 2h_4)F = 0. \tag{1.3.9}$$

Definition 1.3.3. *Let f_1, f_2, $\cdots$, f_m be m quasi-algebraic integrals of* (1.3.1). *If there exists a group of complex number α_1, α_2, $\cdots$, α_m, such that $f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is identically equal a constant, then, f_1, f_2, $\cdots$, f_m is called dependent. Otherwise, f_1, f_2, $\cdots$, f_m is called independent.*

For example, in (1.3.4), we have that $f_5 f_6 = f_3$ and $f_5 f_6^{-1} = f_4^{2i}$, hence, f_3, f_5, f_6 and f_4, f_5, f_6 are dependent, respectively.

Theorem 1.3.1. *The first integral F and the integral factor M of* (1.3.1) *are quasi-algebraic integrals of* (1.3.1).

Proof. By the definition of the first integral, a first integral of (1.3.1) in a region must be a quasi-algebraic integral of (1.3.1).

Let M is an integral factor of (1.3.1) in a region. Then, we have

$$\frac{\partial(M\mathcal{X}_n)}{\partial x} + \frac{\partial(M\mathcal{Y}_n)}{\partial y} = 0, \tag{1.3.10}$$

i.e.,

$$\frac{\partial M}{\partial x}\mathcal{X}_n + \frac{\partial M}{\partial y}\mathcal{Y}_n + \left(\frac{\partial \mathcal{X}_n}{\partial x} + \frac{\partial \mathcal{Y}_n}{\partial y}\right) M = 0. \tag{1.3.11}$$

It follows that

$$\left.\frac{dM}{dt}\right|_{(1.3.1)} = -\left(\frac{\partial \mathcal{X}_n}{\partial x} + \frac{\partial \mathcal{Y}_n}{\partial y}\right) M. \tag{1.3.12}$$

This implies that M is a quasi-algebraic integral of (1.3.1). □

Theorem 1.3.2. *Suppose that f_1, f_2, $\cdots$, f_m are m independent quasi-algebraic integrals of* (1.3.1) *satisfying*

$$\left.\frac{df_k}{dt}\right|_{(1.3.1)} = h_k f_k, \quad k = 1, 2, \cdots, m. \tag{1.3.13}$$

Then, for any group of non-zero complex numbers α_1, α_2, $\cdots$, α_m, the function $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is also a quasi-algebraic integral of (1.3.1) *satisfying*

$$\left.\frac{df}{dt}\right|_{(1.3.1)} = (\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m) f. \tag{1.3.14}$$

We know from the above theorem that

Theorem 1.3.3. *Suppose that f_1, f_2, $\cdots$, f_m are m independent quasi-algebraic integrals of* (1.3.1) *satisfying* (1.3.13). *If there is a group of non-zero complex numbers α_1, α_2, $\cdots$, α_m, such that*

$$\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m = 0. \tag{1.3.15}$$

Then, $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ is a first integral of (1.3.1).

By using Theorem 1.3.2 and (1.3.12), we obtain

Theorem 1.3.4. *Suppose that f_1, f_2, $\cdots$, f_m are m independent quasi-algebraic integrals of* (1.3.1) *satisfying* (1.3.13). *If there is a group of non-zero complex numbers α_1, α_2, $\cdots$, α_m, such that*

$$\alpha_1 h_1 + \alpha_2 h_2 + \cdots + \alpha_1 h_1 + \alpha_m h_m = -\left(\frac{\partial \mathcal{X}_n}{\partial x} + \frac{\partial \mathcal{Y}_n}{\partial y}\right). \tag{1.3.16}$$

Then, $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_m^{\alpha_m}$ are an integral factor of (1.3.1).

Because the set of all polynomials of degree $n-1$ forms a linear space of dimension $n(n+1)/2$. Every polynomial of degree $n-1$ is a vector of this linear space. Thus, by Theorem 1.3.2 and Theorem 1.3.3, if f_1, f_2, $\cdots$, f_m are m independent quasi-algebraic integrals of (1.3.1) satisfying (1.3.13), then

1. If h_1, h_2, $\cdots$, h_m are linear dependent, then, by using f_1, f_2, $\cdots$, f_m, we can construct a first integral of (1.3.1);

2. If h_1, h_2, $\cdots$, h_m are linear independent and

$$h_1, h_2, \cdots, h_m, \frac{\partial \mathcal{X}_n}{\partial x} + \frac{\partial \mathcal{Y}_n}{\partial y}$$

are linear dependent, then, by using f_1, f_2, $\cdots$, f_m, we can construct an integral factor of (1.3.1).

For a given polynomial system, we hope to know that under what parametric conditions, it has a quasi-algebraic integral and we wish to get more quasi-algebraic integrals. In principle, if there exists an algebraic integral of (1.3.1) which satisfies (1.3.2), then, we can obtain f and h, by using the method of undetermined coefficients. Letting

$$f = \sum_{k+j=0}^{m} c_{kj} x^k y^j, \quad h = \sum_{k+j=0}^{n-1} d_{kj} x^k y^j, \tag{1.3.17}$$

and substituting (1.3.17) into (1.3.2), comparing the coefficients of the corresponding terms in the two sides of obtained representation, it gives rise to a linear system of algebraic equations with respect to $c_{kj}, d_{k,j}, k, j = 1, \cdots$. Solving this system, it follows f and h.

Unfortunately, generally, we do not know the existence and its degree m of an algebraic integral f for a given polynomial system. This is a difficult classical problem in the theory of the planar dynamical systems.

We next discuss some special cases.

Consider the following polynomial system of degree $n+m$:

$$\frac{dx}{dt} = G_k(x)\mathcal{X}_n(x, y), \quad \frac{dy}{dt} = \mathcal{Y}_{n+m}(x, y), \tag{1.3.18}$$

where $1 \leqslant k \leqslant m$,

$$G_k(x) = a_0 + a_1 x + \cdots + a_k x^k, \ a_k \neq 0 \tag{1.3.19}$$

Because a polynomial of degree k has exact k complex roots. We have the following conclusion.

Proposition 1.3.2. *If $x = x_0$ is a simple zero of $G_k(x)$, then, system* (1.3.18) *has a quasi-algebraic integral $f = x - x_0$. If $x = x_0$ is a j-multiple zero of $G_k(x)$, then, system* (1.3.18) *has j independent quasi-algebraic integrals as follows:*

$$f_1 = x - x_0, \ f_2 = e^{\frac{1}{x-x_0}}, \ f_3 = e^{\frac{1}{(x-x_0)^2}}, \ \cdots, \ f_j = e^{\frac{1}{(x-x_0)^{j-1}}}. \tag{1.3.20}$$

Proposition 1.3.3. *The system*

$$\frac{dx}{dt} = \mathcal{X}_n(x, y), \quad \frac{dy}{dt} = \mathcal{Y}_{n+m}(x, y) \tag{1.3.21}$$

has the following m independent quasi-algebraic integrals

$$f_1 = e^x, \ f_2 = e^{x^2}, \ \cdots, \ f_m = e^{x^m}. \tag{1.3.22}$$

Proposition 1.3.4. *Suppose that system* (1.3.1) *is real and* $f = f_1 + if_2$ *is a quasi-algebraic integral of* (1.3.1) *satisfying*

$$\left.\frac{df}{dt}\right|_{(1.3.1)} = (h_1 + ih_2)f, \tag{1.3.23}$$

where f_1 *and* f_2 *are two real non-zero functions,* h_1 *and* h_2 *are real coefficient polynomials. Then,* $\bar{f} = f_1 - if_2$, $f_3 = f\bar{f}$ *and* $f_4 = f^i\bar{f}^{-i}$ *are quasi-algebraic integrals of* (1.3.1) *respectively satisfying*

$$\begin{aligned} \left.\frac{d\bar{f}}{dt}\right|_{(1.3.1)} &= (h_1 - ih_2)\bar{f}, \\ \left.\frac{df_3}{dt}\right|_{(1.3.1)} &= 2h_1 f_3, \\ \left.\frac{df_4}{dt}\right|_{(1.3.1)} &= -2h_2 f_4. \end{aligned} \tag{1.3.24}$$

Example 1.3.2. *Consider the following real quadratic system*

$$\frac{dx}{dt} = -y + \delta x + lx^2 + mxy + ny^2, \quad \frac{dy}{dt} = x(1 + by). \tag{1.3.25}$$

This system has the following quasi-algebraic integral

$$f = \begin{cases} (1 + by)^{\frac{1}{b}}, & if\ b \neq 0, \\ e^y, & if\ b = 0. \end{cases} \tag{1.3.26}$$

where e^y *is a limit function of* $(1 + by)^{\frac{1}{b}}$ *as* $b \to 0$.

1.4 Cauchy Majorant and Analytic Properties in a Neighborhood of an Ordinary Point

For the complex differential equations, in order to investigate the convergence of a solution of the power series, Cauchy posed the classical majorant method. It provided an important tool. In this section, we introduce this method.

Definition 1.4.1. *Let* $f = \sum\limits_{\alpha+\beta=0}^{\infty} c_{\alpha\beta}x^\alpha y^\beta$ *and* $F = \sum\limits_{\alpha+\beta=0}^{\infty} C_{\alpha\beta}x^\alpha y^\beta$ *be two power series of* x, y, *where* x, y *are complex variables and for all* $\alpha, \beta \in \mathbf{N}, c_{\alpha\beta}$ *are complex coefficients,* $C_{\alpha\beta}$ *are non-negative real numbers. If* $\forall \alpha, \beta$, *the inequalities* $|c_{\alpha\beta}| \leqslant C_{\alpha\beta}$ *hold, then,* F *is called a majorant of* f, *denoted by* $f \prec F$ *or* $F \succ f$.

Proposition 1.4.1. *Suppose that* f_1, f_2, F_1, F_2 *are power series of* x, y. F_1, F_2 *have non-negative real coefficients and non-zero convergent radius. If* $f_1 \prec F_1$,

$f_2 \prec F_2$, then, f_1, f_2 also have non-zero convergent radius and

$$\begin{aligned} f_1 \pm f_2 &\prec F_1 + F_2, \quad f_1 f_2 \prec F_1 F_2, \\ \frac{\partial f_1}{\partial z} &\prec \frac{\partial F_1}{\partial z}, \quad \frac{\partial f_1}{\partial w} \prec \frac{\partial F_1}{\partial w}, \\ \int f_1 dz &\prec \int F_1 dz, \quad \int f_1 dw \prec \int F_1 dw. \end{aligned} \tag{1.4.1}$$

In addition, if $f_1(0,0) = F_1(0,0) = 0$, then

$$\frac{1}{1 \pm f_1} \prec \frac{1}{1 - F_1} = \sum_{k=0}^{\infty} F_1^k. \tag{1.4.2}$$

Suppose that $f = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} x^\alpha y^\beta$ has a non-zero convergent radius. By Cauchy inequality, there are positive numbers M, r, such that

$$|c_{\alpha\beta}| \leqslant \frac{M}{r^{\alpha+\beta}}. \tag{1.4.3}$$

From (1.4.3) and (1.4.2), we have

Proposition 1.4.2. *If $f = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} x^\alpha y^\beta$ has a non-zero convergent radius, then, there exist positive numbers M, r, such that*

$$f \prec \frac{M}{\left(1 - \frac{x}{r}\right)\left(1 - \frac{y}{r}\right)} \prec \frac{M}{1 - \frac{x+y}{r}}. \tag{1.4.4}$$

Proposition 1.4.3. *If $f = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta} x^\alpha y^\beta$ has a non-zero convergent radius, then, there exist positive numbers M, r, such that for any positive integer m,*

$$\sum_{\alpha+\beta=m}^{\infty} c_{\alpha\beta} x^\alpha y^\beta \prec \frac{M(x+y)^m}{r^{m-1}(r-x-y)}. \tag{1.4.5}$$

Proof. By (1.4.3), for any positive integer k, we have

$$\sum_{\alpha+\beta=k} c_{\alpha\beta} x^\alpha y^\beta \prec M \sum_{\alpha+\beta=k} \frac{x^\alpha y^\beta}{r^k} \prec M\left(\frac{x+y}{r}\right)^k. \tag{1.4.6}$$

Thus,

$$\sum_{\alpha+\beta=m}^{\infty} c_{\alpha\beta} x^\alpha y^\beta \prec M \sum_{k=m}^{\infty} \left(\frac{x+y}{r}\right)^k = \frac{M(x+y)^m}{r^{m-1}(r-x-y)}. \tag{1.4.7}$$

□

We next discuss the analytic properties of the solutions of system (1.1.1) in a neighborhood of an ordinary point. Suppose that the right hand of (1.1.1) are analytic in a neighborhood of (x_0, y_0) and (x_0, y_0) is an ordinary point of (1.1.1). We can see (x_0, y_0) as the origin. Consider system

$$\frac{dx}{dt} = a_0 + a_1x + a_2y + h.o.t.,$$
$$\frac{dy}{dt} = b_0 + b_1x + b_2y + h.o.t., \tag{1.4.8}$$

where

$$|a_0| + |b_0| \neq 0. \tag{1.4.9}$$

Since the origin is an ordinary point of (1.4.8). Cauchy proved the following result.

Theorem 1.4.1. *If $a_0 \neq 0$, then system (1.4.8) has a unique power series solution with the initial condition $y(0) = 0$ as follows:*

$$y = \sum_{k=1}^{\infty} c_k x^k, \tag{1.4.10}$$

which has non-zero convergent radius.

By using the non-singular linear transformation

$$u = -b_0x + a_0y, \quad v = -\bar{a}_0x - \bar{b}_0y, \tag{1.4.11}$$

system (1.4.8) becomes

$$\frac{du}{dt} = U(u, v) = \sum_{k=0}^{\infty} \varphi_k(u)v^k,$$
$$\frac{dv}{dt} = -\Delta + V(u, v) = -\Delta + \sum_{k=0}^{\infty} \psi_k(u)v^k. \tag{1.4.12}$$

where for all k, $\varphi_k(u), \psi_k(u)$ are power series of u and

$$\Delta = |a_0|^2 + |b_0|^2 > 0,$$
$$U(0, 0) = \varphi_0(0) = 0, \quad V(0, 0) = \psi_0(0) = 0. \tag{1.4.13}$$

Definition 1.4.2. *Let $f = \sum_{\alpha+\beta=0}^{\infty} c_{\alpha\beta}x^\alpha y^\beta$ be power series of x, y. If we do not consider the convergence of f, then f is called a formal power series of x, y.*

If $f(x, y)$ is a formal power series of x, y and $f(0, 0) = 1$, then f is called a unitary formal power series of x, y.

Let $\{c_{\alpha_k,\beta_k}\}$ *for some* k *be a subsequence of* $\{c_{\alpha\beta}\}$*, then* $\sum c_{\alpha_k,\beta_k}x^{\alpha_k}y^{\beta_k}$ *is called a subseries of* f.

For two formal series, when we make the algebraic operations and analytic operations of term by term, if we do not consider their convergence, these operations are called the formal operations.

Definition 1.4.3. *Suppose that the functions of right hand of* (1.1.1) *are analytic in a neighborhood of the origin. If a formal series* $F(x, y)$ *of* x, y *satisfies*

$$\left.\frac{dF}{dt}\right|_{(1.1.1)} = \frac{\partial F}{\partial x}X + \frac{\partial F}{\partial y}Y = 0 \tag{1.4.14}$$

and F *is a nonconstant function in a neighborhood of the origin, then,* F *is called a formal first integral of* (1.1.1) *in a neighborhood of the origin.*

If a formal series $M(x, y)$ *of* x, y *satisfies*

$$\frac{\partial(MX)}{\partial x} + \frac{\partial(MY)}{\partial y} = 0 \tag{1.4.15}$$

and M *is a nonconstant function in a neighborhood of the origin, then,* M *is called a formal integral factor of* (1.1.1) *in a neighborhood of the origin.*

Lemma 1.4.1. *For system* (1.4.12)*, one can determine term by term the formal series*

$$H(u, v) = \sum_{k=0}^{\infty} h_k(u)v^k, \quad h_0(u) = u, \tag{1.4.16}$$

such that

$$\left.\frac{dH}{dt}\right|_{(1.4.12)} = 0. \tag{1.4.17}$$

Proof. Using (1.4.16) and (1.4.12) to do formal operation, we have

$$\begin{aligned}
\left.\frac{dH}{dt}\right|_{(1.4.12)} &= \frac{\partial H}{\partial u}U + \frac{\partial H}{\partial v}(-\Delta + V) \\
&= \sum_{k=0}^{\infty} h_k'(u)v^k \sum_{j=0}^{\infty} \varphi_j(u)v^j \\
&\quad + \sum_{k=1}^{\infty} kh_k(u)v^{k-1}\left[-\Delta + \sum_{j=0}^{\infty}\psi_j(u)v^j\right] \\
&= \sum_{m=0}^{\infty}\left[-(m+1)(\Delta - \psi_0)h_{m+1} + g_m\right]v^m,
\end{aligned} \tag{1.4.18}$$

where

$$g_m = \sum_{k=0}^{m}(h_k'\varphi_{m-k} + kh_k\psi_{m-k+1}) \tag{1.4.19}$$

is a polynomial of $h_k, h'_k, \varphi_j, \psi_j$ with positive coefficients. (1.4.17) and (1.4.18) imply the recursion formulas of h_m as follows:

$$h_{m+1} = \frac{g_m}{(m+1)(\Delta - \psi_0)}, \quad m = 0, 1, 2, \cdots. \tag{1.4.20}$$

Because the relationship of $h_0(u) = u$ has been given, by (1.4.20), this lemma holds. □

Lemma 1.4.2. *The function* $H(u,v)$ *defined by* (1.4.16) *has a non-zero convergent radius.*

Proof. Since the functions $U(u,v), V(u,v)$ are analytic in a neighborhood of the origin and $U(0,0) = V(0,0) = 0$, by Proposition 1.4.3, there exist two positive numbers M, r, such that

$$U(u,v) \prec \frac{M(u+v)}{r-u-v}, \quad V(x,y) \prec \frac{M(u+v)}{r-u-v}. \tag{1.4.21}$$

Consider the majorant system

$$\frac{du}{dt} = \frac{M(u+v)}{r-u-v}, \quad \frac{dv}{dt} = -\Delta + \frac{M(u+v)}{r-u-v}. \tag{1.4.22}$$

It is easy to see that system (1.4.22) has the following formal first integral

$$\tilde{H}(u,v) = u + G(u+v), \tag{1.4.23}$$

where

$$\begin{aligned} G(w) &= \frac{-M\Delta r}{(\Delta+2M)^2}\left[\frac{\Delta+2M}{\Delta r}w + \ln\left(1 - \frac{\Delta+2M}{\Delta r}\, w\right)\right] \\ &= \frac{M\Delta r}{(\Delta+2M)^2}\sum_{k=2}^{\infty}\frac{1}{k}\left(\frac{\Delta+2M}{\Delta r}\right)^k w^k = o(w) \end{aligned} \tag{1.4.24}$$

is a power series of w with positive coefficients, which is analytic in the disk $|w| < \Delta r/(\Delta + 2M)$. In order to prove

$$H(u,v) \prec \tilde{H}(u,v), \tag{1.4.25}$$

write that

$$\tilde{H}(u,v) = \sum_{k=0}^{\infty}\tilde{h}_k(u)v^k, \quad \frac{M(u+v)}{r-u-v} = \sum_{k=0}^{\infty}\chi_k(u)v^k, \tag{1.4.26}$$

then,

$$\begin{aligned} &h_0(u) \prec \tilde{h}_0(u) = u + G(u), \\ &\varphi_k(u) \prec \chi_k(u), \quad \psi_k(u) \prec \chi_k(u), \quad k = 0, 1, 2, \cdots. \end{aligned} \tag{1.4.27}$$

Similar to the proof of Lemma 1.4.1, we have the recursion formulas for the computation of $\tilde{h}_k$ as follows:

$$\tilde{h}_{m+1} = \frac{\tilde{g}_m}{(m+1)(\Delta - \chi_0)}, \quad m = 0, 1, 2, \cdots, \tag{1.4.28}$$

where

$$\tilde{g}_m = \sum_{k=0}^{m} (\tilde{h}'_k \chi_{m-k} + k\tilde{h}_k \chi_{m-k+1}). \tag{1.4.29}$$

By using Proposition 1.4.1, (1.4.19), (1.4.20) (1.4.27), (1.4.28), (1.4.29) and mathematical induction, for any positive integer m, we obtain

$$g_m \prec \tilde{g}_m, \quad h_m \prec \tilde{h}_m. \tag{1.4.30}$$

It follows (1.4.25) and this lemma. □

Theorem 1.4.2. *Let*

$$h_0^*(u) = u + h.o.t. \tag{1.4.31}$$

be a power series which is convergence in a neighborhood of $u = 0$. *One can derive successively every term of the following unique power series of* u, v,

$$H^*(u, v) = \sum_{k=0}^{\infty} h_k^*(u) v^k, \tag{1.4.32}$$

with a non-zero convergent radius, such that

$$\left.\frac{dH^*}{dt}\right|_{(1.4.12)} = 0, \tag{1.4.33}$$

and

$$H^*(u, v) = h_0^*(H(u, v)). \tag{1.4.34}$$

Proof. Similar to the proof of Lemma 1.4.1, for any positive integer k, we use h_k^* instead of h_k of (1.4.19) and (1.4.20) to get the the recursion formulas for $h_k^*(u)$. When $h_0^*(u)$ has been obtained, then, there is unique formal series $H^*(u, v)$ satisfying (1.4.33). Write that

$$\mathcal{H}(u, v) = h_0^*(H(u, v)). \tag{1.4.35}$$

By Lemma 1.4.1 and Lemma 1.4.2, $\mathcal{H}(u, v)$ is a power series with a non-zero convergent radius and

$$\mathcal{H}(u, 0) = h_0^*(u), \quad \left.\frac{d\mathcal{H}}{dt}\right|_{(1.4.12)} = 0. \tag{1.4.36}$$

By the uniqueness, we have $H^*(u, v) = \mathcal{H}(u, v)$, i.e., this theorem holds.

□

Theorem 1.4.3. *In a neighborhood of the origin there is a first integral of* (1.4.8) *as follows:*

$$F(x,y) = -b_0x + a_0y + \sum_{k=2}^{\infty} F_k(x,y), \tag{1.4.37}$$

where $F(x,y)$ is convergent in a small neighborhood of the origin, for every k, $F_k(x,y)$ is a homogeneous polynomial of x, y. Especially,

$$F_2(x,y) = \frac{1}{2(|a_0|^2 + |b_0|^2)^2}\,(\bar{a}_0x + \bar{b}_0y)(Ax + By), \tag{1.4.38}$$

where

$$\begin{aligned} A &= 2b_0\bar{b}_0(a_1b_0 - a_0b_1) + \bar{a}_0(a_0a_1b_0 - a_2b_0^2 - a_0^2b_1 + a_0b_0b_2), \\ B &= 2a_0\bar{a}_0(a_2b_0 - a_0b_2) - \bar{b}_0(a_0a_1b_0 - a_2b_0^2 - a_0^2b_1 + a_0b_0b_2). \end{aligned} \tag{1.4.39}$$

Proof. Let

$$F(x,y) = H(-b_0x + a_0y,\ -\bar{a}_0x - \bar{b}_0y). \tag{1.4.40}$$

By Lemma 1.4.1 and Lemma 1.4.2, $F(x,y)$ is a first integral of (1.4.8) and it has a non-zero convergent radius. Write that

$$H(x,y) = u + (c_1u + c_2v)v + h.o.t., \tag{1.4.41}$$

substituting (1.4.41) into (1.4.17), we can determine c_1, c_2. It follows the representation of F_2. □

Theorem 1.4.4. *For system* (1.4.8), *one can derive successively every term of the following power series*

$$\mathcal{T}(x,y) = \frac{\bar{a}_0x + \bar{b}_0y}{|a_0|^2 + |b_0|^2} + h.o.t., \tag{1.4.42}$$

which is convergent in a neighborhood of the origin, such that

$$\left.\frac{d\mathcal{T}}{dt}\right|_{(1.4.8)} = 1. \tag{1.4.43}$$

Proof. By Lemma 1.4.1 and Lemma 1.4.1, the first integral $H(u,v) = u + h.o.t.$ of (1.4.12) has a non-zero convergent radius. Hence, in a small neighborhood of the origin, by using the implicit function equations

$$z = H(u,v), \quad w = v, \tag{1.4.44}$$

we can uniquely solve

$$u = \zeta(z,w) = z + h.o.t., \quad v = w, \tag{1.4.45}$$

where the function $\zeta(z,w)$ has a non-zero convergent radius. By using the transformation (1.4.44), system (1.1.12) becomes

$$\frac{dz}{dt}=0,\quad \frac{dw}{dt}=-\Delta+V(\zeta(z,w),w). \tag{1.4.46}$$

Write that

$$\frac{1}{-\Delta+V(\zeta(z,w),w)}=-\frac{1}{\Delta}+\sum_{k+j=1}^{\infty}C_{kj}z^kw^j,$$

$$\mathcal{T}=\frac{-w}{\Delta}+\sum_{k+j=1}^{\infty}\frac{1}{j+1}C_{kj}z^kw^{j+1}. \tag{1.4.47}$$

By (1.4.11), (1.4.44) and (1.4.47), we obtain (1.4.42) and the convergence of $\mathcal{T}(x,y)$ in a small neighborhood of the origin. By (1.4.46) and (1.4.47), we have

$$\frac{dz}{dt}=0,\quad \frac{d\mathcal{T}}{dt}=1. \tag{1.4.48}$$

□

Theorem 1.4.5. *For system* (1.4.8), *one can derive successively every term of the following unique power series of* x,y

$$\xi=x+h.o.t.,\quad \eta=y+h.o.t., \tag{1.4.49}$$

with a non-zero convergent radius, such that, by the transformation (1.4.49), *system* (1.4.8) *becomes the following normal form*

$$\frac{d\xi}{dt}=a_0,\quad \frac{d\eta}{dt}=b_0. \tag{1.4.50}$$

Proof. For the function $F(x,y)$ in Theorem 1.4.3 and the function $\mathcal{T}(x,y)$ in Theorem 1.4.4, letting

$$\xi=a_0\mathcal{T}(x,y)-\frac{\bar{b}_0}{|a_0|^2+|b_0|^2}F(x,y),$$
$$\eta=b_0\mathcal{T}(x,y)+\frac{\bar{a}_0}{|a_0|^2+|b_0|^2}F(x,y), \tag{1.4.51}$$

then, Theorem 1.4.3 and Theorem 1.4.4 imply (1.4.49) and (1.4.50). □

Finally, we consider the following analytic system

$$\frac{dx}{dt}=y^{m-1}\left[a_0+a_1x^n+a_2y^m+\sum_{k=2}^{\infty}X_k(x^n,y^m)\right],$$
$$\frac{dy}{dt}=x^{n-1}\left[b_0+b_1x^n+b_2y^m+\sum_{k=2}^{\infty}Y_k(x^n,y^m)\right], \tag{1.4.52}$$

where m and n are two positive integers, for all k, $X_k(u,v), Y_k(u,v)$ are homogeneous polynomials of u, v with degree k, $b_0 \neq 0$.

The origin of (1.4.52) may be an ordinary point, an elementary singular point or a multiple singular point.

Theorem 1.4.6. *In a neighborhood of the origin, system* (1.4.52) *has the following formal first integral*:

$$F(x^n, y^m) = -mb_0x^n + na_0y^m + \sum_{k=2}^{\infty} F_k(x^n, y^m), \tag{1.4.53}$$

where $F(u,v)$ is analytic in a neighborhood of the origin. For all k, $F_k(u,v)$ are homogeneous polynomials of u, v.

Proof. By the transformation

$$u = x^n, \quad v = y^m, \quad d\tau = x^{n-1}y^{m-1}dt \tag{1.4.54}$$

system (1.4.52) becomes

$$\begin{aligned} \frac{du}{d\tau} &= n\,(a_0 + a_1u + a_2v) + n\sum_{k=2}^{\infty} X_k(u,v), \\ \frac{dv}{d\tau} &= m\,(b_0 + b_1u + b_2v) + m\sum_{k=2}^{\infty} Y_k(u,v). \end{aligned} \tag{1.4.55}$$

Because $b_0 \neq 0$, the origin is an ordinary point of (1.4.55). Thus, Theorem 1.4.3 follows the conclusion of this theorem. □

For an analytic system, as a corollary of Theorem 1.4.6, we can obtain the symmetric principle to the test of center or focus in the theory of real planar dynamical systems. In fact, if $n = 2, m = 1, a_0 = 0, b_0 \neq 0$, then (1.4.52) becomes

$$\begin{aligned} \frac{dx}{dt} &= a_1x^2 + a_2y + \sum_{k=2}^{\infty} X_k(x^2, y) = a_2y + h.o.t., \\ \frac{dy}{dt} &= x\left[b_0 + b_1x^2 + b_2y + \sum_{k=2}^{\infty} Y_k(x^2, y)\right] = b_0x + h.o.t., \end{aligned} \tag{1.4.56}$$

By the transformation

$$u = x^2, \quad v = y, \quad d\tau = xdt, \tag{1.4.57}$$

system (1.4.56) becomes

$$\begin{aligned} \frac{du}{d\tau} &= 2a_1u + 2a_2v + h.o.t., \\ \frac{dv}{d\tau} &= b_0 + b_1u + b_2v + h.o.t. \end{aligned} \tag{1.4.58}$$

By Theorem 1.4.3 and Theorem 1.4.6, we have

Corollary 1.4.1. *In a neighborhood of the origin, system* (1.4.56) *has the following formal first integral:*

$$F(x,y) = -b_0x^2 + a_2y^2 + \sum_{k=3}^{\infty} F_k(x,y), \tag{1.4.59}$$

where $F(x,y)$ *is analytic in a neighborhood of the origin and for all* k, $F_k(x,y)$ *are homogeneous polynomials of* x, y.

Suppose that (1.4.56) is a real system. Then, the vector field defined by (1.4.56) is symmetric with respect to y-axis. In this case, when $a_2 = 0$, the origin of (1.4.56) is a multiple singular point; When $b_0a_2 > 0$, it is a saddle point. When $b_0a_2 < 0$, by the symmetric principle, it is a center. Corollary 1.4.1 implies that when $b_0a_2 \geqslant 0$ and the coefficients of the right hand of (1.4.56) are complex numbers, in a neighborhood of the origin, system (1.4.56) is integrable.

Corollary 1.4.2. *For system* (1.4.56)*, One can determine successively every term of the following convergent power series of* x, y

$$g(x,y) = y + h.o.t., \tag{1.4.60}$$

such that

$$\left.\frac{dg}{dt}\right|_{(1.4.56)} = b_0x. \tag{1.4.61}$$

Proof. By Theorem 1.4.4, for system (1.4.58), in a neighborhood of the origin, there is a convergent power series

$$\mathcal{T}(u,v) = \frac{v}{b_0} + h.o.t., \tag{1.4.62}$$

such that

$$\left.\frac{d\mathcal{T}}{d\tau}\right|_{(1.4.58)} = 1. \tag{1.4.63}$$

Let $g(x,y) = b_0\mathcal{T}(x^2,y)$. Then, (1.4.57), (1.4.62) and (1.4.63) follows this lemma. □

This corollary means that if (1.4.56) is a polynomial system, then, e^g is a quasi-algebraic integral of (1.4.56) in a neighborhood of the origin.

1.5 Classification of Elementary Singular Points and Linea-rized Problem

Suppose that system (1.1.1) is analytic in a neighborhood of the origin and the origin is an elementary singular point of (1.1.1). We consider the complex system

$$\frac{dx}{dt} = ax + by + h.o.t., \quad \frac{dy}{dt} = cx + dy + h.o.t.. \tag{1.5.1}$$

The coefficient matrix of the linearized system of (1.5.1) at the origin has the characteristic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0. \tag{1.5.2}$$

Let λ_1, λ_2 be two roots of (1.5.2). If $\lambda_1\lambda_2 = ad - bc \neq 0$, the origin is an elementary singular point.

For an isolated singular point of the complex equations, the following result is useful.

Theorem 1.5.1 (Briot-Bouquet theorem). *Let $F(u,v) = Au + Bv + h.o.t.$ be a power series of u, v with a non-zero convergent radius. If B is not a positive integer, then, in a neighborhood of the origin, equation*

$$u\frac{dv}{du} = F(u,v), \quad v|_{u=0} = 0 \tag{1.5.3}$$

has the unique solution:

$$v = f(u) = \frac{A}{1-B}u + h.o.t., \tag{1.5.4}$$

where $f(u)$ is a power series of u with a non-zero convergent radius.

We next consider

$$\frac{dx}{dt} = \lambda_1 x + \sum_{k=2}^{\infty} X_k(x,y), \quad \frac{dy}{dt} = \lambda_2 y + \sum_{k=2}^{\infty} Y_k(x,y), \tag{1.5.5}$$

where the functions of right hand of (1.5.5) are analytic and $X_2(0,1) = A,\ Y_2(1,0) = B$.

Theorem 1.5.2. *If $\lambda_1 \neq 0$, λ_2/λ_1 is not a positive integer, then, system* (1.5.5) *has a solution*

$$y = \psi(x) = \frac{B}{2\lambda_1 - \lambda_2}x^2 + h.o.t., \tag{1.5.6}$$

satisfying $\psi(0) = 0$, where $\psi(x)$ a power series of x with a non-zero convergent radius.

Proof. Let $y = xv$. Then (1.5.5) becomes

$$x\frac{dv}{dx} = -v + \frac{\lambda_2 v + \sum_{k=2}^{\infty} x^{k-1} Y_k(1,v)}{\lambda_1 + \sum_{k=2}^{\infty} x^{k-1} X_k(1,v)} = \frac{Bx + (\lambda_2 - \lambda_1)v}{\lambda_1} + h.o.t.. \tag{1.5.7}$$

(1.5.7) and Theorem 1.5.1 follows this theorem. □

Similarly, we have

Theorem 1.5.3. *If $\lambda_1 \neq 0$, λ_2/λ_1 is not a positive integer, then, system* (1.5.5) *has a solution*

$$x = \varphi(y) = \frac{A}{2\lambda_2 - \lambda_1} y^2 + h.o.t., \tag{1.5.8}$$

satisfying $\varphi(0) = 0$ where $\varphi(y)$ is a power series of y with a non-zero convergent radius.

The above two theorems imply the following result.

Theorem 1.5.4. *If $\lambda_1\lambda_2 \neq 0$, λ_1/λ_2 and λ_2/λ_1 are not positive integers, then, system* (1.5.5) *has two analytic solutions $x = \varphi(y)$ and $y = \psi(x)$, satisfying $\varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = 0$. By transformation*

$$u = x - \varphi(y), \quad v = y - \psi(x) \tag{1.5.9}$$

system (1.5.5) *can be reduced to*

$$\frac{du}{dt} = \lambda_1 u F_1(u, v), \quad \frac{dv}{dt} = \lambda_2 v F_2(u, v). \tag{1.5.10}$$

where $F_1(u, v)$ and $F_2(u, v)$ are two power series of u, v with non-zero convergent radius and $F_1(0, 0) = F_2(0, 0) = 1$.

Definition 1.5.1. *If there exist two convergent power series*

$$\xi = x + \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} x^\alpha y^\beta, \quad \eta = y + \sum_{\alpha+\beta=2}^{\infty} d_{\alpha\beta} x^\alpha y^\beta, \tag{1.5.11}$$

such that by transformation (1.5.11), *system* (1.5.1) *becomes*

$$\frac{d\xi}{dt} = a\xi + b\eta, \quad \frac{d\eta}{dt} = c\xi + d\eta, \tag{1.5.12}$$

Then, we say that system (1.5.1) *is linearizable in a neighborhood of the origin.* (1.5.11) *is called a linearized transformation of* (1.5.1) *in a neighborhood of the origin.*

Remark 1.5.1. *Suppose that* (1.5.1) *is linearizable in a neighborhood of the origin and* (1.5.11) *is a linearized transformation of* (1.5.1). *The function $F(\xi, \eta)$ is continuously differentiable and satisfies*

$$\left.\frac{dF}{dt}\right|_{(1.5.12)} = 0. \tag{1.5.13}$$

Let

$$\tilde{\xi} = \xi F(\xi, \eta), \quad \tilde{\eta} = \eta F(\xi, \eta). \tag{1.5.14}$$

We see from (1.5.13) *and* (1.5.14) *that*

$$\frac{d\tilde{\xi}}{dt} = a\tilde{\xi} + b\tilde{\eta}, \quad \frac{d\tilde{\eta}}{dt} = c\tilde{\xi} + d\tilde{\eta}. \tag{1.5.15}$$

Thus, if $F(\xi, \eta)$ *is power series of* ξ, η *with non-zero convergent radius and* $F(0,0) = 1$*, then,* (1.5.14) *is also a linearized transformation of* (1.5.1) *in a neighborhood of the origin.*

Remark 1.5.2. *Suppose that* (1.5.1) *is linearizable in a neighborhood of the origin and* (1.5.11) *is a linearized transformation of* (1.5.1)*. System* (1.5.1) *is a real coefficient system. Then, we can see* x, y, t *as real variable and write*

$$\xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2, \tag{1.5.16}$$

where $\xi_1, \eta_1, \xi_2, \eta_2$ *are power series of* x, y *with non-zero convergent radius and*

$$\xi_1 = x + h.o.t., \quad \eta_1 = y + h.o.t.. \tag{1.5.17}$$

Substituting (1.5.16) *into* (1.5.12)*, separating the real and imaginary parts, we have*

$$\begin{aligned} \frac{d\xi_1}{dt} &= a\xi_1 + b\eta_1, \quad \frac{d\eta_1}{dt} = c\xi_1 + d\eta_1, \\ \frac{d\xi_2}{dt} &= a\xi_2 + b\eta_2, \quad \frac{d\eta_2}{dt} = c\xi_2 + d\eta_2. \end{aligned} \tag{1.5.18}$$

In this case, (1.5.17) *is a real linearized transformation of* (1.5.1) *in a neighborhood of the origin.*

In [Qin Y.X., 1985], the author introduced the following definition.

Definition 1.5.2. *For system* (1.5.1):

(1) *If* $\lambda_1\lambda_2 \neq 0$, $\mathrm{Im}(\lambda_1/\lambda_2) \neq 0$, *then the origin is called a focus type singular point*;

(2) *If* $\lambda_1\lambda_2 \neq 0$, $\mathrm{Im}(\lambda_1/\lambda_2) = 0$, $\mathrm{Re}(\lambda_1/\lambda_2) > 0$, *then the origin is called a node type singular point*;

(3) *If* $\lambda_1\lambda_2 \neq 0$, $\mathrm{Im}(\lambda_1/\lambda_2) = 0$, $\mathrm{Re}(\lambda_1/\lambda_2) < 0$, *then the origin is called a critical type singular point.*

In details, we have

Definition 1.5.3. *Suppose that the origin of* (1.5.1) *is a node type singular point.*

If $\lambda_1 = \lambda_2$ *and* $b = c = 0$*, then the origin of* (1.5.1) *is called a starlike node.*

If $\lambda_1 = \lambda_2$ *and* $|b| + |c| \neq 0$*, then the origin of* (1.5.1) *is called a degenerate node.*

If λ_1/λ_2 *or* λ_2/λ_1 *is a positive integer more than* 1*, then the origin of* (1.5.1) *is called an integer-ratio node.*

If λ_1/λ_2 *and* λ_2/λ_1 *are not positive integer, then the origin of* (1.5.1) *is called an ordinary node.*

Definition 1.5.4. *Suppose that the origin of* (1.5.1) *is a critical type singular point.*

If $\lambda_1/\lambda_2 = -1$, *then the origin of* (1.5.1) *is called a weak critical singular point.*

If $\lambda_1/\lambda_2 = -p/q$, *where* p *and* q *are irreducible positive integers and* $p/q \neq -1$, *then the origin of* (1.5.1) *is called a* $p:q$ *resonance singular point.*

If λ_1/λ_2 *is a negative irrational number, then the origin of* (1.5.1) *is called an irrational singular point.*

If the origin is an elementary singular point of (1.5.1), but is not a degenerate node, then, by a linear transformation, (1.5.1) can be reduced to as follows:

$$\frac{dx}{dt} = \lambda_1 x + \sum_{k=2}^{\infty} X_k(x,y), \quad \frac{dy}{dt} = \lambda_2 y + \sum_{k=2}^{\infty} Y_k(x,y), \tag{1.5.19}$$

where $\lambda_1\lambda_2 \neq 0$ and for all k, $X_k(x,y), Y_k(x,y)$ are homogeneous polynomials of x, y of degree k.

By the theory of classical complex analysis, we know that

Theorem 1.5.5. *If the origin of* (1.5.19) *is not an integer-ratio node, a weak critical singular point or a resonance singular point, then, one can determine successively every term of the following formal series*

$$\xi = x + \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} x^\alpha y^\beta, \quad \eta = y + \sum_{\alpha+\beta=2}^{\infty} d_{\alpha\beta} x^\alpha y^\beta, \tag{1.5.20}$$

such that, by this formal transformation, (1.5.19) *becomes the following linear system:*

$$\frac{d\xi}{dt} = \lambda_1 \xi, \quad \frac{d\eta}{dt} = \lambda_2 \eta. \tag{1.5.21}$$

Proof. For any positive integer k more than 1, letting $f_k(x,y), g_k(x,y)$ be two homogeneous polynomials of x, y of degree k given by

$$f_k(x,y) = \sum_{\alpha+\beta=k} c_{\alpha\beta} x^\alpha y^\beta, \quad g_k(x,y) = \sum_{\alpha+\beta=k} d_{\alpha\beta} x^\alpha y^\beta. \tag{1.5.22}$$

Clearly,

$$\begin{aligned}
\frac{d\xi}{dt} - \lambda_1 \xi &= \sum_{m=2}^{\infty} \left[\left(\lambda_1 \frac{\partial f_m}{\partial x} x + \lambda_2 \frac{\partial f_m}{\partial y} y - \lambda_1 f_m \right) + \Phi_m(x,y) \right], \\
\frac{d\eta}{dt} - \lambda_2 \eta &= \sum_{m=2}^{\infty} \left[\left(\lambda_1 \frac{\partial g_m}{\partial x} x + \lambda_2 \frac{\partial g_m}{\partial y} y - \lambda_2 g_m \right) + \Psi_m(x,y) \right],
\end{aligned} \tag{1.5.23}$$

where for all m, $\Phi_m(x,y), \Psi_m(x,y)$ are defined by the following homogeneous polynomials:

$$\Phi_m(x,y) = X_m(x,y) + \sum_{k=2}^{m-1}\left(\frac{\partial f_k}{\partial x}X_{m-k+1} + \frac{\partial f_k}{\partial y}Y_{m-k+1}\right),$$
$$\Psi_m(x,y) = Y_m(x,y) + \sum_{k=2}^{m-1}\left(\frac{\partial g_k}{\partial x}X_{m-k+1} + \frac{\partial g_k}{\partial y}Y_{m-k+1}\right). \tag{1.5.24}$$

From (1.5.22), we have

$$\lambda_1\frac{\partial f_m}{\partial x}x + \lambda_2\frac{\partial f_m}{\partial y}y - \lambda_1 f_m = \sum_{\alpha+\beta=m}^{\infty}(\alpha\lambda_1+\beta\lambda_2-\lambda_1)c_{\alpha\beta}x^\alpha y^\beta,$$
$$\lambda_1\frac{\partial g_m}{\partial x}x + \lambda_2\frac{\partial g_m}{\partial y}y - \lambda_2 g_m = \sum_{\alpha+\beta=m}^{\infty}(\alpha\lambda_1+\beta\lambda_2-\lambda_2)d_{\alpha\beta}x^\alpha y^\beta. \tag{1.5.25}$$

By the conditions of this theorem, because λ_1/λ_2 and λ_2/λ_1 are not a positive integer more than 1 and are not a negative irrational number. So that, for any positive integers α, β, when $\alpha+\beta \geqslant 2$,

$$\alpha\lambda_1+\beta\lambda_2-\lambda_1 \neq 0, \quad \alpha\lambda_1+\beta\lambda_2-\lambda_2 \neq 0. \tag{1.5.26}$$

Write that

$$\Phi_m(x,y) = \sum_{\alpha+\beta=m} A_{\alpha\beta}x^\alpha y^\beta, \quad \Psi_m(x,y) = \sum_{\alpha+\beta=m} B_{\alpha\beta}x^\alpha y^\beta. \tag{1.5.27}$$

Thus, (1.5.23), (1.5.25) and (1.5.27) imply (1.5.21) if and only if for any positive integer α, β, when $\alpha+\beta \geqslant 2$,

$$(\alpha\lambda_1+\beta\lambda_2-\lambda_1)c_{\alpha\beta} = -A_{\alpha\beta}, \quad (\alpha\lambda_1+\beta\lambda_2-\lambda_2)d_{\alpha\beta} = -B_{\alpha\beta}. \tag{1.5.28}$$

Obviously, (1.5.28) is just the recursion formulas to compute $c_{\alpha\beta}, d_{\alpha\beta}$. Namely, $c_{\alpha\beta}, d_{\alpha\beta}$ can be uniquely determined by (1.5.28). □

For the convergence of the formal transformation (1.5.20), in [Qin Y.X., 1985], by using Cauchy majorant method, the author proved that if the origin of (1.5.19) is a nfocus type singular point, ordinary node or starlike node, then in (1.5.20), the power series of ξ, η with respect to x, y have non-zero convergent radius. Therefore, we have

Theorem 1.5.6. *If the origin of* (1.5.1) *is a focus type singular point, ordinary node or starlike node, then, system* (1.5.1) *is linearizable in a neighborhood of the origin. In addition, the linearized transformation is unique.*

If the origin of (1.5.19) is a irrational singular point, so that λ_1/λ_2 is an negative irrational number. Then, in (1.5.28), $\alpha\lambda_1+\beta\lambda_2-\lambda_1$ and $\alpha\lambda_1+\beta\lambda_2-\lambda_2$ can be taken as very small number, such that the convergence problem of the formal series (1.5.20) becomes very difficult problem.

1.6 Node Value and Linearized Problem of the Integer-Ratio Node

Let the origin of system (1.5.1) be an integer-ratio node. By using a suitable linear transformation, system (1.5.1) can be reduced to

$$\frac{dx}{dt}=\lambda x+\sum_{\alpha+\beta=2}^{\infty}a_{\alpha\beta}x^{\alpha}y^{\beta},\quad \frac{dy}{dt}=n\lambda y+\sum_{\alpha+\beta=2}^{\infty}b_{\alpha\beta}x^{\alpha}y^{\beta}, \tag{1.6.1}$$

where $\lambda\neq 0$ and n is an integer greater than 1. System (1.6.1) is a special case of system (1.5.19) under the condition $\lambda_1=\lambda,\lambda_2=n\lambda$. For system (1.6.1), (1.5.28) becomes

$$(\alpha+n\beta-1)c_{\alpha\beta}\lambda=-A_{\alpha\beta},\quad (\alpha+n\beta-n)d_{\alpha\beta}\lambda=-B_{\alpha\beta}. \tag{1.6.2}$$

Obviously, for any natural numbers α and β, $\alpha+\beta\geqslant 2$ leads $\alpha+n\beta-1\neq 0$. $\alpha+n\beta-n=0$ holds if and only if $\alpha=n,\beta=0$. Hence, all $c_{\alpha\beta},d_{\alpha\beta}$ can be determined uniquely by (1.6.2) except d_{n0}. Consequently, there is a formal transformation (1.5.20) such that system (1.6.1) becomes linear system if and only if $B_{n0}=0$, and when $B_{n0}=0$, d_{n0} can take any value.

By cited Theorem 2.3 in [Qin Y.X., 1985], we have the following conclusion.

Theorem 1.6.1. *For system* (1.6.1), *one can find series* (1.5.20) *which are convergent in a neighborhood of the origin, such that system* (1.6.1) *reduced to the normal form*

$$\frac{d\xi}{dt}=\lambda\xi,\quad \frac{d\eta}{dt}=n\lambda\,\eta+\sigma\,\lambda\,\xi^{n}. \tag{1.6.3}$$

In (1.6.3), $\sigma=B_{n0}/\lambda$ is determined uniquely by the coefficients of system (1.6.1).

Theorem 1.6.2. *If there are formal series $\tilde{\xi}=x+h.o.t.$, $\tilde{\eta}=y+h.o.t.$, such that*

$$\left.\frac{d\tilde{\xi}}{dt}\right|_{(1.6.1)}=\lambda\tilde{\xi},\quad \left.\frac{d\tilde{\eta}}{dt}\right|_{(1.6.1)}=n\lambda\tilde{\eta}+\sigma\lambda\,(\tilde{\xi})^{n}, \tag{1.6.4}$$

then

$$\tilde{\xi}=\xi,\quad \tilde{\eta}=\eta+C\xi^{n}, \tag{1.6.5}$$

where, C is an arbitrary constant.

Proof. We take $\tilde{\xi}, \tilde{\eta}$ as the power series of ξ, η of the form

$$\tilde{\eta} = \eta + C\xi^n + h(\xi, \eta), \quad h(\xi, \eta) = \sum_{k=m}^{\infty} h_k(\xi, \eta), \tag{1.6.6}$$

where $h_k(\xi, \eta)$ are homogeneous polynomials of degree k of ξ, η, m is a positive integer. It is easy to see that all $c_{\alpha\beta}, d_{\alpha\beta}$ in (1.5.20) except d_{n0} are determined uniquely by (1.6.2). Thus, we have

$$\tilde{\xi} = \xi, \quad m > n. \tag{1.6.7}$$

From (1.6.3), (1.6.4) and (1.6.6), we obtain

$$\left.\frac{d\tilde{\eta}}{dt}\right|_{(1.6.3)} - n\lambda\tilde{\eta} - \sigma\lambda(\tilde{\xi})^n = \left.\frac{dh}{dt}\right|_{(1.6.3)} - n\lambda h = 0. \tag{1.6.8}$$

Let us prove (1.6.5) by using reductio ad absurdum, i.e., we prove that h is equivalent to zero. Suppose that

$$h_m(\xi, \eta) = \sum_{\alpha+\beta=m} e_{\alpha\beta}\xi^\alpha\eta^\beta \tag{1.6.9}$$

is not zero. From (1.6.8) and (1.6.9) we have

$$\begin{aligned} 0 &= \left.\frac{dh}{dt}\right|_{(1.6.3)} - n\lambda h \\ &= \lambda\left(\xi\frac{\partial h_m}{\partial \xi} + n\eta\frac{\partial h_m}{\partial \eta} - nh_m\right) + h.o.t. \\ &= \lambda \sum_{\alpha+\beta=m} (\alpha + n\beta - n)e_{\alpha\beta}\xi^\alpha\eta^\beta + h.o.t.. \end{aligned} \tag{1.6.10}$$

When $\alpha + \beta = m > n$, $\alpha + n\beta - n$ is a positive integer. From (1.6.10), it reduces that all $e_{\alpha\beta}$ in (1.6.9) equal zero, which contradicts with h_m is not identically zero. Hence, the conclusion of Theorem 1.6.2 holds. □

System (1.6.3) has a first integral of the form

$$\frac{\eta}{\xi^n} - \sigma \ln \xi = \text{constant}. \tag{1.6.11}$$

Obviously, the fact of σ is zero or nonzero is concerned with the linearized problem system (1.6.1) and the analytic property of (1.6.11). We need to introduce the following definition given in [Liu Y.R., 2002].

Definition 1.6.1. *σ is called node value of the origin of system* (1.6.1).

From Theorem 1.6.1 we obtain

Theorem 1.6.3. *System* (1.6.1) *is linearizable in a neighborhood of the origin if and only if the node value $\sigma = 0$.*

Let the origin be an integer-ratio node. In order to know if system is linearizable. We need to compute node values.

Theorem 1.6.4. *For every α and β satisfying $2 \leqslant \alpha + \beta \leqslant n-1$, $d_{\alpha\beta}$ defined by* (1.5.20) *are determined uniquely by the recurrent formula*

$$d_{\alpha\beta} = \frac{1}{\lambda(n-\alpha-n\beta)} \left[b_{\alpha\beta} + \sum_{k+j=2}^{\alpha+\beta-1} (\alpha-k+1) a_{k,j} d_{\alpha-k+1,\beta-j} \right. \\ \left. + \sum_{k+j=2}^{\alpha+\beta-1} (\beta-j+1) b_{k,j} d_{\alpha-k,\beta-j+1} \right]. \tag{1.6.12}$$

Furthermore, σ is given uniquely by

$$\sigma = \frac{1}{\lambda} \left[b_{n0} + \sum_{k=2}^{n-1} (n-k+1) a_{k0} d_{n-k+1,0} + b_{k0} d_{n-k,1} \right]. \tag{1.6.13}$$

Proof. From (1.5.20) and (1.6.3), we have

$$\begin{aligned} 0 &= \left.\frac{d\eta}{dt}\right|_{(1.6.1)} - n\lambda\eta \\ &= \sum_{\alpha+\beta=2}^{\infty} \alpha d_{\alpha\beta} x^{\alpha-1} y^{\beta} \left(\lambda x + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j \right) \\ &\quad + \left(1 + \sum_{\alpha+\beta=2}^{\infty} \beta d_{\alpha\beta} x^{\alpha} y^{\beta-1} \right) \left(n\lambda y + \sum_{k+j=2}^{\infty} b_{kj} x^k y^j \right) \\ &\quad - n\lambda \left(y + \sum_{\alpha+\beta=2}^{\infty} d_{\alpha\beta} x^{\alpha} y^{\beta} \right). \end{aligned} \tag{1.6.14}$$

It implies that

$$\begin{aligned} 0 = \lambda & \sum_{\alpha+\beta=2}^{\infty} (\alpha + n\beta - n) d_{\alpha\beta} x^{\alpha} y^{\beta} + \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} x^{\alpha} y^{\beta} \\ & + \sum_{\alpha+\beta=2}^{\infty} \sum_{k+j=2}^{\infty} \alpha d_{\alpha\beta} a_{kj} x^{\alpha+k-1} y^{\beta+j} \\ & + \sum_{\alpha+\beta=2}^{\infty} \sum_{k+j=2}^{\infty} \beta d_{\alpha\beta} b_{kj} x^{\alpha+k} y^{\beta+j-1}. \end{aligned} \tag{1.6.15}$$

Thus, we have

$$
\begin{aligned}
0=\lambda &\sum_{\alpha+\beta=2}^{\infty}(\alpha+n\beta-n)d_{\alpha\beta}x^{\alpha}y^{\beta}+\sum_{\alpha+\beta=2}^{\infty}b_{\alpha\beta}x^{\alpha}y^{\beta}\\
&+\sum_{\alpha+\beta=2}^{\infty}\sum_{k+j=2}^{\alpha+\beta-1}(\alpha-k+1)a_{kj}d_{\alpha-k+1,\beta-j}x^{\alpha}y^{\beta}\\
&+\sum_{\alpha+\beta=2}^{\infty}\sum_{k+j=2}^{\alpha+\beta-1}(\beta-j+1)b_{kj}d_{\alpha-k,\beta-j+1}x^{\alpha}y^{\beta}. \qquad (1.6.16)
\end{aligned}
$$

Because $\xi = x+h.o.t.$ and when $\alpha+\beta \geqslant 2$, $\alpha+n\beta-n=0$ if and only if $\alpha=n, \beta=0$. (1.6.16) follows the conclusion of Theorem 1.6.4. □

From the recursive formulas (1.6.12) and (1.6.13) we have

Theorem 1.6.5. *When $n = 2, 3$ and 4, the node values of the origin of system* (1.6.1) *are as follows*

$$
\begin{aligned}
\sigma|_{n=2} &= \frac{1}{\lambda}b_{20},\\
\sigma|_{n=3} &= \frac{1}{\lambda^2}[(2a_{20}-b_{11})b_{20}+b_{30}\lambda],\\
\sigma|_{n=4} &= \frac{1}{4\lambda^3}[b_{20}(12a_{20}^2-10a_{20}b_{11}+2b_{11}^2-2a_{11}b_{20}+b_{02}b_{20})\\
&\quad +2(2a_{30}b_{20}-b_{20}b_{21}+6a_{20}b_{30}-2b_{11}b_{30})\lambda+4b_{40}\lambda^2]. \qquad (1.6.17)
\end{aligned}
$$

Theorem 1.6.6. *If*

$$b_{20}=b_{30}=\cdots=b_{n-1,0}=0, \qquad (1.6.18)$$

then the node value of the origin of system (1.6.1) *is*

$$\sigma=\frac{1}{\lambda}b_{n0}. \qquad (1.6.19)$$

Proof. Theorem 1.6.4 follows that if (1.6.18) holds, then $d_{20}=0$ and when $\alpha = 3, 4, \cdots, n-1$, we have

$$d_{\alpha0}=\frac{1}{\lambda(n-\alpha)}\sum_{k=2}^{\alpha-1}(\alpha-k+1)a_{k0}d_{\alpha-k+1,0}. \qquad (1.6.20)$$

By using the mathematical induction, we obtain

$$d_{20}=d_{30}=\cdots=d_{n-1,0}=0. \qquad (1.6.21)$$

By (1.6.18), (1.6.21) and (1.6.13), the conclusion of Theorem 1.6.6 holds. □

Theorem 1.6.6 tell us that if $b_{20} = b_{30} = \cdots = b_{n0} = 0$, then the node value of the origin of system (1.6.1) is zero.

Corollary 1.6.1. *If $y = 0$ is a solution of system* (1.6.1)*, then at the origin, the node value $\sigma = 0$.*

Notice that in some special cases, other singular points can become an integer-ratio nodes. Therefore, at these singular points, the integrability and linearized problem of systems can be solved by computing node values in an integer-ratio nodes. For example, we have

Theorem 1.6.7. *System*

$$\begin{aligned}\frac{dz}{dT} &= z + 2b_3z^3w + a_2z^2w^2 + a_1zw^3,\\ \frac{dw}{dT} &= -w - b_1w^4 - 2a_2zw^3 - b_3z^2w^2\end{aligned} \tag{1.6.22}$$

is linearizable in a neighborhood of the origin.

Proof. By the transformation

$$z_1 = zw^2, \quad w_1 = w^3, \tag{1.6.23}$$

system (1.6.22) can become a special quadratic system

$$\begin{aligned}\frac{dz_1}{dT} &= -z_1 - 3a_2z_1^2 + (a_1 - 2b_1)z_1w_1,\\ \frac{dw_1}{dT} &= -3w_1 - 3b_3z_1^2 - 6a_2z_1w_1 - 3b_1w_1^2.\end{aligned} \tag{1.6.24}$$

The origin of system (1.6.24) is an integer-ratio node with $n = 3$. Theorem 1.6.5 implies that the node value $\sigma = 0$. Notice that $z_1 = 0$ is a solution of system (1.6.24), Theorem 1.6.1 follows that there are two convergent power series

$$\xi = z_1f_1(z_1, w_1), \quad \eta = w_1 + \sum_{k=2}^{\infty} \eta_k(z_1, w_1), \tag{1.6.25}$$

in a neighborhood of the origin, where $f_1(0, 0) = 1$, and $\eta_k(z_1, w_1)$ are homogeneous polynomials of degree k of z_1, w_1, such that system (1.6.24) becomes

$$\frac{d\xi}{dT} = -\xi, \quad \frac{d\eta}{dT} = -3\eta. \tag{1.6.26}$$

Let

$$z_2 = \xi\eta^{\frac{-2}{3}}, \quad w_2 = \eta^{\frac{1}{3}}, \tag{1.6.27}$$

then from (1.6.26) we have

$$\frac{dz_2}{dT} = z_2, \quad \frac{dw_2}{dT} = -w_2. \tag{1.6.28}$$

We next prove that z_2, w_2 are power series in z, w. Write that

$$f_2(z,w) = 1 + \sum_{k=2}^{\infty} w^{2k-3}\eta_k(z,w). \tag{1.6.29}$$

From (1.6.23) and (1.6.25), we have

$$\xi = zw^2 f_1(zw^2, w^3), \quad \eta = w^3 f_2(z,w). \tag{1.6.30}$$

Hence (1.6.27) and (1.6.30) follows that

$$\begin{aligned} z_2 &= z f_1(zw^2, w^3) f_2^{\frac{-2}{3}}(z,w) = z + h.o.t., \\ w_2 &= w f_2^{\frac{1}{3}}(z,w) = w + h.o.t.. \end{aligned} \tag{1.6.31}$$

This means that z_2, w_2 are power series of z, w having nonzero radius of convergence. So from (1.6.28), it is obtained that system (1.6.22) is linearizable in a neighborhood of the origin. □

Similarly, we have

Theorem 1.6.8. *System*

$$\begin{aligned} \frac{dz}{dT} &= z(1 + a_1 w^3 + a_2 w^2 z), \\ \frac{dw}{dT} &= -w(1 + b_1 w^3 + b_2 w^2 z) \end{aligned} \tag{1.6.32}$$

is linearizable in a neighborhood of the origin.

1.7 Linearized Problem of the Degenerate Node

If the origin is a degenerate node of system (1.5.1), we can find a suitable linear transformation such that system (1.5.1) becomes

$$\frac{dx}{dt} = \lambda x + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} x^\alpha y^\beta, \quad \frac{dy}{dt} = \mu x + \lambda y + \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} x^\alpha y^\beta, \tag{1.7.1}$$

where $\lambda \mu \neq 0$.

In the classical complex analytic theory, the linearized problem of the degenerate node is still an open problem. In this section, we discuss this problem.

Lemma 1.7.1. *In a neighborhood of the origin, there is a convergent power series solution of system* (1.7.1) *as follows*:

$$x = \varphi(y) = \frac{a_{02}}{\lambda}y^2 + h.o.t., \tag{1.7.2}$$

where all coefficients of power series of $\varphi(y)$ *can be determined uniquely by the coefficients of* (1.7.1).

Proof. Let $x = yv$, from (1.7.1) we have

$$\begin{aligned} y\frac{dv}{dy} &= \frac{\lambda v + \sum\limits_{\alpha+\beta=2}^{\infty} a_{\alpha\beta}v^{\alpha}y^{\alpha+\beta-1}}{\mu v + \lambda + \sum\limits_{\alpha+\beta=2}^{\infty} b_{\alpha\beta}v^{\alpha}y^{\alpha+\beta-1}} - v \\ &= \frac{a_{02}}{\lambda}y + h.o.t.. \end{aligned} \tag{1.7.3}$$

According to Theorem 1.5.1, equation (1.7.3) has a unique and convergent power series solution in a neighborhood of the origin

$$v = v(y) = \frac{a_{02}}{\lambda}y + h.o.t., \tag{1.7.4}$$

which follows Lemma 1.7.1. □

Let $x = \varphi(y)$ given by (1.7.2) be a convergent power series solution of system (1.7.1) in a neighborhood of the origin, then by the transformation

$$u = x - \varphi(y), \quad v = y, \tag{1.7.5}$$

system (1.7.1) becomes the following analytic system:

$$\begin{aligned} \frac{du}{dt} &= \lambda u\left(1 + \sum_{\alpha+\beta=1}^{\infty} a'_{\alpha\beta}u^{\alpha}v^{\beta}\right), \\ \frac{dv}{dt} &= \mu u + \lambda v + \sum_{\alpha+\beta=2}^{\infty} b'_{\alpha\beta}u^{\alpha}v^{\beta}. \end{aligned} \tag{1.7.6}$$

Letting

$$u = w^2, \quad v = v. \tag{1.7.7}$$

System (1.7.6) changes to

$$\begin{aligned} \frac{dw}{dt} &= \frac{\lambda}{2}w\left(1 + \sum_{\alpha+\beta=1}^{\infty} a'_{\alpha\beta}w^{2\alpha}v^{\beta}\right), \\ \frac{dv}{dt} &= \mu w^2 + \lambda v + \sum_{\alpha+\beta=2}^{\infty} b'_{\alpha\beta}w^{2\alpha}v^{\beta}, \end{aligned} \tag{1.7.8}$$

where the origin of system (1.7.8) is an integer-ratio node with $n = 2$. By Theorem 1.6.5, the node value $\sigma = 2\mu/\lambda$. From Theorem 1.6.1 and Theorem 1.6.2, we have

Lemma 1.7.2. *There are two power series of w and v*

$$f(w, v) = w + h.o.t., \quad g(w, v) = v + h.o.t., \tag{1.7.9}$$

having a nonzero convergent radius, such that system (1.7.8) *becomes*

$$\frac{df}{dt} = \frac{\lambda}{2} f, \quad \frac{dg}{dt} = \lambda g + \mu f^2. \tag{1.7.10}$$

Moreover, if there exist two formal series of w and v of the form

$$\tilde{f} = w + h.o.t., \quad \tilde{g} = v + h.o.t., \tag{1.7.11}$$

such that system (1.7.8) *changes to*

$$\frac{d\tilde{f}}{dt} = \frac{\lambda}{2} \tilde{f}, \quad \frac{d\tilde{g}}{dt} = \lambda \tilde{g} + \mu \tilde{f}^2. \tag{1.7.12}$$

Then

$$\tilde{f} = f, \quad \tilde{g} = g + Cf^2, \tag{1.7.13}$$

where C is a constant.

Remark 1.7.1. *Lemma* 1.7.2 *implies that we can assume that the coefficient of w^2 in the power series of the g given by* (1.7.9) *is zero.*

Lemma 1.7.3. *The function $f = f(w, v)$ given by* (1.7.9) *is an odd function of w, i.e.,*

$$f(w, v) = w\, h(w^2, v), \tag{1.7.14}$$

where $h(u, v)$ is a power series of u and v with nonzero convergent radius and $h(0, 0) = 1$.

Proof. Write that

$$f(w, v) = f_1(w, v) + f_2(w, v), \tag{1.7.15}$$

where

$$\begin{aligned} f_1(w, v) &= \frac{f(w, v) - f(-w, v)}{2}, \\ f_2(w, v) &= \frac{f(w, v) + f(-w, v)}{2}. \end{aligned} \tag{1.7.16}$$

Clearly, f_1 is an odd function of w and f_2 is an even function of w. We see from (1.7.8) that

$$\left.\frac{df_1}{dt}\right|_{(1.7.8)} = \frac{df_1}{dw}\frac{dw}{dt} + \frac{df_1}{dv}\frac{dv}{dt} \tag{1.7.17}$$

is an odd function of w and

$$\left.\frac{df_2}{dt}\right|_{(1.7.8)} = \frac{df_2}{dw}\frac{dw}{dt} + \frac{df_2}{dv}\frac{dv}{dt} \tag{1.7.18}$$

is an even function of w. (1.7.10) and (1.7.15) follows that

$$\left.\frac{df_1}{dt}\right|_{(1.7.8)} + \left.\frac{df_2}{dt}\right|_{(1.7.8)} = \frac{\lambda}{2}f_1 + \frac{\lambda}{2}f_2. \tag{1.7.19}$$

Comparing the functions in the right and left sides of (1.7.19), we have

$$\left.\frac{df_1}{dt}\right|_{(1.7.8)} = \frac{\lambda}{2}f_1, \quad \left.\frac{df_2}{dt}\right|_{(1.7.8)} = \frac{\lambda}{2}f_2. \tag{1.7.20}$$

Because the function f in Lemma 1.7.2 is unique and $f_1 = w + h.o.t.$. Therefore, we obtain $f = f_1$. It give rise to this lemma. □

Similarly, we have

Lemma 1.7.4. *The function $g(w, v)$ in Lemma 1.7.2 is an even function of w.*

Theorem 1.7.1. *There are two power series of x and y with a nonzero convergent radius of the form*

$$\xi = x + h.o.t., \quad \eta = y + h.o.t., \tag{1.7.21}$$

such that system (1.7.1) becomes the following linear system:

$$\frac{d\xi}{dt} = \lambda\xi, \quad \frac{d\eta}{dt} = \mu\xi + \lambda\eta. \tag{1.7.22}$$

In addition, if there are another two formal series of x and y

$$\tilde{\xi} = x + h.o.t., \quad \tilde{\eta} = y + h.o.t., \tag{1.7.23}$$

such that system (1.7.1) reduces to

$$\frac{d\tilde{\xi}}{dt} = \lambda\tilde{\xi}, \quad \frac{d\tilde{\eta}}{dt} = \mu\tilde{\xi} + \lambda\tilde{\eta}, \tag{1.7.24}$$

then

$$\tilde{\xi} = \xi, \quad \tilde{\eta} = \eta + C\xi, \tag{1.7.25}$$

where C is a constant.

Proof. By Lemma 1.7.4 and Remark 1.7.1, the function $g(w, v)$ in Lemma 1.7.2 can be written as

$$g(w, v) = \eta(w^2, v), \tag{1.7.26}$$

where $\eta(u,v)$ is a power series of u and v having a nonzero convergent radius:

$$\eta(u,v) = v + \sum_{\alpha+\beta=2}^{\infty} e_{\alpha\beta} u^{\alpha} v^{\beta}. \tag{1.7.27}$$

Let

$$\xi(u,v) = uh^2(u,v). \tag{1.7.28}$$

From (1.7.14) and (1.7.28) we have

$$\xi(w^2, v) = f^2(w,v). \tag{1.7.29}$$

Thus, (1.7.7), (1.7.10), (1.7.26) and (1.7.29) follow that

$$\left.\frac{d\xi}{dt}\right|_{(1.7.6)} = \lambda\xi, \quad \left.\frac{d\eta}{dt}\right|_{(1.7.6)} = \mu\xi + \lambda\eta. \tag{1.7.30}$$

(1.7.5), (1.7.30) and Lemma 1.7.2 implies the result of Theorem 1.7.1. □

1.8 Integrability and Linearized Problem of Weak Critical Singular Point

Let the origin of system (1.5.1) be a weak critical singular point. By using a suitable linear transformation, system (1.5.1) can become the following second-order complex differential autonomous system

$$\begin{aligned}\frac{dx}{dt} &= -y + \sum_{k=2}^{\infty} X_k(x,y) = X(x,y),\\ \frac{dy}{dt} &= x + \sum_{k=2}^{\infty} Y_k(x,y) = Y(x,y),\end{aligned} \tag{1.8.1}$$

which is analytic in a neighborhood of the origin, where $X_k(x,y), Y_k(x,y)$ are polynomials of degree k of x and y:

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta}, \quad Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta}. \tag{1.8.2}$$

Making the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{1.8.3}$$

system (1.8.1) becomes

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),\end{aligned} \tag{1.8.4}$$

where

$$Z_k = Y_k - iX_k = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta,$$
$$W_k = Y_k + iX_k = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta \tag{1.8.5}$$

are homogeneous polynomials of degree k of z and w ($k = 2, 3, \cdots$), z, w, T are independent complex variables, $a_{\alpha\beta}, b_{\alpha\beta}$ are independent complex constants.

We call that system (1.8.1) is the associated system of (1.8.4) and vice versa. We see from (1.8.5) that $\forall(\alpha,\beta)$, $A_{\alpha,\beta}, B_{\alpha\beta}$ are real coefficients if and only if $\forall(\alpha,\beta)$, $b_{\alpha\beta} = \bar{a}_{\alpha\beta}$.

If $\forall(\alpha,\beta)$, $A_{\alpha,\beta}, B_{\alpha\beta}$ are real coefficients and x, y, t are all real variables, then system (1.8.1) is a real planar differential autonomous system, for which the origin is a center or a focus. While if $\forall(\alpha,\beta)$, $a_{\alpha\beta}, b_{\alpha\beta}$ are real coefficients and z, w, T are all real variables, then system (1.8.4) is a real planar differential autonomous system, for which the origin is a weak saddle. The monograph [Amelikin etc, 1982] proved that

Theorem 1.8.1. *For any given $\tilde{c}_{k+1,k}$ and $\tilde{d}_{k+1,k}$, $k = 1, 2, \cdots$, one can determine successively other $\tilde{c}_{k,j}$ and $\tilde{d}_{k,j}$ and derive uniquely the formal series*

$$\tilde{\xi} = z + \sum_{k+j=2}^{\infty} \tilde{c}_{kj} z^k w^j,$$
$$\tilde{\eta} = w + \sum_{k+j=2}^{\infty} \tilde{d}_{kj} w^k z^j, \tag{1.8.6}$$

such that by formal variable transformation (1.8.6), *system* (1.8.4) *reduces to the following normal form*

$$\frac{d\tilde{\xi}}{dT} = \tilde{\xi} + \tilde{\xi}\sum_{k=1}^{\infty} \tilde{p}_k(\tilde{\xi}\tilde{\eta})^k,$$
$$\frac{d\tilde{\eta}}{dT} = -\tilde{\eta} - \tilde{\eta}\sum_{k=1}^{\infty} \tilde{q}_k(\tilde{\xi}\tilde{\eta})^k. \tag{1.8.7}$$

Proof. We denote

$$\tilde{\xi} = \sum_{k=1}^{\infty} f_k(z,w), \quad \tilde{\eta} = \sum_{k=1}^{\infty} g_k(z,w), \tag{1.8.8}$$

where $f_1 = z$, $g_1 = w$, $f_k(z,w), g_k(z,w)$ are homogeneous polynomials of degree k of z, w. Write that

$$\tilde{\xi}\sum_{k=1}^{\infty}\tilde{p}_k(\tilde{\xi}\tilde{\eta})^k=\sum_{k=1}^{\infty}\tilde{p}_k z^{k+1}w^k+\sum_{k=3}^{\infty}\Phi_k(z,w),$$
$$\tilde{\eta}\sum_{k=1}^{\infty}\tilde{q}_k(\tilde{\xi}\tilde{\eta})^k=\sum_{k=1}^{\infty}\tilde{q}_k w^{k+1}z^k+\sum_{k=3}^{\infty}\Psi_k(z,w), \tag{1.8.9}$$

where $\Phi_k(z,w),\Psi_k(z,w)$ are homogeneous polynomials of degree k of z,w. From (1.8.4) and (1.8.8), we have

$$\frac{d\tilde{\xi}}{dT}-\tilde{\xi}=\sum_{m=2}^{\infty}\left[\left(\frac{\partial f_m}{\partial z}z-\frac{\partial f_m}{\partial w}w-f_m\right)+F_m\right],$$
$$\frac{d\tilde{\eta}}{dT}+\tilde{\eta}=\sum_{m=2}^{\infty}\left[\left(\frac{\partial g_m}{\partial z}z-\frac{\partial g_m}{\partial w}w+g_m\right)-G_m\right], \tag{1.8.10}$$

where

$$F_m=Z_m+\sum_{j=2}^{m-1}\left(\frac{\partial f_j}{\partial z}Z_{m-j+1}-\frac{\partial f_j}{\partial w}W_{m-j+1}\right),$$
$$G_m=W_m+\sum_{j=2}^{m-1}\left(\frac{\partial g_j}{\partial w}W_{m-j+1}-\frac{\partial g_j}{\partial z}Z_{m-j+1}\right) \tag{1.8.11}$$

are homogeneous polynomials of degree m of z,w. From (1.8.9), (1.8.10) and (1.8.7), we obtain

$$\sum_{m=2}^{\infty}\left(\frac{\partial f_m}{\partial z}z-\frac{\partial f_m}{\partial w}w-f_m\right)=\sum_{m=2}^{\infty}(\Phi_m-F_m)+\sum_{k=1}^{\infty}\tilde{p}_k z^{k+1}w^k,$$
$$\sum_{m=2}^{\infty}\left(\frac{\partial g_m}{\partial w}w-\frac{\partial g_m}{\partial z}z-g_m\right)=\sum_{m=2}^{\infty}(\Psi_m-G_m)+\sum_{k=1}^{\infty}\tilde{q}_k w^{k+1}z^k, \tag{1.8.12}$$

where

$$\frac{\partial f_m}{\partial z}z-\frac{\partial f_m}{\partial w}w-f_m=\sum_{\alpha+\beta=m}(\alpha-\beta-1)\tilde{c}_{\alpha\beta}z^{\alpha}w^{\beta},$$
$$\frac{\partial g_m}{\partial w}w-\frac{\partial g_m}{\partial z}z-g_m=\sum_{\alpha+\beta=m}(\alpha-\beta-1)\tilde{d}_{\alpha\beta}w^{\alpha}z^{\beta}. \tag{1.8.13}$$

For F_m,G_m,Φ_m,Ψ_m in (1.8.12), we see from (1.8.8) (1.8.9) and (1.8.11) that for any positive integer k, Φ_{2k}, Φ_{2k+1} Ψ_{2k} and Ψ_{2k+1} are polynomials of f_1, f_2, $\cdots$, f_{2k-1}, g_1, g_2, $\cdots$, g_{2k-1}, p_1, p_2, $\cdots$, p_{k-1}, q_1, q_2, $\cdots$, q_{k-1}, which have positive rational coefficients. In addition, for any m, F_m,G_m only depend on $f_1,f_2,\cdots,f_{m-1}$, $g_1,g_2,\cdots,g_{m-1}$. Write that

$$\Phi_m - F_m = \sum_{\alpha+\beta=m} C_{\alpha\beta} z^\alpha w^\beta, \quad \Psi_m - G_m = \sum_{\alpha+\beta=m} D_{\alpha\beta} w^\alpha z^\beta. \tag{1.8.14}$$

From (1.8.13) and (1.8.14), we know that (1.8.12) holds if and only if for any positive integer k,

$$\begin{aligned} \sum_{\alpha+\beta=2k} (\alpha-\beta-1)\tilde{c}_{\alpha\beta} z^\alpha w^\beta &= \sum_{\alpha+\beta=2k} C_{\alpha\beta} z^\alpha w^\beta, \\ \sum_{\alpha+\beta=2k} (\alpha-\beta-1)\tilde{d}_{\alpha\beta} w^\alpha z^\beta &= \sum_{\alpha+\beta=2k} D_{\alpha\beta} w^\alpha z^\beta \end{aligned} \tag{1.8.15}$$

and

$$\begin{aligned} \sum_{\alpha+\beta=2k+1} (\alpha-\beta-1)\tilde{c}_{\alpha\beta} z^\alpha w^\beta &= \tilde{p}_k z^{k+1} w^k + \sum_{\alpha+\beta=2k+1} C_{\alpha\beta} z^\alpha w^\beta, \\ \sum_{\alpha+\beta=2k+1} (\alpha-\beta-1)\tilde{d}_{\alpha\beta} w^\alpha z^\beta &= \tilde{q}_k w^{k+1} z^k + \sum_{\alpha+\beta=2k+1} D_{\alpha\beta} w^\alpha z^\beta. \end{aligned} \tag{1.8.16}$$

Because in (1.8.16), all coefficients of $\tilde{c}_{k+1,k}, \tilde{d}_{k+1,k}$ are zeros. Hence, all $\tilde{c}_{k+1,k}$ and $\tilde{d}_{k+1,k}$ can be given as arbitrary constants. By (1.8.15) and (1.8.16), for any two natural numbers α, β satisfying $\alpha+\beta \geqslant 2$ and $\alpha-\beta-1 \neq 0$, $\tilde{c}_{\alpha\beta}$, $\tilde{d}_{\alpha\beta}$ can be uniquely determined by the recursive formulas

$$\tilde{c}_{\alpha\beta} = \frac{C_{\alpha\beta}}{\alpha-\beta-1}, \quad \tilde{d}_{\alpha\beta} = \frac{D_{\alpha\beta}}{\alpha-\beta-1}. \tag{1.8.17}$$

In addition, $\tilde{p}_k$, $\tilde{q}_k$ can be derived uniquely by the recursive formulas

$$\tilde{p}_k = -C_{k+1,k}, \quad \tilde{q}_k = -D_{k+1,k}. \tag{1.8.18}$$

This completes the proof of this theorem. □

Definition 1.8.1. *Suppose that by means of formal transformation* (1.8.6), *system* (1.8.4) *can be reduced to the normal form* (1.8.7). *Then, the transformation* (1.8.6) *is called a normal transformation in a neighborhood of the origin of system* (1.8.4). *System* (1.8.7) *is called a normal form corresponding to the transformation* (1.8.6).

Let (1.8.6) *be a normal transformation in a neighborhood of the origin of system* (1.8.4) *and* $\tilde{c}_{k+1,k} = \tilde{d}_{k+1,k} = 0$, $k = 1, 2, \cdots$. *Then* (1.8.6) *is called a standard normal transformation in a neighborhood of the origin of system* (1.8.4), *which is written by*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j = \xi(z,w), \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j = \eta(z,w). \tag{1.8.19}$$

The normal form derived by standard normal transformation is called standard normal form, which is written by

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \xi\sum_{k=1}^{\infty} p_k(\xi\eta)^k,\\ \frac{d\eta}{dT} &= -\eta - \eta\sum_{k=1}^{\infty} q_k(\xi\eta)^k.\end{aligned} \tag{1.8.20}$$

From Theorem 1.8.1 and its proof, we have

Corollary 1.8.1. *Let*

$$\xi^* = z + \sum_{k+j=2}^{\infty} c_{kj}^* z^k w^j, \quad \eta* = w + \sum_{k+j=2}^{\infty} d_{kj}^* w^k z^j \tag{1.8.21}$$

and (1.8.6) *be two standard normal transformations in a neighborhood of the origin of system* (1.8.4)*. If for any positive integer k, we have $c_{k+1,k}^* = \tilde{c}_{k+1,k}$, $d_{k+1,k}^* = \tilde{d}_{k+1,k}$. Then $\xi^* = \tilde{\xi}$, $\eta^* = \tilde{\eta}$.*

Remark 1.8.1. *From Corollary* 1.8.1 *and the proof of Theorem* 1.8.1*, we know that the standard normal transformation is unique in a neighborhood of the origin of system* (1.8.4)*. Moreover, all c_{kj}, d_{kj}, p_k and q_k in* (1.8.19) *and* (1.8.20) *are polynomials of $a_{\alpha\beta}$'s, $b_{\alpha\beta}'s$. Their coefficients are all rational numbers.*

[Amelikin etc, 1982] proved that

Theorem 1.8.2. *If for any positive integer k, we have $p_k = q_k$. Then the formal series of ξ, η in the standard normal transformation have nonzero convergent radius.*

Theorem 1.8.3. *Let $H = \xi\eta$ and $F(H)$, $G(H)$ be any unit formal power series of H. Then,*

$$\tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H) \tag{1.8.22}$$

is a normal transformation in a neighborhood of the origin of system (1.8.4).

Proof. From equations (1.8.20) and (1.8.22), we have

$$\frac{d\tilde{\xi}}{dT} = \tilde{\xi}\Phi(H), \quad \frac{d\tilde{\eta}}{dT} = -\tilde{\eta}\Psi(H), \tag{1.8.23}$$

where $\Phi(H), \Psi(H)$ are the following unit formal power series of H:

$$\begin{aligned}\Phi(H) &= 1 + \sum_{k=1}^{\infty} p_k H^k + \frac{F'(H)}{F(H)}\sum_{k=1}^{\infty}(p_k - q_k)H^{k+1},\\ \Psi(H) &= 1 + \sum_{k=1}^{\infty} q_k H^k - \frac{G'(H)}{G(H)}\sum_{k=1}^{\infty}(p_k - q_k)H^{k+1}.\end{aligned} \tag{1.8.24}$$

Denote that

$$\tilde{H} = \tilde{\xi}\tilde{\eta} = HF(H)G(H) = H + h.o.t.. \tag{1.8.25}$$

Thus, H can be written as the formal series of $\tilde{H}$

$$H = \tilde{H} + o(\tilde{H}). \tag{1.8.26}$$

(1.8.23) and (1.8.26) follow thee assertion of this theorem. □

Theorem 1.8.4. *Let* (1.8.6) *be a normal transformation in a neighborhood of the origin of system* (1.8.4)*. Then, there exists unit formal series of* H *as follows*:

$$F(H) = 1 + \sum_{k=1}^{\infty} A_k H^k, \quad G(H) = 1 + \sum_{k=1}^{\infty} B_k H^k, \tag{1.8.27}$$

such that $\tilde{\xi} = \xi F(H), \tilde{\eta} = \eta G(H)$*, where* A_k, B_k *are the given constant coefficients.*

Proof. Denote that

$$f = \xi F(H) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j,$$

$$g = \eta G(H) = w + \sum_{k+j=2}^{\infty} d'_{kj} w^k z^j. \tag{1.8.28}$$

Since the functions of ξ, η in the standard normal transformation are determined uniquely, we only need to find A_k, B_k of f and g $(k = 1, 2, \cdots)$. From (1.8.27) and (1.8.28), f and g can can be written as

$$f(z, w) = z + \sum_{k=1}^{\infty} A_k z^{k+1} w^k + \sum_{k=2}^{\infty} f_k(z, w),$$

$$g(z, w) = w + \sum_{k=1}^{\infty} B_k w^{k+1} z^k + \sum_{k=2}^{\infty} g_k(z, w), \tag{1.8.29}$$

where $f_k(z, w), g_k(z, w)$ are homogeneous polynomials of degree k of z, w. For any positive integer k, f_{2k+1} only depend on $A_1, A_2, \cdots, A_{k-1}$, while g_{2k+1} only depend on $B_1, B_2, \cdots, B_{k-1}$. From (1.8.28) and (1.8.29), we can take appropriately A_k, B_k, such that for any positive integer k, $c'_{k+1,k} = \tilde{c}_{k+1,k}$, $d'_{k+1,k} = \tilde{d}_{k+1,k}$ hold. By Corollary 1.8.1, we obtain the conclusion of this theorem. □

From equation (1.8.20) and Proposition 1.1.2, we obtain the following three important formulas

Theorem 1.8.5. *Denote that*

$$\mu_k = p_k - q_k, \quad \tau_k = p_k + q_k, \quad k = 1, 2, \cdots. \tag{1.8.30}$$

For system (1.8.4)*, we have*

$$\frac{dH}{dT} = \sum_{k=1}^{\infty} \mu_k H^{k+1},$$
$$\frac{d\Omega}{dT} = \frac{1}{2i}\left(2 + \sum_{k=1}^{\infty} \tau_k H^k\right) \tag{1.8.31}$$

and

$$\frac{\partial}{\partial z}(JZ) - \frac{\partial}{\partial w}(JW) = J\sum_{k=1}^{\infty}(k+1)\mu_k(\xi\eta)^k, \tag{1.8.32}$$

where

$$H = \xi\eta, \quad \Omega = \frac{1}{2i}\ln\frac{\xi}{\eta}, \quad J(z,w) = \begin{vmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial w} \\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial w} \end{vmatrix}. \tag{1.8.33}$$

In [Liu Y.R. et al, 1989] and [Liu Y.R. et al, 2003a], we introduced the following definition.

Definition 1.8.2. *Let* $\mu_0 = \tau_0 = 0$*. For any positive integers* k*,* $\mu_k = p_k - q_k$ *is called the* k*-th singular point value of the origin of system* (1.8.4)*, while* $\tau_k = p_k + q_k$ *is called the* k*-th period constant of the origin of system* (1.8.4)*.*

If there exists a positive integer m*, such that* $\mu_0 = \mu_1 = \cdots = \mu_{m-1} = 0, \mu_m \neq 0$*, then the origin of is called a* m *order weak critical singular point of system* (1.8.4)*. If for all* k*, we have* $\mu_k = 0$*. Then the origin of system* (1.8.4) *is called a complex center.*

Theorem 1.8.6. *Let* (1.8.6) *be any normal transformation in a neighborhood of the origin of system* (1.8.4)*. Denote that* $\tilde{H} = \tilde{\xi}\tilde{\eta}$*. When the origin of system* (1.8.4) *is a* m*-order weak critical singular point, we have*

$$\frac{d\tilde{H}}{dT} = \mu_m \tilde{H}^{m+1} + h.o.t.. \tag{1.8.34}$$

When the origin of system (1.8.4) *is a complex center, we have*

$$\frac{d\tilde{H}}{dT} = 0. \tag{1.8.35}$$

Proof. Let (1.8.6) be any normal transformation in a neighborhood of the origin of system (1.8.4). By Theorem 1.8.4, there exist two unit formal series $F(H), G(H)$ of H, such that

$$\tilde{H} = HF(H)G(H) = H + h.o.t.. \tag{1.8.36}$$

From (1.8.36) and Theorem 1.8.5, we have

$$\begin{aligned}\frac{d\tilde{H}}{dT} &= (FG + HF'G + HFG')\frac{dH}{dT}\\ &= (FG + HF'G + HFG')\sum_{k=1}^{\infty}\mu_k H^{k+1},\end{aligned} \tag{1.8.37}$$

where $FG + HF'G + HFG'$ is a unit formal series of H. By using (1.8.36), we can represent H as a formal series of $\tilde{H}$:

$$H = \tilde{H} + h.o.t.. \tag{1.8.38}$$

Hence, (1.8.37) and (1.8.38) follows the conclusion of this theorem. □

Lemma 1.8.1. *Let F be a formal first integral in a neighborhood of the origin of system* (1.8.4)*. Then, F can be written as a formal power series of ξ, η as follows*:

$$F = C_{mm}(\xi\eta)^m + \sum_{\alpha+\beta=2m+1}^{\infty} C_{\alpha\beta}\xi^\alpha\eta^\beta, \quad C_{mm} \neq 0, \tag{1.8.39}$$

where m is a positive integer.

Proof. Solving z and w from (1.8.19), we obtain

$$z = z(\xi,\eta) = \xi + h.o.t., \quad w = w(\xi,\eta) = \eta + h.o.t.. \tag{1.8.40}$$

Hence, F can be written as the following formal power series of ξ, η:

$$F = \sum_{k=n} F_n(\xi,\eta) = \sum_{\alpha+\beta=n}^{\infty} C_{\alpha\beta}\xi^\alpha\eta^\beta, \tag{1.8.41}$$

where n is a positive integer. $F_k(\xi,\eta)$ is a homogeneous polynomial of degree k of ξ, η. F_n is a non-zero polynomial. From (1.8.41) and (1.8.20), we have

$$\begin{aligned}0 &= \frac{dF}{dT} = \frac{\partial F}{\partial \xi}\frac{d\xi}{dT} + \frac{\partial F}{\partial \eta}\frac{d\eta}{dT}\\ &= \frac{\partial F_n}{\partial \xi}\xi - \frac{\partial F_n}{\partial \eta}\eta + h.o.t.\\ &= \sum_{\alpha+\beta=n}(\alpha-\beta)C_{\alpha\beta}\xi^\alpha\eta^\beta + h.o.t.\end{aligned} \tag{1.8.42}$$

It implies that

$$\sum_{\alpha+\beta=n}(\alpha-\beta)C_{\alpha\beta}\xi^\alpha\eta^\beta = 0. \tag{1.8.43}$$

Since F_n is a non-zero polynomial, we see from (1.8.43) that $n = 2m$, $\mathcal{H}_{2m} = C_{mm}\xi^m\eta^m$, $C_{mm} \neq 0$. Thus, the assertion of this lemma holds.

□

Theorem 1.8.7. *System* (1.8.4) *has a formal first integral in a neighborhood of the origin if and only if all singular point values of the origin are zero.*

Proof. First, we prove the sufficiency of theorem. If all singular point values are zeros , then by Theorem 1.8.2, the power series of ξ, η have a nonzero convergent radius. Theorem 1.8.5 implies that $H = \xi\eta$ is a first integral in a neighborhood of the origin, which is a power series of z, w with a a nonzero convergent radius.

Second, we prove the necessity of theorem. Suppose that system (1.8.4) has a formal first integral F in a neighborhood of the origin. By Lemma 1.8.1, F can be written as the form of (1.8.41). From (1.8.40) and (1.8.20), we have

$$\begin{aligned}\frac{dF}{dT} = &\left(mC_{mm}\xi^m\eta^m + \sum_{\alpha+\beta=2m+1}^{\infty} \alpha C_{\alpha\beta}\xi^\alpha\eta^\beta\right)\left(1+\sum_{k=1}^{\infty} p_k\xi^k\eta^k\right)\\ &-\left(mC_{mm}\xi^m\eta^m + \sum_{\alpha+\beta=2m+1}^{\infty} \beta C_{\alpha\beta}\xi^\alpha\eta^\beta\right)\left(1+\sum_{k=1}^{\infty} q_k\xi^k\eta^k\right).\end{aligned} \tag{1.8.44}$$

It can be represented by a formal power series of ξ, η as follows:

$$\frac{dF}{dT} = \sum_{\alpha+\beta=2m}^{\infty} D_{\alpha\beta}\xi^\alpha\eta^\beta. \tag{1.8.45}$$

Since F is a formal first integral in a neighborhood of the origin for system (1.8.4), therefore, all $D_{\alpha\beta}$ must be zeros. From (1.8.44) and (1.8.45), we have

$$\begin{aligned}0 &= \sum_{k=m}^{\infty} D_{kk}\xi^k\eta^k\\ &= \left(mC_{mm}\xi^m\eta^m + \sum_{j=m+1}^{\infty} jC_{jj}\xi^j\eta^j\right)\left(1+\sum_{k=1}^{\infty} p_k\xi^k\eta^k\right)\\ &\quad -\left(mC_{mm}\xi^m\eta^m + \sum_{j=m+1}^{\infty} jC_{jj}\xi^j\eta^j\right)\left(1+\sum_{k=1}^{\infty} q_k\xi^k\eta^k\right)\\ &= \left(mC_{mm}\xi^m\eta^m + \sum_{j=m+1}^{\infty} jC_{jj}\xi^j\eta^j\right)\sum_{k=1}^{\infty} \mu_k\xi^k\eta^k \ = \ 0.\end{aligned} \tag{1.8.46}$$

Because of $C_{mm} \neq 0$. (1.8.46) follows that

$$\sum_{k=1}^{\infty} \mu_k\xi^k\eta^k = 0. \tag{1.8.47}$$

It means that for all k, $\mu_k = 0$. □

Theorem 1.8.8. *If the the origin of system* (1.8.4) *is a complex center, then, in a neighborhood of the origin, any formal first integral* $\mathcal{F}$ *of system* (1.8.4) *can be represented by*

$$\mathcal{F} = \mathcal{F}(H), \tag{1.8.48}$$

where $\mathcal{F}(H)$ *is a formal series of* H.

Proof. Let the origin of system (1.8.4) be a complex center. Then, $H(z, w) = \xi\eta$ is an analytic first integral in a neighborhood of the origin. Suppose that $\mathcal{F}$ is a first integral in a neighborhood of the origin of system (1.8.4), which is represented as a formal series of ξ, η:

$$\mathcal{F} = \sum_{\alpha+\beta=1}^{\infty} C_{\alpha\beta}\xi^{\alpha}\eta^{\beta}. \tag{1.8.49}$$

Write that

$$\mathcal{F}^* = \sum_{k=1}^{\infty} C_{kk}(\xi\eta)^k, \quad \tilde{\mathcal{F}} = \mathcal{F} - \mathcal{F}^*. \tag{1.8.50}$$

Clearly, $\mathcal{F}^*$ is also a formal first integral in a neighborhood of the origin of system (1.8.4). Lemma 1.8.1 follows that $\tilde{\mathcal{F}}$ is not a formal first integral in a neighborhood of the origin of system (1.8.4). Since $\tilde{\mathcal{F}}$ is the difference of two formal first integrals. So that, $\tilde{\mathcal{F}} \equiv 0$. □

Theorem 1.8.8 gives rise to the following conclusion.

Theorem 1.8.9. *If the the origin of system* (1.8.4) *is a complex center, then in a neighborhood of the origin, any analytic first integral of system* (1.8.4) *can be written as a power series of* H *with a nonzero convergent radius.*

Theorem 1.8.10. *The origin of system* (1.8.4) *is a complex center if and only if there exists an analytic integrating factor* $M(z, w)$ *in a neighborhood of the origin with* $M(0, 0) \neq 0$.

Proof. The sufficiency of the conclusion is obvious. We prove the necessity. If the origin of system (1.8.4) is a complex center, Theorem 1.8.2 and Theorem 1.8.5 tell us that the Jacobian determinant $J(z, w)$ of ξ, η with respect to z, w is an analytic integral factor in a neighborhood of the origin and $J(0, 0) = 1$. □

Theorem 1.8.11. *In a neighborhood of the origin, system* (1.8.4) *is linearizable if and only if*

$$p_k = q_k = 0, \quad k = 1, 2, \cdots. \tag{1.8.51}$$

Proof. If (1.8.51) holds, then system (1.8.20) is just a linear system.

Suppose that system (1.8.4) can be linearized in a neighborhood of the origin. Thus, there exists a normal transformation (1.8.6) in a neighborhood of origin, such that system (1.8.4) is reduced to the linear system

$$\frac{d\tilde{\xi}}{dT} = \tilde{\xi}, \quad \frac{d\tilde{\eta}}{dT} = -\tilde{\eta}. \tag{1.8.52}$$

Denote that

$$\tilde{H} = \tilde{\xi}\tilde{\eta}, \quad \tilde{\Omega} = \frac{1}{2i} \ln \frac{\tilde{\xi}}{\tilde{\eta}}. \tag{1.8.53}$$

From (1.8.52), we have

$$\frac{d\tilde{H}}{dT} \equiv 0, \quad \frac{d\tilde{\Omega}}{dT} \equiv -i. \tag{1.8.54}$$

It follows that $\tilde{H}$ is a first integral in a neighborhood of the origin of system (1.8.4). Hence, by Theorem 1.8.7, we obtain

$$\mu_k = 0, \quad k = 1, 2, \cdots. \tag{1.8.55}$$

From Theorem 1.8.4, there are two unit formal series $F(H)$ and $G(H)$ of H, such that

$$\tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H). \tag{1.8.56}$$

From (1.8.33), (1.8.53) and (1.8.56), we have

$$\tilde{\Omega} - \Omega = \frac{1}{2i} \ln \frac{F(H)}{G(H)}. \tag{1.8.57}$$

Because the right side of (1.8.57) is a formal power series of H, from (1.8.31), (1.8.55) and (1.8.57), we obtain

$$\frac{d\tilde{\Omega}}{dT} = \frac{d\Omega}{dT} = \frac{1}{2i}\left(2 + \sum_{k=1}^{\infty} \tau_k H^k\right). \tag{1.8.58}$$

From (1.8.54) and (1.8.58), we have

$$\tau_k = 0, \quad k = 1, 2, \cdots. \tag{1.8.59}$$

Thus, (1.8.30), (1.8.55) and (1.8.59) give rise to (1.8.51). □

Theorem 1.8.12. *In system* (1.8.4)*, if for all* α *and* β*, the relationships* $b_{\alpha\beta} = \bar{a}_{\alpha\beta}$ *hold. Then in* (1.8.19) *and* (1.8.20)*, we have that* $\forall k, j$, $c_{kj} = \bar{d}_{kj}$, $p_k = \bar{q}_k$.

Proof. The relationships $\forall(\alpha, \beta)$, $b_{\alpha\beta} = \bar{a}_{\alpha\beta}$ imply that $A_{\alpha\beta}, B_{\alpha\beta}$ are real numbers in (1.8.2). Let x, y, t be real variables. Then (1.8.1) is real planar differential system. From (1.8.3), we have

$$\bar{z} = w, \quad \bar{w} = z, \quad T^* = -T. \tag{1.8.60}$$

By (1.8.19) and (1.8.60), we obtain

$$\bar{\eta} = z + \sum_{k+j=2}^{\infty} \bar{d}_{kj} z^k w^j, \quad \bar{\xi} = w + \sum_{k+j=2}^{\infty} \bar{c}_{kj} w^k z^j. \tag{1.8.61}$$

Denote that

$$\xi^* = \bar{\eta}, \quad \eta^* = \bar{\xi}. \tag{1.8.62}$$

Making the conjugated transformation on the two sides of (1.8.20), from (1.8.60) and (1.8.62), we have

$$\frac{d\xi^*}{dT^*} = \xi^* + \xi^* \sum_{k=1}^{\infty} \bar{q}_k (\xi^* \eta^*)^k, \quad \frac{d\eta^*}{dT^*} = -\eta^* - \eta^* \sum_{k=1}^{\infty} \bar{p}_k (\xi^* \eta^*)^k. \tag{1.8.63}$$

(1.8.61) and (1.8.63) follows that (1.8.62) is a standard normal transformation in a neighborhood of the origin of system (1.8.4). The uniqueness of the standard normal transformation gives that

$$\eta = \bar{\xi}, \quad \bar{q}_k = p_k, \quad k = 1, 2, \cdots. \tag{1.8.64}$$

It follows the conclusion of this theorem. □

For system (1.8.1), consider the normal transformation

$$\begin{aligned} u &= \frac{\xi(x+iy, x-iy) + \eta(x+iy, x-iy)}{2} = x + \sum_{k+j=2}^{\infty} c'_{kj} x^k y^j, \\ v &= \frac{\xi(x+iy, x-iy) - \eta(x+iy, x-iy)}{2i} = y + \sum_{k+j=2}^{\infty} d'_{kj} x^k y^j. \end{aligned} \tag{1.8.65}$$

Theorem 1.8.12 implies that if all coefficients on the right side of system (1.8.1) are real numbers, then u, v are power series of x, y having real coefficients. (1.8.20) and (1.8.3) follow the following conclusion given by [Amelikin etc, 1982].

Theorem 1.8.13. *By using formal transformation* (1.8.65), *complex autonomous differential system* (1.8.1) *can become a normal form as follows*:

$$\begin{aligned} \frac{du}{dt} &= -v + \frac{1}{2} \sum_{k=1}^{\infty} (\sigma_k u - \tau_k v)(u^2 + v^2)^k = U(u, v), \\ \frac{dv}{dt} &= u + \frac{1}{2} \sum_{k=1}^{\infty} (\tau_k u + \sigma_k v)(u^2 + v^2)^k = V(u, v), \end{aligned} \tag{1.8.66}$$

where

$$\sigma_k = i(p_k - q_k), \quad \tau_k = p_k + q_k \tag{1.8.67}$$

and all σ_k, τ_k, c'_{kj} *and* d'_{kj} *are polynomials of* $A_{\alpha\beta}, B_{\alpha\beta}$ *with rational coefficients.*

Proposition 1.1.2 and Theorem 1.8.13 imply the following three important formulas.

Theorem 1.8.14. *For system* (1.8.1), *we have*

$$\frac{d\mathcal{H}}{dt} = \sum_{k=1}^{\infty} \sigma_k \mathcal{H}^{k+1}, \quad \frac{d\omega}{dt} = 1 + \frac{1}{2}\sum_{k=1}^{\infty} \tau_k \mathcal{H}^k \tag{1.8.68}$$

and

$$\frac{\partial(\mathcal{J}X)}{\partial x} + \frac{\partial(\mathcal{J}Y)}{\partial y} = \mathcal{J}\left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right) = \mathcal{J}\sum_{k=1}^{\infty}(k+1)\sigma_k \mathcal{H}^k, \tag{1.8.69}$$

where

$$\mathcal{H} = u^2 + v^2, \quad \omega = \arctan\frac{v}{u}, \quad \mathcal{J} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}. \tag{1.8.70}$$

Theorem 1.8.7 gives the following conclusion.

Theorem 1.8.15. *System* (1.8.4) *has a formal first integral in a neighborhood of the origin if and only if all* $\sigma_k = 0$.

From Theorem 1.8.8 and Theorem 1.8.9, we have

Theorem 1.8.16. *If the origin of system* (1.8.1) *is a complex center, then any first integral* $\mathcal{F}$ *in a neighborhood of the origin of system* (1.8.1) *can be written by*

$$\mathcal{F} = \mathcal{F}(\mathcal{H}), \tag{1.8.71}$$

where $\mathcal{F}(\mathcal{H})$ *is a formal series of* $\mathcal{H}$. *In addition, any analytic first integral in a neighborhood of the origin of system* (1.8.1) *can be represented as a power series of* $\mathcal{H}$ *with a nonzero convergent radius.*

Theorem 1.8.10 derives the following conclusion.

Theorem 1.8.17. *The origin of system* (1.8.1) *is a complex center if and only if there exists an analytic integral factor* $\mathcal{M}(x, y)$ *in a neighborhood of the origin and* $\mathcal{M}(0, 0) \neq 0$.

By Theorem 1.8.11, we have

Theorem 1.8.18. *System* (1.8.1) *is linearizable in a neighborhood of the origin if and only if for all positive integer* k, $\sigma_k = 0$, $\tau_k = 0$.

Definition 1.8.3. *Suppose that the functions of the right side of system* (1.8.1) *satisfy*

$$X(x, -y) = -X(x, y), \quad Y(x, -y) = Y(x, y). \tag{1.8.72}$$

We say that the functions of the right side of system (1.8.1) *are symmetric with respect to the coordinate* x.

Suppose that the functions of the right side of system (1.8.1) *satisfy*

$$X(-x, y) = X(x, y), \quad Y(-x, y) = -Y(x, y), \tag{1.8.73}$$

we say that the functions of the right side of system (1.8.1) *are symmetric with respect to the coordinate* y.

If one of (1.8.72) *and* (1.8.73) *satisfies, we say that system* (1.8.1) *is a symmetric system with respect to a coordinate.*

From Corollary 1.4.1, we have

Theorem 1.8.19 (The symmetric principle). *Suppose that* (1.8.1) *is a symmetric system with respect to a coordinate. Then, it has a analytic first integral in a neighborhood of the origin.*

Since the coefficients of the right side of system (1.8.1) can be complex, Theorem 1.8.19 expands the symmetric principle for the center-focus problem in real planar differential autonomous systems.

Theorem 1.8.20 (The anti-symmetric principle). *Suppose that the origin of system* (1.8.1) *is a complex center. Then, there exist two power series* u, v *of* x, y:

$$u = x + h.o.t., \quad v = y + h.o.t.. \tag{1.8.74}$$

with a nonzero convergent radius, such that by transformation (1.8.74), *system* (1.8.1) *becomes a symmetric system.*

Proof. Since the origin of system (1.8.1) is a complex center, by Theorem 1.8.2, the functions of (1.8.65) are power series of x, y with nonzero convergent radius. From Theorem 1.8.13 and Theorem 1.8.15, we see that by the transformation (1.8.65), system (1.8.1) can become the following symmetric system:

$$\begin{aligned}
\frac{du}{dt} &= -v - \frac{1}{2}v\sum_{k=1}^{\infty}\tau_k(u^2+v^2)^k,\\
\frac{dv}{dt} &= u + \frac{1}{2}u\sum_{k=1}^{\infty}\tau_k(u^2+v^2)^k.
\end{aligned} \tag{1.8.75}$$

□

By using the above two theorems, we obtain a method to check if the origin is a complex center. In fact, for a given system of the form (1.8.1), if we find a suitable transformation to make this system become a symmetric system, then the origin of the system is a complex center.

Example 1.8.1. *Consider the real planar differential system*

$$\begin{aligned}\frac{dx}{dt} &= -y + x^2,\\ \frac{dy}{dt} &= x + 2x^3 - 5ax^8 - 2(1-4ax^5)xy - 4ax^2y^3 + ay^4.\end{aligned} \tag{1.8.76}$$

By the transformation $u = x, v = y - x^2$*, system* (1.8.76) *becomes*

$$\frac{du}{dt} = -v, \quad \frac{dv}{dt} = u - 6au^4v^2 + av^4. \tag{1.8.77}$$

Letting $\xi = v^2$*, the above system reduces to the Riccati equation*

$$\frac{d\xi}{du} = -2(u - 6au^4\xi + a\xi^2). \tag{1.8.78}$$

The functions of the right side of system (1.8.77) *is symmetric with respect to the variable* v*. Therefore, Theorem* 1.8.19 *follows that the origin of system* (1.8.76) *is a center.*

We now consider the existence of integrating factor in a neighborhood of the origin when the origin of system (1.8.20) is a m-order weak critical singular point. It is easy to show that the following conclusion holds.

Theorem 1.8.21. *Let the origin of system* (1.8.20) *be a* m*-order weak critical singular point. Then, in a neighborhood of the origin, system* (1.8.20) *has the following integrating factor* :

$$M(\xi,\eta) = \frac{1}{H^{m+1}\left(1 + \sum_{k=1}^{\infty} \frac{\mu_{m+k}}{\mu_m} H^k\right)} = \frac{1}{H^{m+1}}\left(1 + h.o.t.\right), \tag{1.8.79}$$

where $H = \xi\eta$*.*

From Proposition 1.1.4 and Theorem 1.8.21, we have

Theorem 1.8.22. *Let the origin of system* (1.8.4) *be a* m*-order weak critical singular point. Then, in a neighborhood of the origin, system* (1.8.4) *has the following integrating factor* :

$$\mathcal{M}(z,w) = JM = \frac{1}{(zw + h.o.t.)^{m+1}}, \tag{1.8.80}$$

where H *and* J *are given by* (1.8.33)*,* M *is given by* (1.8.79)*.*

Similarly, we have

Theorem 1.8.23. *Let the origin of system* (1.8.66) *be a m-order weak critical singular point. Then, in a neighborhood of the origin, system* (1.8.66) *has the following integrating factor* :

$$M^*(u,v) = \frac{1}{\mathcal{H}^{m+1}\left(1+\sum_{k=1}^{\infty}\frac{\sigma_{m+k}}{\sigma_m}\mathcal{H}^k\right)} = \frac{1}{\mathcal{H}^{m+1}}(1+h.o.t.)\,. \tag{1.8.81}$$

where $\mathcal{H} = u^2 + v^2$.

Theorem 1.8.24. *Let the origin of system* (1.8.1) *be a m-order weak focus. Then, in a neighborhood of the origin, system* (1.8.1) *has the following integrating factor*:

$$\mathcal{M}^*(x,y) = \mathcal{J}M^* = \frac{1}{\mathcal{H}^{m+1}}(1+h.o.t.) = \frac{1}{(x^2+y^2+h.o.t.)^{m+1}}, \tag{1.8.82}$$

where $\mathcal{H}$ and $\mathcal{J}$ are given by (1.8.70), *M^* is given by* (1.8.81).

Theorem 1.8.25. *Suppose that in a neighborhood of the origin, system* (1.8.1) *has an integrating factor $\tilde{\mathcal{M}}^*(x,y)$ with the form $f^s(x,y)G(x,y)$ and $s+1$ is not a negative integer, where*

$$f(x,y) = x^2+y^2+h.o.t., \quad G(x,y) = 1+h.o.t. \tag{1.8.83}$$

are two formal series of x,y. Then, the origin of (1.8.1) *is a complex center.*

Proof. We use reductio ad absurdum. Suppose that the origin of system (1.8.1) is not a complex center but a m order weak focus. Then, Theorem 1.8.24 follows that there is a first integral of (1.8.1)

$$F(x,y) = \frac{\tilde{\mathcal{M}}^*(x,y)}{\mathcal{M}^*(x,y)} = f^s(x,y)G(x,y)(x^2+y^2+h.o.t.)^{m+1} \tag{1.8.84}$$

in a neighborhood of the origin. From (1.8.83) and (1.8.84), $F(r\cos\theta, r\sin\theta)$ has the form

$$F(r\cos\theta, r\sin\theta) = r^{2(s+m+1)}\left[1+\sum_{k=1}^{\infty}\zeta_k(\theta)r^k\right], \tag{1.8.85}$$

where for all k, $\zeta_k(\theta)$ are polynomials of $\cos\theta, \sin\theta$ $(k=1,2,\cdots)$. Because $s+1$ is not a negative integer, hence, $s+m+1 \neq 0$. (1.8.85) implies that the origin of system (1.8.1) is a complex center which is in contradiction to the original hypothesis. □

We next consider the case of $s+1$ is a negative integer. Let $s+1=-k$, where k is a positive integer. Then, $\tilde{\mathcal{M}}^*(x,y)$ in Theorem 1.8.25 has the form:

$$\tilde{\mathcal{M}}^*(x,y) = \frac{G(x,y)}{f^{k+1}(x,y)}. \tag{1.8.86}$$

Theorem 1.8.26. *Suppose that system* (1.8.1) *has an integrating factor* $\tilde{\mathcal{M}}^*(x,y)$ *with the form* (1.8.86) *in a neighborhood of the origin. If the origin is not a k order weak focus. Then the origin of* (1.8.1) *is a complex center.*

Proof. We use reductio ad absurdum. Suppose that the origin of system (1.8.1) is not a complex center but a m order weak focus, where $k \neq m$. Then, Theorem 1.8.24 follows that in a neighborhood of the origin of system (1.8.1), there is the following first integral

$$F(x,y) = \frac{\tilde{\mathcal{M}}^*(x,y)}{\mathcal{M}^*(x,y)} = \frac{G(x,y)(x^2+y^2+h.o.t.)^{m+1}}{f^{k+1}(x,y)}. \tag{1.8.87}$$

Since $k \neq m$, (1.8.87) implies the origin of system (1.8.1) is a complex center. It is in contradiction to the original hypothesis. □

Similarly, we have

Theorem 1.8.27. *If system* (1.8.4) *has an integrating factor* $\tilde{\mathcal{M}}(z,w)$ *of the form* $f^s(z,w)G(z,w)$ *in a neighborhood of the origin, where $s+1$ is not a negative integer and*

$$f(z,w) = zw + h.o.t., \quad G(z,w) = 1 + h.o.t. \tag{1.8.88}$$

are two formal series of z, w. Then the origin of (1.8.4) *is a complex center.*

If $s+1=-k$, where k is a positive integer. Then $\tilde{\mathcal{M}}(z,w)$ given by Theorem 1.8.27 becomes

$$\tilde{\mathcal{M}}(z,w) = \frac{G(z,w)}{f^{k+1}(z,w)}. \tag{1.8.89}$$

Theorem 1.8.28. *Suppose that system* (1.8.4) *has an integrating factor* $\tilde{\mathcal{M}}(z,w)$ *of the form* (1.8.89) *in a neighborhood of the origin. If the origin of* (1.8.4) *is not a k-order weak critical singular point. Then, the origin of* (1.8.4) *is a complex center.*

Example 1.8.2. *System*

$$\frac{dz}{dT} = z + 3zw(az^2+bw^2), \quad \frac{dw}{dT} = -w \tag{1.8.90}$$

has an integrating factor

$$\tilde{\mathcal{M}}(z,w) = \frac{1}{(zw)^3 e^{2bw^3}}\ . \tag{1.8.91}$$

This is a singular factor. However, we have the first two singular point values $\mu_1 = \mu_2 = 0$ of the origin of system (1.8.90). *The origin is not a 2-order weak critical singular point. By Theorem* 1.8.28, *the origin is a complex center.*

Example 1.8.2 tells us that Theorem 1.8.25 ~ Theorem 1.8.28 are useful for solving the center problem.

Definition 1.8.4. *If there are a constant* $\gamma \neq 0$ *and three formal series of* (z^*, w^*):

$$\varphi(z^*, w^*) = \gamma w^* + h.o.t., \quad \psi(z^*, w^*) = \frac{1}{\gamma} z^* + h.o.t., \quad G(z^*, w^*) = 1 + h.o.t., \tag{1.8.92}$$

such that by using the transformation

$$z = \varphi(z^*, w^*), \quad w = \psi(z^*, w^*) \tag{1.8.93}$$

system (1.8.4) *becomes*

$$\frac{dz^*}{dT} = -Z(z^*, w^*)G(z^*, w^*), \quad \frac{dw^*}{dT} = W(z^*, w^*)G(z^*, w^*). \tag{1.8.94}$$

Then, system (1.8.4) *is called generalized time-reversible system.*

Let the origin of system (1.8.4) is a complex center. Then, in a neighborhood of the origin, the standard normal form of (1.8.4) has the form

$$\begin{aligned}
\frac{d\xi}{dT} &= \xi \left[1 + \frac{1}{2} \sum_{k=1}^{\infty} \tau_k (\xi\eta)^k \right] = \Phi(\xi, \eta), \\
\frac{d\eta}{dT} &= -\eta \left[1 + \frac{1}{2} \sum_{k=1}^{\infty} \tau_k (\xi\eta)^k \right] = -\Psi(\xi, \eta).
\end{aligned} \tag{1.8.95}$$

By using the transformation $\xi = \eta^*, \eta = \xi^*$, system (1.8.94) can become the following system:

$$\frac{d\xi^*}{dT} = -\Phi(\xi^*, \eta^*), \quad \frac{d\eta^*}{dT} = \Psi(\xi^*, \eta^*), \tag{1.8.96}$$

It implies that system (1.8.95) is generalized time-reversible system.

Theorem 1.8.29. *If the origin of system* (1.8.4) *is a complex center, then, by using a suitable analytic transformation, system* (1.8.4) *can become a generalized time-reversible system.*

The following is the converse theorem of Theorem 1.8.29.

Theorem 1.8.30. *If system* (1.8.4) *is a generalized time-reversible system, then the origin of system* (1.8.4) *is a complex center.*

Proof. We use reductio ad absurdum. Suppose that system (1.8.4) is a generalized time-reversible system and its origin is not a complex center. Then, there is a positive integer m, such that $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0$, $\mu_m \neq 0$. Thus, there is a polynomial $F(z, w) = zw + h.o.t.$, such that

$$\begin{aligned}
\left.\frac{dF(z, w)}{dT}\right|_{(1.8.4)} &= \frac{\partial F(z, w)}{\partial z} Z(z, w) - \frac{\partial F(z, w)}{\partial w} W(z, w) \\
&= \mu_m (zw)^{m+1} + h.o.t..
\end{aligned} \tag{1.8.97}$$

By using (1.8.93), in a neighborhood of the origin, we can solve z^* and w^* as follows:

$$z^* = \varphi^*(z,w) = \gamma w + h.o.t., \quad w^* = \psi^*(z,w) = \frac{1}{\gamma} z + h.o.t.. \tag{1.8.98}$$

Let

$$F^*(z,w) = F(\varphi^*(z,w), \quad \psi^*(z,w)) = zw + h.o.t.. \tag{1.8.99}$$

We see from (1.8.97) that

$$\begin{aligned}
&\left.\frac{dF^*(z,w)}{dT}\right|_{(1.8.4)} = \left.\frac{dF(z^*,w^*)}{dT}\right|_{(1.8.94)} \\
=&-G(z^*,w^*)\left[\frac{\partial F(z^*,w^*)}{\partial z^*} Z(z^*,w^*) - \frac{\partial F(z^*,w^*)}{\partial w^*} W(z^*,w^*)\right] \\
=&-\mu_m (zw)^{m+1} + h.o.t..
\end{aligned} \tag{1.8.100}$$

(1.8.97) and (1.8.100) follows that $\mu_m = 0$. It gives the conclusion of this theorem. □

Example 1.8.3. *Consider the following system*

$$\begin{aligned}
&\frac{dz}{dT} = Z(z,w) \\
=&z + 9(7-8\lambda)[3(1+4\lambda)z^3 + 9\lambda w z^2 + (7-8\lambda)w^2 z - 3(2-\lambda)w^3]z, \\
&\frac{dw}{dT} = -W(z,w) \\
=&-w + 9(7-8\lambda)[3(1+4\lambda)w^3 + (4+7\lambda)w^2 z - 9wz^2 + 3(2-\lambda)z^3]w.
\end{aligned} \tag{1.8.101}$$

This system has a algebraic integral

$$f(z,w) = 1 + 27(1+4\lambda)(7-8\lambda)(w+z)^2(z-w). \tag{1.8.102}$$

Let

$$z^* = \frac{w}{f^{\frac{1}{3}}(z,w)}, \quad w^* = \frac{z}{f^{\frac{1}{3}}(z,w)} . \tag{1.8.103}$$

Then

$$z = \frac{w^*}{f^{\frac{1}{3}}(z^*,w^*)}, \quad w = \frac{z^*}{f^{\frac{1}{3}}(z^*,w^*)}. \tag{1.8.104}$$

By transformation (1.8.103), *system* (1.8.101) *becomes*

$$\frac{dz^*}{dT} = -\frac{Z(z^*,w^*)}{f(z^*,w^*)}, \quad \frac{dw^*}{dT} = \frac{W(z^*,w^*)}{f(z^*,w^*)}. \tag{1.8.105}$$

Thus, system (1.8.101) *is a generalized time-reversible system. By Theorem* 1.8.30, *the origin of system* (1.8.101) *is a complex center.*

1.9 Integrability and Linearized Problem of the Resonant Singular Point

Let the origin of the system (1.5.1) be a p, q resonant singular point. By using a suitable linear transformation, system (1.5.1) become

$$\begin{aligned}\frac{dz}{dT} &= pz + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w),\\ \frac{dw}{dT} &= -qw - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),\end{aligned} \tag{1.9.1}$$

where p, q are two irreducible integers, $Z(z,w), W(z,w)$ are two power series of z, w having nonzero convergent radius. For all k, $Z_k(z,w), W_k(z,w)$ are homogeneous polynomials of degree k of z, w:

$$Z_k(z,w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta}, \quad W_k(z,w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}. \tag{1.9.2}$$

We now cite the definition of a normal form of system(1.9.1) given by [Christopher ect, 2003].

Definition 1.9.1. *Suppose that there are two formal series of z, w*

$$\tilde{\xi} = z + h.o.t., \quad \tilde{\eta} = w + h.o.t., \tag{1.9.3}$$

such that by transformation (1.9.3), *system* (1.9.1) *reduces to the form:*

$$\begin{aligned}\frac{d\tilde{\xi}}{dT} &= \ \ p\tilde{\xi}\left[1 + \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\xi}^q \tilde{\eta}^p)^k\right],\\ \frac{d\tilde{\eta}}{dT} &= -q\tilde{\eta}\left[1 + \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\xi}^q \tilde{\eta}^p)^k\right].\end{aligned} \tag{1.9.4}$$

Then, we say that (1.9.3) *is a normal transformation in a neighborhood of the origin of the system* (1.9.1). *System* (1.9.4) *is a normal form corresponding to the transformation* (1.9.3).

A resonant singular point can be transformed to a weak critical singular point by a suitable transformation. Actually, from Theorem 1.5.4, there are two power series $\varphi(w), \psi(z)$ with nonzero convergent radius, satisfying $\varphi(0) = \psi(0) = \varphi'(0) = \psi'(0) = 0$, such that by the transformation

$$u = z - \varphi(w), \quad v = w - \psi(z), \tag{1.9.5}$$

system (1.9.1) becomes

$$\begin{aligned}\frac{du}{dT} &= p\,u\,U(u,v) = p\,u\left[1+\sum_{k=1}^{\infty}U_k(u,v)\right],\\ \frac{dv}{dT} &= -q\,v\,V(u,v) = -q\,v\left[1+\sum_{k=1}^{\infty}V_k(u,v)\right],\end{aligned} \tag{1.9.6}$$

where $U(u,v), V(u,v)$ are two power series with nonzero convergent radius of u, v. $U_k(u,v)$, $V_k(u,v)$ are homogeneous polynomials of degree k of u, v.

Suppose that $z=\varphi(w)$, $w=\psi(z)$ are analytic solutions of the system (1.9.1), passing through the origin. By using the transformation

$$u = x^p, \quad v = y^q \tag{1.9.7}$$

system (1.9.6) becomes the following special system having weak critical singular point $O(0,0)$:

$$\begin{aligned}\frac{dx}{dT} &= xU(x^p,y^q) = x\left[1+\sum_{k=1}^{\infty}U_k(x^p,y^q)\right],\\ \frac{dy}{dT} &= -yV(x^p,y^q) = -y\left[1+\sum_{k=1}^{\infty}V_k(x^p,y^q)\right].\end{aligned} \tag{1.9.8}$$

Since $x=0$ and $y=0$ are two solutions of system (1.9.8), hence, any normal transformation in a neighborhood of the origin of system (1.9.8) has the form

$$\chi^* = x\left(1+\sum_{\alpha+\beta=1}^{\infty}c^*_{\alpha\beta}x^\alpha y^\beta\right), \quad \zeta^* = y\left(1+\sum_{\alpha+\beta=1}^{\infty}d^*_{\alpha\beta}y^\alpha x^\beta\right). \tag{1.9.9}$$

By Theorem 1.8.1, for the coefficients of the formal series (1.9.9), first, c^*_{kk}, d^*_{kk} are taken as any constant numbers. Then, the other coefficients can be determined uniquely. Corresponding to (1.9.9), the normal form of system (1.9.8) is as follows:

$$\begin{aligned}\frac{d\chi^*}{dT} &= \chi^*\left[1+\sum_{k=1}^{\infty}p^*_k(\chi^*\zeta^*)^k\right],\\ \frac{d\zeta^*}{dT} &= -\zeta^*\left[1+\sum_{k=1}^{\infty}q^*_k(\chi^*\zeta^*)^k\right].\end{aligned} \tag{1.9.10}$$

Compare with system (1.8.4), the right sides functions of the system (1.9.8) have the following properties:

(1) $x=0$ and $y=0$ are two solutions of (1.9.8).

(2) $U(x^p, y^q), V(x^p, y^q)$ are two power series of x^p, y^q.

These properties of (1.9.8) make it have a particular normal transformation and a special normal form.

Definition 1.9.2. *Let* (1.9.9) *be a normal transformation of system* (1.9.8) *in a neighborhood of the origin satisfying for* $m/(pq)$ *are not positive integers,* $c^*_{mm} = d^*_{mm} = 0$. *We say that* (1.9.9) *is a* p, q *resonant normal transformation. The corresponding normal form is called the* p, q *resonant normal form.*

Obviously, the standard normal transformation in a neighborhood of the origin of system (1.9.8) is p, q resonant.

Theorem 1.9.1. *For all given* $\tilde{c}_{kq,\ kp}$ *and* $\tilde{d}_{kp,\ kq}$ $(k = 1, 2, \cdots)$, *one can derive uniquely and successively the terms of the formal series*

$$\tilde{\chi} = x\left(1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{c}_{\alpha\beta} x^{\alpha p} y^{\beta q}\right) = x\tilde{f}(x^p, y^q),$$
$$\tilde{\zeta} = y\left(1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{d}_{\alpha\beta} y^{\alpha q} x^{\beta p}\right) = y\tilde{g}(x^p, y^q), \tag{1.9.11}$$

such that by transformation (1.9.11), *system* (1.9.8) *becomes the following normal form*

$$\frac{d\tilde{\chi}}{dT} = \tilde{\chi}\left[1 + \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\chi}\tilde{\zeta})^{kpq}\right],$$
$$\frac{d\tilde{\zeta}}{dT} = -\tilde{\zeta}\left[1 + \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\chi}\tilde{\zeta})^{kpq}\right]. \tag{1.9.12}$$

.

Proof. Write that

$$\tilde{f}(u, v) = 1 + \sum_{m=1}^{\infty} \tilde{f}_m(u, v),$$
$$\tilde{g}(u, v) = 1 + \sum_{m=1}^{\infty} \tilde{g}_m(u, v), \tag{1.9.13}$$

where $\tilde{f}_m(u, v), \tilde{g}_m(u, v)$ are homogeneous polynomials of degree m of u, v:

$$\tilde{f}_m(u, v) = \sum_{\alpha+\beta=m} \tilde{c}_{\alpha\beta} u^\alpha v^\beta, \quad \tilde{g}_m(u, v) = \sum_{\alpha+\beta=m} \tilde{d}_{\alpha\beta} v^\alpha u^\beta. \tag{1.9.14}$$

From (1.9.8), (1.9.11) and (1.9.13), we have

$$\begin{aligned} \frac{d\tilde{\chi}}{dT} - \tilde{\chi} &= x \sum_{m=1}^{\infty} \left[pu \frac{\partial \tilde{f}_m}{\partial u} - qv \frac{\partial \tilde{f}_m}{\partial v} + F_m(u,v) \right], \\ \frac{d\tilde{\zeta}}{dT} + \tilde{\zeta} &= y \sum_{m=1}^{\infty} \left[pu \frac{\partial \tilde{g}_m}{\partial u} - qv \frac{\partial \tilde{g}_m}{\partial v} - G_m(u,v) \right], \end{aligned} \tag{1.9.15}$$

where $F_m(u,v), G_m(u,v)$ are homogeneous polynomials of degree m of u, v:

$$\begin{aligned} F_m &= U_m + \sum_{k=1}^{m-1} \left[\left(\tilde{f}_k + pu \frac{\partial \tilde{f}_k}{\partial u} \right) U_{m-k} - qv \frac{\partial \tilde{f}_k}{\partial v} V_{m-k} \right], \\ G_m &= V_m + \sum_{k=1}^{m-1} \left[\left(\tilde{g}_k + qv \frac{\partial \tilde{g}_k}{\partial v} \right) V_{m-k} - pu \frac{\partial \tilde{g}_k}{\partial u} U_{m-k} \right]. \end{aligned} \tag{1.9.16}$$

By using (1.9.11), we obtain

$$\begin{aligned} \tilde{\chi} \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\chi}\tilde{\zeta})^{kpq} &= x \sum_{k=1}^{\infty} \tilde{p}_k u^{kq} v^{kp} \tilde{f}^{kpq+1}(u,v) \tilde{g}^{kpq}(u,v), \\ \tilde{\zeta} \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\chi}\tilde{\zeta})^{kpq} &= y \sum_{k=1}^{\infty} \tilde{q}_k u^{kq} v^{kp} \tilde{g}^{kpq+1}(u,v) \tilde{f}^{kpq}(u,v). \end{aligned} \tag{1.9.17}$$

Let

$$\begin{aligned} \tilde{\chi} \sum_{k=1}^{\infty} \tilde{p}_k (\tilde{\chi}\tilde{\zeta})^{kpq} &= x \left[\sum_{k=1}^{\infty} \tilde{p}_k u^{kq} v^{kp} + \sum_{m=1}^{\infty} \Phi_m(u,v) \right], \\ \tilde{\zeta} \sum_{k=1}^{\infty} \tilde{q}_k (\tilde{\chi}\tilde{\zeta})^{kpq} &= y \left[\sum_{k=1}^{\infty} \tilde{q}_k u^{kq} v^{kp} + \sum_{m=1}^{\infty} \Psi_m(u,v) \right]. \end{aligned} \tag{1.9.18}$$

Then, by (1.9.13), (1.9.17) and (1.9.18), we see that for any integer m, Φ_m, Ψ_m only depend on $\tilde{f}_1, \tilde{f}_2, \cdots, \tilde{f}_{m-1}$ and $\tilde{g}_1, \tilde{g}_2, \cdots, \tilde{g}_{m-1}$. For any positive integer k, when $(k-1)(p+q) < m \leqslant k(p+q)$, Φ_m, Ψ_m only depend on $\tilde{p}_1, \tilde{p}_2, \cdots, \tilde{p}_{k-1}$ and $\tilde{q}_1, \tilde{q}_2, \cdots, \tilde{q}_{k-1}$. We know from (1.9.15) and (1.9.18) that (1.9.12) holds if and only if

$$\begin{aligned} \sum_{m=1}^{\infty} \left(pu \frac{\partial \tilde{f}_m}{\partial u} - qv \frac{\partial \tilde{f}_m}{\partial v} \right) &= \sum_{m=1}^{\infty} (\Phi_m - F_m) + \sum_{k=1}^{\infty} \tilde{p}_k u^{kq} v^{kp}, \\ \sum_{m=1}^{\infty} \left(qv \frac{\partial \tilde{g}_m}{\partial v} - pu \frac{\partial \tilde{g}_m}{\partial u} \right) &= \sum_{m=1}^{\infty} (\Psi_m - G_m) + \sum_{k=1}^{\infty} \tilde{q}_k u^{kq} v^{kp}. \end{aligned} \tag{1.9.19}$$

From (1.9.14), for any positive integer m, we have

$$
\begin{aligned}
pu\frac{\partial \tilde{f}_m}{\partial u} - qv\frac{\partial \tilde{f}_m}{\partial v} &= \sum_{\alpha+\beta=m} (\alpha p - \beta q)\tilde{c}_{\alpha\beta} u^\alpha v^\beta, \\
qv\frac{\partial \tilde{g}_m}{\partial v} - pu\frac{\partial \tilde{g}_m}{\partial u} &= \sum_{\alpha+\beta=m} (\beta q - \alpha p)\tilde{d}_{\beta\alpha} u^\alpha v^\beta.
\end{aligned} \tag{1.9.20}
$$

Denote that

$$
\Phi_m - F_m = \sum_{\alpha+\beta=m} C_{\alpha\beta} u^\alpha v^\beta, \quad \Psi_m - G_m = \sum_{\alpha+\beta=m} D_{\alpha\beta} v^\alpha u^\beta. \tag{1.9.21}
$$

Then, from (1.9.19), (1.9.20) and (1.9.21) we get

$$
\begin{aligned}
\sum_{m=1}^{\infty}\sum_{\alpha+\beta=m} (\alpha p - \beta q)\tilde{c}_{\alpha\beta} u^\alpha v^\beta &= \sum_{m=1}^{\infty}\sum_{\alpha+\beta=m} C_{\alpha\beta} u^\alpha v^\beta + \sum_{k=1}^{\infty} \tilde{p}_k u^{kq} v^{kp}, \\
\sum_{m=1}^{\infty}\sum_{\alpha+\beta=m} (\beta q - \alpha p)\tilde{d}_{\beta\alpha} u^\alpha v^\beta &= \sum_{m=1}^{\infty}\sum_{\alpha+\beta=m} D_{\beta\alpha} u^\alpha v^\beta + \sum_{k=1}^{\infty} \tilde{q}_k u^{kq} v^{kp}.
\end{aligned} \tag{1.9.22}
$$

Since p and q are two irreducible integers, hence for any natural numbers α, β, when $\alpha + \beta \geqslant 1$, $\alpha p - \beta q = 0$ if and only if there exists a positive integer k, such that $\alpha = kp, \beta = kq$. Thus, (1.9.22) follows that for any natural numbers α, β, when $\alpha p - \beta q \neq 0$, all $\tilde{c}_{\alpha\beta}, \tilde{d}_{\beta\alpha}$ are determined uniquely by the following recursive formulas

$$
\tilde{c}_{\alpha\beta} = \frac{1}{\alpha p - \beta q} C_{\alpha\beta}, \quad \tilde{d}_{\beta\alpha} = \frac{1}{\beta q - \alpha p} D_{\beta\alpha}. \tag{1.9.23}
$$

Moreover, for any positive integer k, $\tilde{p}_k, \tilde{q}_k$ are determined uniquely by the following recursive formulas

$$
\tilde{p}_k = -C_{kq,\ kp}, \quad \tilde{q}_k = -D_{kp,\ kq}. \tag{1.9.24}
$$

Because all coefficients of $\tilde{c}_{kq,kp}$ and $\tilde{d}_{kp,kq}$ in (1.9.22) are zeros. So that, $\tilde{c}_{kq,kp}$ and $\tilde{d}_{kp,kq}$ can be given as arbitrary constants in advance. □

From Theorem 1.9.1 and Corollary 1.8.1 we have

Theorem 1.9.2. *In a neighborhood of the origin, any p, q resonant normal transformation of system* (1.9.8) *has the form of* (1.9.11). *The corresponding normal form has the form of* (1.9.12).

In a neighborhood of the origin, the standard normal transformation and the standard normal form of the origin of system (1.9.8) can be written respectively as follows:

$$\chi = x\left(1 + \sum_{\alpha+\beta=1}^{\infty} c_{\alpha\beta} x^{\alpha p} y^{\beta q}\right) = xf(x^p, y^q),$$
$$\zeta = y\left(1 + \sum_{\alpha+\beta=1}^{\infty} d_{\alpha\beta} y^{\alpha q} x^{\beta p}\right) = yg(x^p, y^q), \tag{1.9.25}$$

$$\frac{d\chi}{dT} = \chi\left[1 + \sum_{k=1}^{\infty} p_k (\chi\zeta)^{kpq}\right],$$
$$\frac{d\zeta}{dT} = -\zeta\left[1 + \sum_{k=1}^{\infty} q_k (\chi\zeta)^{kpq}\right], \tag{1.9.26}$$

where

$$c_{kq,\ kp} = d_{kp,\ kq} = 0, \quad k = 1, 2, \cdots. \tag{1.9.27}$$

Theorem 1.9.3. *Let* (1.9.11) *be a* p, q *resonant normal transformation in a neighborhood of the origin of system* (1.9.8) *and its corresponding normal form be* (1.9.12). *Then,*

$$\tilde{\xi} = (z-\varphi)\tilde{f}^p(z-\varphi, w-\psi), \quad \tilde{\eta} = (w-\psi)\tilde{g}^q(z-\varphi, w-\psi) \tag{1.9.28}$$

is a resonant normal transformation in a neighborhood of the origin of system (1.9.1). *By transformation* (1.9.28), *system* (1.9.1) *can be reduced to* (1.9.4).

Proof. From (1.9.5), (1.9.7), (1.9.11) and (1.9.28), we have

$$\tilde{\xi} = \tilde{\chi}^p, \quad \tilde{\eta} = \tilde{\zeta}^q. \tag{1.9.29}$$

(1.9.12) and (1.9.29) follows (1.9.4). □

Theorem 1.9.4. *Let* (1.9.3) *be a normal transformation in a neighborhood of the origin of system* (1.9.1) *and corresponding normal form be* (1.9.4). *Then, there exist two unit formal series* $\tilde{f}(u, v), \tilde{g}(u, v)$ *of* u, v, *such that* $\tilde{\xi}, \tilde{\eta}$ *can be expressed as the form of* (1.9.28). *By using transformation*

$$\tilde{\chi} = x\tilde{f}(x^p, y^q), \quad \tilde{\zeta} = y\tilde{g}(x^p, y^q) \tag{1.9.30}$$

system (1.9.8) *becomes the* p, q *resonance normal form* (1.9.12).

Proof. Let (1.9.3) be a normal transformation in a neighborhood of the origin of system (1.9.1). Then, by transformation (1.9.5), $\tilde{\xi}$ and $\tilde{\eta}$ can be represented as two formal series of u, v. Since $u = 0$ and $v = 0$ are two solutions of system (1.9.6)hence, there are two unit formal series $\tilde{F}(u, v), \tilde{G}(u, v)$ of u, v, such that,

$$\tilde{\xi} = u\tilde{F}(u, v), \quad \tilde{\eta} = v\tilde{G}(u, v). \tag{1.9.31}$$

Denote that

$$\tilde{f}(u,v) = \tilde{F}^{\frac{1}{p}}(u,v), \quad \tilde{g}(u,v) = \tilde{G}^{\frac{1}{q}}(u,v), \tag{1.9.32}$$

where the functions of the right hands take their principal values. Then, $\tilde{f}(u,v)$, $\tilde{g}(u,v)$ are unit formal series of u, v. From (1.9.5), (1.9.31) and (1.9.32) we obtain the representations (1.9.28) of $\tilde{\xi}, \tilde{\eta}$.

From (1.9.5), (1.9.7),(1.9.28) and (1.9.30), we have (1.9.29).

(1.9.4) and (1.9.29) follows (1.9.12). □

Remark 1.9.1. *Theorem* (1.9.3) *and theorem* (1.9.4) *imply that in a neighborhood of the origin, the* p, q *resonance normal transformation* (1.9.11) *of system* (1.9.8) *and the following normal transformation of system* (1.9.1)

$$\begin{aligned}
\tilde{\xi} &= (z-\varphi)\left[1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{c}_{\alpha\beta}(z-\varphi)^{\alpha}(w-\psi)^{\beta}\right]^{p} = z + h.o.t.,\\
\tilde{\eta} &= (w-\psi)\left[1 + \sum_{\alpha+\beta=1}^{\infty} \tilde{d}_{\alpha\beta}(w-\psi)^{\alpha}(z-\varphi)^{\beta}\right]^{q} = w + h.o.t.
\end{aligned} \tag{1.9.33}$$

have the one-to-one correspondence relation. Moreover, (1.9.5) *and* (1.9.7) *imply* (1.9.29).

Definition 1.9.3. *We say that in a neighborhood of the origin,*

$$\begin{aligned}
\xi &= (z-\varphi)\left[1 + \sum_{\alpha+\beta=1}^{\infty} c_{\alpha\beta}(z-\varphi)^{\alpha}(w-\psi)^{\beta}\right]^{p} = z + h.o.t.,\\
\eta &= (w-\psi)\left[1 + \sum_{\alpha+\beta=1}^{\infty} d_{\alpha\beta}(w-\psi)^{\alpha}(z-\varphi)^{\beta}\right]^{q} = w + h.o.t.
\end{aligned} \tag{1.9.34}$$

is the standard normal transformation of system (1.9.1), *where* $c_{kq,\ kp} = d_{kp,\ kq} = 0,\ k = 1, 2, \cdots$. *Corresponding to transformation* (1.9.34), *system*

$$\begin{aligned}
\frac{d\xi}{dT} &= \ \ p\xi\left[1 + \sum_{k=1}^{\infty} p_k(\xi^q\eta^p)^k\right] = \Phi(\xi,\eta),\\
\frac{d\eta}{dT} &= -q\eta\left[1 + \sum_{k=1}^{\infty} q_k(\xi^q\eta^p)^k\right] = -\Psi(\xi,\eta)
\end{aligned} \tag{1.9.35}$$

is called the standard normal form in a neighborhood of the origin of system (1.9.1).

From Remark 1.9.1 and Theorem 1.8.2, we have

Theorem 1.9.5. *In a neighborhood of the origin, the standard normal transformation of system* (1.9.1) *and the standard normal transformation of system* (1.9.8) *have the following relation*

$$\xi = \chi^p, \quad \eta = \zeta^q. \tag{1.9.36}$$

Moreover, if for all k, $p_k = q_k$, then ξ, η are two power series of z, w having nonzero convergent radius.

Theorem 1.9.6. *Let*

$$\mu_k = p_k - q_k, \quad \tau_k = p_k + q_k, \quad k = 1, 2, \cdots . \tag{1.9.37}$$

For system (1.9.1)*, we have*

$$\begin{aligned} \frac{dH}{dT} &= pq \sum_{k=1}^{\infty} \mu_k H^{k+1}, \\ \frac{d\Omega}{dT} &= \frac{pq}{2i} \left(2 + \sum_{k=1}^{\infty} \tau_k H^k \right) \end{aligned} \tag{1.9.38}$$

and

$$\frac{\partial}{\partial z}(MZ) - \frac{\partial}{\partial w}(MW) = pqM \sum_{k=1}^{\infty} (k+1)\mu_k (H)^k, \tag{1.9.39}$$

where

$$H = \xi^q \eta^p, \quad \Omega = \frac{1}{2i} \ln \frac{\xi^q}{\eta^p}, \quad M = \xi^{q-1}\eta^{p-1} \begin{vmatrix} \dfrac{\partial \xi}{\partial z} & \dfrac{\partial \xi}{\partial w} \\ \dfrac{\partial \eta}{\partial z} & \dfrac{\partial \eta}{\partial w} \end{vmatrix}. \tag{1.9.40}$$

Proof. By using (1.9.35) and (1.9.40) to do computations directly, we obtain (1.9.38). Let $M = \xi^{q-1}\eta^{p-1} J$, where J is the Jacobian of ξ, η with respect to z, w. Then, Proposition 1.1.3 follows that

$$\begin{aligned} &\frac{\partial}{\partial z}(MZ) - \frac{\partial}{\partial w}(MW) \\ &= \frac{\partial}{\partial z}\left(\xi^{p-1}\eta^{q-1} JZ\right) - \frac{\partial}{\partial w}\left(\xi^{p-1}\eta^{q-1} JW\right) \\ &= J\left[\frac{\partial}{\partial \xi}\left(\xi^{p-1}\eta^{q-1}\Phi\right) - \frac{\partial}{\partial \eta}\left(\xi^{p-1}\eta^{q-1}\Psi\right)\right]. \end{aligned} \tag{1.9.41}$$

(1.9.41) implies (1.9.39). □

Similar to Theorem 1.8.3, we have

Theorem 1.9.7. *Let $F(H)$ and $G(H)$ be two unit formal power series of H. Then,*

$$\tilde{\xi} = \xi F(H), \quad \tilde{\eta} = \eta G(H) \tag{1.9.42}$$

gives the normal transformation in a neighborhood of the origin of system (1.9.1).

Theorem 1.9.8. *If* (1.9.3) *is a normal transformation in a neighborhood of the origin of system* (1.9.1), *then there exist two units of formal series of H of the form*

$$F(H) = 1 + \sum_{k=1}^{\infty} A_k H^k, \quad G(H) = 1 + \sum_{k=1}^{\infty} B_k H^k, \tag{1.9.43}$$

such that $\tilde{\xi} = \xi F(H), \tilde{\eta} = \eta G(H)$.

Proof. Let (1.9.3) be a normal form in a neighborhood of the origin of system (1.9.1). Then system (1.9.1) becomes the normal form (1.9.4) by transformation (1.9.3). By Theorem 1.9.4, there are two unit formal series $\tilde{f}(u,v), \tilde{g}(u,v)$ of u, v, such that (1.9.28) holds. System (1.9.8) becomes the normal form (1.9.12) by transformation (1.9.30). From Theorem 1.8.4, $\tilde{\chi}, \tilde{\zeta}$ can be written as the following formal series of χ, ζ:

$$\tilde{\chi} = \chi \left[1 + \sum_{m=1}^{\infty} \tilde{A}_m (\chi\zeta)^m\right], \quad \tilde{\zeta} = \zeta \left[1 + \sum_{m=1}^{\infty} \tilde{B}_m (\chi\zeta)^m\right]. \tag{1.9.44}$$

From (1.9.25) and (1.9.44), we have

$$\begin{aligned} \tilde{\chi} &= xf \left[1 + \sum_{m=1}^{\infty} \tilde{A}_m (xy)^m (fg)^m\right], \\ \tilde{\zeta} &= yg \left[1 + \sum_{m=1}^{\infty} \tilde{B}_m (xy)^m (fg)^m\right], \end{aligned} \tag{1.9.45}$$

where $f = f(x^p, y^q), g = g(x^p, y^q)$. By (1.9.30), $\tilde{\chi}/x = \tilde{f}(x^p, y^q)$ and $\tilde{\zeta}/y = \tilde{g}(x^p, y^q)$ are two formal series of x^p and y^q. Thus, when $m/(pq)$ is not a positive integer, we have $\tilde{A}_m = \tilde{B}_m = 0$. Now (1.9.45) can be become

$$\tilde{\chi} = \chi \left[1 + \sum_{k=1}^{\infty} \tilde{A}_{kpq} (\chi\zeta)^{kpq}\right], \quad \tilde{\zeta} = \zeta \left[1 + \sum_{k=1}^{\infty} \tilde{B}_{kpq} (\chi\zeta)^{kpq}\right]. \tag{1.9.46}$$

(1.9.29), (1.9.36) and (1.9.46) follow that

$$\tilde{\xi} = \xi \left[1 + \sum_{k=1}^{\infty} \tilde{A}_{kpq} (\xi^q \eta^p)^k\right]^p, \quad \tilde{\eta} = \eta \left[1 + \sum_{k=1}^{\infty} \tilde{B}_{kpq} (\xi^q \eta^p)^k\right]^q. \tag{1.9.47}$$

This gives the conclusion. ∎

In [Xiao P., 2005], the author gave the following definition.

Definition 1.9.4. *For any positive integer k, $\mu_k = p_k - q_k$ is called the k-th resonant singular point value of the origin of system* (1.9.1) *and $\tau_k = p_k + q_k$ is called the k-th resonant period constant of the origin of system* (1.9.1).

Define that $\mu_0 = 0$. If there is a positive integer k, such that $\mu_0 = \mu_1 = \cdots = \mu_{k-1} = 0$, but $\mu_k \neq 0$, then the origin is called the k-order resonant singular point;

If for any positive integer k, there are $\mu_k = 0$, then the origin is called a complex resonant center.

Remark 1.9.2. *The k-th resonant singular point value is the k-th saddle quatity given by* [Christopher ect, 2003]

Remark 1.9.3. *Theorem* 1.9.5 *and Theorem* 1.9.6 *imply that if the origin of system*(1.9.1) *is a complex resonant center, then $H = \xi^p \eta^q$ is an analytic first integral of system* (1.9.1), *and H a is power series in z, w having nonzero convergent radius.*

Similar to the proofs of Theorem 1.8.7∼ Theorem 1.8.11, we have the following results.

Theorem 1.9.9. *System* (1.9.1) *has an analytic first integral in a neighborhood of the origin if and only if all resonant singular point values of the origin are zeros.*

Theorem 1.9.10. *System* (1.9.1) *in a neighborhood of the origin is linearizable if and only if for all k, $p_k = 0$ and $q_k = 0$.*

Theorem 1.9.11. *If the origin of system*(1.9.1) *is a complex resonant center, then any first integral in a neighborhood of the origin of system*(1.9.1) *can be expressed as a formal series of H. In addition, any analytic first integral in a neighborhood of the origin of system* (1.9.1) *can be expressed as power series of H with a nonzero convergent radius.*

Because system (1.9.8) can be reduced to system (1.9.26) by using standard normal transformation (1.9.25). Therefore, we have

Theorem 1.9.12. *The origin of system* (1.9.1) *is a complex resonant center if and only if in a neighborhood of the origin there is an analytic integral factor* :

$$M(z, w) = z^{q-1} w^{p-1} + h.o.t.. \tag{1.9.48}$$

Remark 1.9.4. *In* [Simon etc 2000], *the conditions of Theorem* 1.9.9 *are taken as the definition of the integrability. While the conditions given by Theorem* 1.9.10 *are taken as the definition of the linearizable systems in a neighborhood of the origin of system* (1.9.1).

Theorem 1.9.13. *For any positive integer k, the k-th resonant singular value and resonant period constant of the origin of system* (1.9.1) *are the kpq-th singular value and the kpq-th period constant of the origin of system* (1.9.8), *respectively. In addition, if $m/(pq)$ isn't a positive integer, then the m-th singular value and the m-th period constant of the origin of* (1.9.8) *are zeros.*

Finally, we have

Theorem 1.9.14. *For system* (1.9.1), *if the origin is a m-order resonant singular point, then its standard normal form* (1.9.35) *has the following integrating factor in a neighborhood of the origin*:

$$M = \frac{1}{\xi\eta H^m\left(1+\displaystyle\sum_{k=1}^{\infty}\frac{\mu_{m+k}}{\mu_m}H^k\right)}. \tag{1.9.49}$$

Moreover, system (1.9.1) *has an integrating factor JM in a neighborhood of the origin, where $H=\xi^q\eta^p$, $J=1+h.o.t.$ is the Jacobian of ξ,η with respect to z,w.*

Bibliographical Notes

The materials of this chapter are taken from [Amelikin etc, 1982; Qin Y.X., 1985; Griffiths, 1985; Liu Y.R. etc, 1989; Liu Y.R. etc, 1995; Shen L.R., 1998; Liu Y.R., 1999; Simon etc, 2000; Xiao P., 2005; Christopher etc, 2003].

For the linearized problem, a great of number of papers had been published. For instance, see [Zhu D.M., 1987; Chavarriga etc, 1996; Schlomiuk, 1993a; Lloyd etc, 1996; Chavarriga etc, 1997; Chen X.W. etc, 2008; Llibre etc, 2009a; Llibre etc, 2009c; Zhang Q. etc; 2011; Giné ect, 2011] et al.

Chapter 2

Focal Values, Saddle Values and Singular Point Values

In this chapter, we consider a class of real planar autonomous differential systems, for which the functions of the right hand are analytic in a neighborhood of the origin and the origin is a focus or a center. We introduce the elementary theory to solve the center problem.

2.1 Successor Functions and Properties of Focal Values

By making a linear change of the space coordinates and a rescaling of the time variable if necessary, a planar differential system can be written as

$$
\begin{aligned}
\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x,y) = X(x,y),\\
\frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x,y) = Y(x,y),
\end{aligned}
\tag{2.1.1}
$$

where $X(x,y)$, $Y(x,y)$ are analytic in a sufficiently small neighborhood of the origin and

$$
\begin{aligned}
X_k(x,y) &= \sum_{\alpha+\beta=k} A_{\alpha\beta} x^{\alpha} y^{\beta},\\
Y_k(x,y) &= \sum_{\alpha+\beta=k} B_{\alpha\beta} x^{\alpha} y^{\beta}
\end{aligned}
\tag{2.1.2}
$$

are homogeneous polynomials of order k.

It is well know that the origin of system (2.1.1) is a rough focus when $\delta \neq 0$ and it is either a weak focus or a center when $\delta = 0$. The problem of determining whether a non-degenerate singular point (it has purely imaginary eigenvalues) is a center or a weak focus is called the center-focus problem (or simply, center problem). This is one of the most important topics in the qualitative theory of planar dynamical systems. [Poincaré, 1891-1897], [Lyapunov, 1947] and [Bautin, 1952-1954] had done pioneering fundamental work. In last century, many mathematicians also made some important contributions in this direction. We first introduce the method of Poincaré successor function.

Under the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2.1.3}$$

system (2.1.1) is become

$$\begin{aligned} \frac{dr}{dt} &= r\left[\delta + \sum_{k=1}^{\infty}\varphi_{k+2}(\theta)r^k\right], \\ \frac{d\theta}{dt} &= 1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta)r^k, \end{aligned} \tag{2.1.4}$$

where

$$\begin{aligned} \varphi_k(\theta) &= \cos\theta X_{k-1}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta), \\ \psi_k(\theta) &= \cos\theta Y_{k-1}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta). \end{aligned} \tag{2.1.5}$$

We see from (2.1.4)that

$$\frac{dr}{d\theta} = r\frac{\delta + \sum_{k=1}^{\infty}\varphi_{k+2}(\theta)r^k}{1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta)r^k}. \tag{2.1.6}$$

To study the solutions of this equation, we discuss a class of general differential equations

$$\frac{dr}{d\theta} = r\sum_{k=0}^{\infty}R_k(\theta)r^k = R(r, \theta). \tag{2.1.7}$$

Where we assume that there exists a positive real numbers r_0, such that $R(r, \theta)$ is analytic with respect to r in the region $\{|r| < r_0,\ |\theta| < 4\pi\}$, and it is continuously differentiable with respect to the real variable θ. In addition,

$$R_k(\theta + \pi) = (-1)^k R_k(\theta), \quad k = 0, 1, \cdots. \tag{2.1.8}$$

We next use the small parameter method given by Poincaré (see [Poincaré, 1892]). Suppose that (2.1.7) has the following solution of convergent power series

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty}\nu_k(\theta)h^k, \tag{2.1.9}$$

satisfying the initial condition $r|_{\theta=0} = h$, where h is sufficiently small and

$$\nu_1(0) = 1, \quad \nu_k(0) = 0, \quad k = 2,\ 3, \cdots . \tag{2.1.10}$$

Substituting (2.1.9)) into (2.1.7) and equating coefficients of the same powers of h, it follows that

$$\begin{aligned}
&\nu_1'(\theta) = R_0(\theta)\nu_1(\theta),\\
&\nu_2'(\theta) = R_0(\theta)\nu_2(\theta) + R_1(\theta)\nu_1^2(\theta),\\
&\qquad\cdots\cdots\\
&\nu_m'(\theta) = R_0(\theta)\Omega_{1,m}(\theta) + R_1(\theta)\Omega_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\Omega_{m,m}(\theta),\\
&\qquad\cdots\cdots
\end{aligned} \tag{2.1.11}$$

where $\Omega_{k,m}(\theta)$ is given by

$$\Omega_{k,m}(\theta) = \sum_{j_1+j_2+\cdots+j_k=m} \frac{m!}{j_1!j_2!\cdots j_k!}\nu_{j_1}(\theta)\nu_{j_2}(\theta)\cdots\nu_{j_k}(\theta). \tag{2.1.12}$$

Particularly,

$$\Omega_{1,m}(\theta) = \nu_m(\theta), \quad \Omega_{m,m}(\theta) = \nu_1^m(\theta). \tag{2.1.13}$$

Thus, (2.1.10) and (2.1.11) follow that

$$\begin{aligned}
&\nu_1(\theta) = e^{\int_0^\theta R_0(\varphi)d\varphi},\\
&\qquad\cdots\cdots\\
&\nu_m(\theta) = \nu_1(\theta)\int_0^\theta \frac{R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)}{\nu_1(\varphi)}\, d\varphi,\\
&\qquad\cdots\cdots
\end{aligned} \tag{2.1.14}$$

We see from $R_0(\theta + \pi) = R_0(\theta)$, and (2.1.14) that

Lemma 2.1.1. *For equation* (2.1.7), *we have*

$$\nu_1^2(\pi) = \nu_1(2\pi). \tag{2.1.15}$$

For system (2.1.1), if $\delta = 0$, then (2.1.14) becomes

$$\begin{aligned}
&\nu_1(\theta) = 1,\\
&\nu_m(\theta) = \int_0^\theta [R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)]\, d\varphi,\\
&\ m = 1,\ 2,\ \cdots .
\end{aligned} \tag{2.1.16}$$

Thus, we have

Lemma 2.1.2. *For system* (2.1.1), *if* $\delta = 0$, *then all* $\nu_k(\theta)$ *are polynomials of* θ, $\sin\theta$ *and* $\cos\theta$, *whose coefficients are polynomials of* $A_{\alpha\beta}$, $B_{\alpha\beta}$. *Particularly, for all* k, $\nu_k(\pi)$, $\nu_k(2\pi)$ *are polynomials of* $A_{\alpha\beta}, B_{\alpha\beta}$.

Lemma 2.1.3. *For a sufficiently small* h, *we have*

$$-\tilde{r}(\theta+\pi,h) \equiv \tilde{r}(\theta,-\tilde{r}(\pi,h)). \tag{2.1.17}$$

Proof. The condition (2.1.8) follows that under the transformations $\rho = -r$, $\omega = \theta+\pi$, (2.1.7) is invariant, i.e.,

$$\frac{d\rho}{d\omega} = \rho\sum_{k=0}^{\infty} R_k(\omega)\rho^k = R(\rho,\omega). \tag{2.1.18}$$

Therefore, a solution of (2.1.18) satisfying the initial condition $\rho|_{\omega=0} = h$ is $\rho = \tilde{r}(\omega,h)$, i.e., $r = -\tilde{r}(\theta+\pi,h)$ is one solution of (2.1.7) satisfying the initial condition $r|_{\theta=0} = -\tilde{r}(\pi,h)$. On the other hand, $r = \tilde{r}(\theta,-\tilde{r}(\pi,h))$ is also a solution of (2.1.7) satisfying the same initial condition. By the uniqueness of solution, (2.1.17) holds. □

Similarly, the following conclusion holds.

Lemma 2.1.4. *For a sufficiently small* h, *we have*

$$\tilde{r}(\theta+2\pi,h) \equiv \tilde{r}(\theta,\tilde{r}(2\pi,h)). \tag{2.1.19}$$

In order to study the stability of the zero solution of (2.1.7), Poincaré introduced the following successor function:

$$\Delta(h) = \tilde{r}(2\pi,h) - h = [\nu_1(2\pi)-1]h + \sum_{k=2}^{\infty}\nu_k(2\pi)h^k, \tag{2.1.20}$$

where $\tilde{r}(2\pi,h)$ is called Poincaré return map.

The following theorem is given by [Liu Y.R., 2001].

Theorem 2.1.1. *For any positive integer* m *and* $\nu_{2m}(2\pi)$ *given by* (2.1.20), *we have*

$$[1+\nu_1(\pi)]\nu_{2m}(2\pi) = \zeta_m^{(0)}[\nu_1(2\pi)-1] + \sum_{k=1}^{m-1}\zeta_m^{(k)}\nu_{2k+1}(2\pi). \tag{2.1.21}$$

where for all k, $k = 1,\cdots,m-1$, $\zeta_m^{(k)}$ *are polynomials of* $\nu_1(\pi)$, $\nu_2(\pi),\cdots,$ $\nu_{2m}(\pi)$, $\nu_1(2\pi),\nu_2(2\pi),$ $\cdots,$ $\nu_{2m}(2\pi)$ *with rational coefficients.*

Proof. Taking $\theta = 2\pi$ in (2.1.17) and $\theta = \pi$ in (2.1.19), we have

$$\tilde{r}(\pi, \tilde{r}(2\pi, h)) + \tilde{r}(2\pi, -\tilde{r}(\pi, h)) \equiv 0. \tag{2.1.22}$$

We see from (2.1.9) and (2.1.23) that

$$\sum_{k=1}^{\infty} [\nu_k(\pi)\tilde{r}^k(2\pi, h) + (-1)^k \nu_k(2\pi)\tilde{r}^k(\pi, h)] \equiv 0. \tag{2.1.23}$$

It can be written as the power series of h, for which the coefficient of the term h^{2m} satisfies

$$\sum_{k=1}^{2m} [\nu_k(\pi)\Omega_{k,2m}(2\pi) + (-1)^k \nu_k(2\pi)\Omega_{k,2m}(\pi)] = 0. \tag{2.1.24}$$

Therefore, we have

$$[1 + \nu_1(\pi)]\nu_{2m}(2\pi) = G_1 + G_2, \tag{2.1.25}$$

where

$$G_1 = -[\nu_1^{2m}(\pi) - 1]\nu_{2m}(2\pi) - [\nu_1^{2m}(2\pi) - \nu_1(2\pi)]\nu_{2m}(\pi), \tag{2.1.26}$$

$$G_2 = -\sum_{k=2}^{2m-1} [\nu_k(\pi)\Omega_{k,2m}(2\pi) + (-1)^k \nu_k(2\pi)\Omega_{k,2m}(\pi)]. \tag{2.1.27}$$

According to (2.1.26) and Lemma 2.1.1, G_1 has the factor $\nu_1(2\pi) - 1$. Hence, we have from (2.1.25),(2.1.27) and (2.1.12)that

$$[1 + \nu_1(\pi)]\nu_{2m}(2\pi) = \xi_m^{(0)}[\nu_1(2\pi) - 1] + \sum_{k=2}^{2m-1} \xi_m^{(k)}\nu_k(2\pi), \tag{2.1.28}$$

where all $\xi_m^{(k)}$ are polynomials of $\nu_1(\pi)$, $\nu_2(\pi)$, $\cdots$, $\nu_{2m}(\pi)$, $\nu_1(2\pi)$, $\nu_2(2\pi)$, $\cdots$, $\nu_{2m}(2\pi)$ with rational coefficients. By using the mathematical induction, we obtain

$$[1 + \nu_1(\pi)]^m\nu_{2m}(2\pi) = \eta_m^{(0)}[\nu_1(2\pi) - 1] + \sum_{k=1}^{m-1} \eta_m^{(k)}\nu_{2k+1}(2\pi), \tag{2.1.29}$$

where all $\eta_m^{(k)}$ are polynomials in $\nu_1(\pi)$, $\nu_2(\pi), \cdots$, $\nu_{2m}(\pi)$, $\nu_1(2\pi)$, $\nu_2(2\pi)$, $\cdots$, $\nu_{2m}(2\pi)$ with rational coefficients. Denote that

$$(1 + x)^m = 2^{m-1}(1 + x) + (x^2 - 1)f(x). \tag{2.1.30}$$

Then $f(x)$ is a polynomial of x with rational coefficients. By using (2.1.30) and Lemma 2.1.1, we have

$$[1 + \nu_1(\pi)]^m = 2^{m-1}[1 + \nu_1(\pi)] + [\nu_1(2\pi) - 1]f(\nu_1(\pi)). \tag{2.1.31}$$

Thus, (2.1.29) and (2.1.31) follow the conclusion of Theorem 2.1.1. □

This theorem is important in the studies of the properties of successor function and focal values as well as in the discussion of the multiple Hopf bifurcation of limit cycles.

Theorem 2.1.1 has the following corollary.

Corollary 2.1.1. *If $\nu_1(2\pi) = 1$, then, the first positive integer k satisfying $\nu_k(2\pi) \neq 0$ is an odd number.*

Corollary 2.1.1 and the definition of *Poincaré* successor function (2.1.20) give rise to the following result.

Theorem 2.1.2. *Consider equation* (2.1.7).

(1) *If $\nu_1(2\pi) < 1$ (> 1), then the zero solution $r = 0$ is stable (unstable).*

(2) *If $\nu_1(2\pi) = 1$ and there exists an integer $k > 1$, such that $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0$ and $\nu_{2k+1}(2\pi) \neq 0$, then when $\nu_{2k+1}(2\pi) < 0$ (> 0), the zero solution $r = 0$ is stable (unstable).*

(3) *If $\nu_1(2\pi) = 1$ and all positive integers k, we have $\nu_{2k+1}(2\pi) = 0$, then for a sufficiently small h , all solutions satisfying initial condition $r|_{\theta=0} = h$ are 2π-periodical solutions.*

Definition 2.1.1. *Consider system* (2.1.1).

(1) *If $\nu_1(2\pi) \neq 1$, then the origin is called rough focus.*

(2) *If $\nu_1(2\pi) = 1$ and there exists a positive integer k, such that $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0$ and $\nu_{2k+1}(2\pi) \neq 0$, then the origin is called the k-order weak focus, $\nu_{2k+1}(2\pi)$ is called $k-th$ focal value.*

(3) *If $\nu_1(2\pi) = 1$ and for all positive integers k, we have $\nu_{2k+1}(2\pi) = 0$, then the origin is called a center.*

2.2 Poincaré Formal Series and Algebraic Equivalence

When $\delta = 0$, systems (2.1.1) have the following forms

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{k=1}^{\infty} X_k(x,y) = -y + \sum_{\alpha+\beta=2}^{\infty} A_{\alpha\beta}x^\alpha y^\beta = X(x,y), \\ \frac{dy}{dt} &= x + \sum_{k=1}^{\infty} Y_k(x,y) = x + \sum_{\alpha+\beta=2}^{\infty} B_{\alpha\beta}x^\alpha y^\beta = Y(x,y), \end{aligned} \tag{2.2.1}$$

where $X_k(x,y)$, $Y_k(x,y)$ are given by (2.1.2).

Definition 2.2.1. *Let λ_1, λ_2, $\cdots$, λ_m, and $\tilde{\lambda}_m$, be polynomials with respect to $A_{\alpha\beta}'s$ and $B_{\alpha\beta}'s$. If for a positive integer m, there exist polynomials $\xi_1^{(m)}$, $\xi_2^{(m)}$, $\cdots$, $\xi_{m-1}^{(m)}$, with respect to $A_{\alpha\beta}'s$ and $B_{\alpha\beta}'s$, such that*

$$\lambda_m = \tilde{\lambda}_m + (\xi_1^{(m)}\lambda_1 + \xi_2^{(m)}\lambda_2 + \cdots + \xi_{m-1}^{(m)}\lambda_{m-1}), \tag{2.2.2}$$

then, we say that λ_m is algebraic equivalent to $\tilde{\lambda}_m$, written by $\lambda_m \sim \tilde{\lambda}_m$. Furthermore, if for any positive integer m, we have $\lambda_m \sim \tilde{\lambda}_m$, then we say that sequences of functions $\{\lambda_m\}$ is algebraic equivalent to $\{\tilde{\lambda}_m\}$, written as $\{\lambda_m\} \sim \{\tilde{\lambda}_m\}$.

Remark 2.2.1. *It is easy to see from Definition* 2.2.1 *that the following conclusions hold*:

(1) *The algebraic equivalent relationship of the sequences of functions is self-reciprocal, symmetric and transmissible.*

(2) *If for some positive integer m, $\lambda_m \sim \tilde{\lambda}_m$, then, when $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0$, we have $\lambda_m = \tilde{\lambda}_m$.*

(3) *The relationship $\lambda_1 \sim \tilde{\lambda}_1$ implies that $\lambda_1 = \tilde{\lambda}_1$.*

Definition 2.2.2. *Suppose that*

$$G(h) = 1 + \sum_{k=1}^{\infty} c_k h^k \tag{2.2.3}$$

is a formal power series of h, where for any k, c_k is a polynomial with respect to $A_{\alpha\beta}'s$ and $B_{\alpha\beta}'s$, then $G(h)$ is called a unit formal power series of h. In addition, if $G(h)$ is a unit formal power series, and $G(h)$ has non-zero convergent radius, then $G(h)$ is called a unit power series.

Theorem 2.2.1. *Let λ_m and $\tilde{\lambda}_m$ be polynomials with respect to $A_{\alpha\beta}'s$ and $B_{\alpha\beta}'s$, $G_m(h)$ and $\tilde{G}_m(h)$ be unit formal power series of h, $m = 1, 2, \cdots$. If*

$$\sum_{m=1}^{\infty} \lambda_m h^m G_m(h) = \sum_{m=1}^{\infty} \tilde{\lambda}_m h^m \tilde{G}_m(h), \tag{2.2.4}$$

then $\{\lambda_m\} \sim \{\tilde{\lambda}_m\}$.

Proof. For any positive integer m, comparing the coefficients of h^m at the right-hand side and the left-hand side of (2.2.4), we then have the conclusion of this theorem. □

By using Lemma 2.1.2 and Theorem 2.1.1, we have the following conclusion.

Theorem 2.2.2. *For system* (2.2.1), *we have*

$$\nu_{2m}(2\pi) \sim 0, \quad m = 1,\ 2, \cdots. \tag{2.2.5}$$

Moreover, the equivalence relations $\nu_{2m+1}(2\pi) \sim \tilde{\nu}_{2m+1}$ hold if and only if for any positive integer m, there exist $\eta_1, \eta_2, \cdots, \eta_{m-1}$, such that

$$\nu_{2m+1}(2\pi) = \sum_{k=1}^{m-1} \eta_k \nu_{2k+1}(2\pi) + \tilde{\nu}_{2m+1}, \tag{2.2.6}$$

where all η_k are polynomials of $A_{\alpha\beta}'s$,$B_{\alpha\beta}'s$.

We next introduce Poincaré's formal series method.

Theorem 2.2.3. *For system* (2.2.1), *one can construct successively a formal power series*

$$F(x,y) = \sum_{k=2}^{\infty} F_k(x,y), \tag{2.2.7}$$

where $F_k(x,y)$ *is a homogeneous polynomial of order* k *of* x, y *and* $F_2(x,y) = x^2+y^2$, *such that*

$$\frac{dF}{dt} = \sum_{m=1}^{\infty} V_{2m+1}(x^2+y^2)^{m+1}. \tag{2.2.8}$$

Definition 2.2.3. *For any positive integer* m, V_{2m+1} *is called the* m*-th Liapunov constant of system* (2.2.1).

Remark 2.2.2. *We do not consider the convergence for the formal power series. When we realize some operations such as addition, subtraction, multiplication, division, differentiate and integration on a formal series, we only deal with its coefficients, do not consider its convergence. Such operation is called formal operation.*

It is clear that for a given system (2.2.1), all $\nu_{2k+1}(2\pi)$ can be uniquely determined. But the coefficients of formal series F in (2.2.7) are not unique. In fact, for any positive integer m, when $F_2,\ F_3,\ \cdots,\ F_{2m-1}$ have been determined, the coefficient of one term of F_{2m} can be arbitrarily chosen. So that, this value will effect the latter *Liapunov* constants.

Each time when the first nonzero *Liapunov* constant is determined for the given system, it seems that we have solved the center-focus problem. But when we study the multiple Hopf bifurcation of limit cycles from a weak focus, only considering the first nonzero *Liapunov* constant is not enough. It is necessary to investigate the zero roots and their distributions of the Poincaré successor function.

We notice that the relationship between the focal values and the Liapunov constants was studied in [Gobber etc, 1979] and the algebraic equivalent relation between the *Liapunov* constants and the focal values was proved in [Liu Y.R., 2001].

[Liu Y.R., 2001] proved

Theorem 2.2.4. *For system* (2.2.1), *we have*

$$\{V_{2m+1}\} \sim \left\{\frac{1}{\pi}\nu_{2m+1}(2\pi)\right\}. \tag{2.2.9}$$

Proof. By using the polar coordinate to F given by Theorem 2.2.3, we have

$$\tilde{F}(\theta) = F(\tilde{r}(\theta,h)\cos\theta, \tilde{r}(\theta,h)\sin\theta) = \sum_{k=2}^{\infty} F_k(\cos\theta,\sin\theta)\tilde{r}^k(\theta,h). \tag{2.2.10}$$

Thus, we obtain

$$\begin{aligned}\Delta\tilde{F} &= \tilde{F}(2\pi) - \tilde{F}(0) = \sum_{k=2}^{\infty} F_k(1,0)[\tilde{r}^k(2\pi,h) - h^k] \\ &= 2h[\tilde{r}(2\pi,h) - h]G(h) = 2h\sum_{k=2}^{\infty} v_k(2\pi)h^k G(h), \end{aligned} \tag{2.2.11}$$

where $G(h)$ is a unit formal series of h.

On the other hand, (2.1.4) and (2.2.8)) can be transformed into

$$\begin{aligned}\Delta\tilde{F} &= \int_0^{2\pi} \frac{dF}{dt}\frac{dt}{d\theta}d\theta \\ &= \int_0^{2\pi} \frac{V_{2k+1}\tilde{r}^{2k+2}(\theta,h)d\theta}{1 + \sum\limits_{k=1}^{\infty}\psi_{k+2}(\theta)\tilde{r}^k(\theta,h)} \\ &= 2\pi\sum_{k=2}^{\infty} V_{2k+1}h^{2k+2}G_k(h), \end{aligned} \tag{2.2.12}$$

where for any positive integer k,

$$G_k(h) = \frac{1}{2\pi h^{2k+2}}\int_0^{2\pi} \frac{\tilde{r}^{2k+2}(\theta,h)d\theta}{1 + \sum\limits_{k=1}^{\infty}\psi_{k+2}(\theta)\tilde{r}^k(\theta,h)} \tag{2.2.13}$$

is a unit formal series of h. From (2.2.11) and (2.2.12). we have

$$h\sum_{k=2}^{\infty} v_k(2\pi)h^k G(h) = \pi\sum_{k=2}^{\infty} V_{2k+1}h^{2k+2}G_k(h). \tag{2.2.14}$$

From (2.2.14), Theorem 2.2.1 and Theorem 2.2.2, we then have the conclusion of this theorem. □

Theorem 2.2.4 has important application in the study of successor function and focal value as well as in that of multiple Hopf bifurcations of limit cycles created by higher-order weak focus.

Theorem 2.2.5. *Let*

$$H_{2m+2}(x,y) = (x^2+y^2)^{m+1} + h.o.t., \quad m = 1, 2, \cdots, \tag{2.2.15}$$

be given formal series of x *and* y*, which coefficients be all polynomials with respect to* $A_{\alpha\beta}'s$ *and* $B_{\alpha\beta}'s$*. For system* (2.2.1)*, one can construct successively a formal power series* $F = x^2 + y^2 + h.o.t.$*, such that*

$$\frac{dF}{dt} = \sum_{k=1}^{\infty} V'_{2m+1}H_{2m+2}(x,y). \tag{2.2.16}$$

Furthermore, we have

$$\{V'_{2m+1}\} \sim \left\{\frac{1}{\pi}\nu_{2m+1}(2\pi)\right\}. \tag{2.2.17}$$

The prove of (2.2.17) is similar to Theorem 2.2.4.

Form Theorem 1.8.13, there exist two formal series of x, y,

$$u = x + h.o.t., \quad v = y + h.o.t., \tag{2.2.18}$$

and by means of the transformations (2.2.18), system (2.2.1) can be transformed into the following normal form

$$\begin{aligned}\frac{du}{dt} &= -v + \frac{1}{2}\sum_{k=1}^{\infty}(\sigma_k u - \tau_k v)(u^2+v^2)^k = U(u,v),\\ \frac{dv}{dt} &= u + \frac{1}{2}\sum_{k=1}^{\infty}(\tau_k u + \sigma_k v)(u^2+v^2)^k = V(u,v).\end{aligned} \tag{2.2.19}$$

Let

$$H = u^2 + v^2, \tag{2.2.20}$$

then

$$\frac{dH}{dt} = \sum_{m=1}^{\infty}\sigma_m H^{m+1}. \tag{2.2.21}$$

From Theorem 2.2.5, we have

Theorem 2.2.6. $$\{\sigma_m\} \sim \{V_{2m+1}\} \sim \left\{\frac{1}{\pi}\nu_{2m+1}(2\pi)\right\}. \tag{2.2.22}$$

2.3 Linear Recursive Formulas for the Computation of Singular Point Values

In this section, we assume that $A_{\alpha\beta}'s$ and $B_{\alpha\beta}'s$ are all complex coefficients, t is a complex variable. By using the transformations

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{2.3.1}$$

system (2.2.1) is transformed to system (1.8.4), i.e.,

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta} = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta} = -W(z,w).\end{aligned} \tag{2.3.2}$$

where $a_{\alpha\beta}'s$ and $b_{\alpha\beta}'s$ are all complex coefficients, T is a complex variable.

For system (2.3.2), the singular point values $\{\mu_m\}$ of the origin is defined in Definition 1.8.2. From Theorem 1.8.5, Theorem 1.8.13 and Theorem 2.2.6, we have

Theorem 2.3.1. *For system* (2.2.1) *and* (2.3.2), *we have*

$$\{V_{2m+1}\} \sim \{i\mu_m\}, \quad \{\nu_{2m+1}(2\pi)\} \sim \{i\pi\mu_m\}. \tag{2.3.3}$$

From Theorem 2.2.3, we have

Theorem 2.3.2. *For system* (2.3.1), *one can determine successively a formal series*

$$F(z,w) = \sum_{\alpha+\beta=2}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta} = zw + h.o.t., \tag{2.3.4}$$

where $c_{11} = 1,\ c_{20} = c_{02} = 0$, *such that*

$$\frac{dF}{dT} = \sum_{m=1}^{\infty} \lambda_m (zw)^{m+1}, \tag{2.3.5}$$

and $\{\lambda_m\} \sim \{\mu_m\}$.

Theorem 2.3.3. *In* (2.3.4) *and* (2.3.5), *let* $c_{11} = 1,\ c_{20} = c_{02} = 0,\ c_{k,k} = 0,\ k = 2,\ 3,\ \cdots$, *then when* $\alpha + \beta \geqslant 3$ *and* $\alpha \neq \beta$, $c_{\alpha\beta}$ *and* λ_m *have the recursive formulas given by*

$$c_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha - k + 1)a_{k,j-1} - (\beta - j + 1)b_{j,k-1}] c_{\alpha-k+1,\beta-j+1}, \tag{2.3.6}$$

and

$$\lambda_m = \sum_{k+j=3}^{2m+4} [(m - k + 2)a_{k,j-1} - (m - j + 2)b_{j,k-1}] c_{m-k+2,m-j+2}, \tag{2.3.7}$$

where $c_{\alpha\beta} = 0$ *when* $\alpha < 0$ *or* $\beta < 0$.

Proof. Denote

$$\begin{aligned} Z(z,w) &= z + \sum_{k+j=3}^{\infty} a_{k,j-1} z^k w^{j-1}, \\ W(z,w) &= w + \sum_{k+j=3}^{\infty} b_{j,k-1} z^{k-1} w^j, \end{aligned} \tag{2.3.8}$$

then

$$\begin{aligned}\frac{dF}{dT}&=\sum_{\alpha+\beta=2}^{\infty}[\alpha c_{\alpha\beta}z^{\alpha-1}w^{\beta}Z-\beta c_{\alpha\beta}z^{\alpha}w^{\beta-1}W]\\&=(\alpha-\beta)\sum_{\alpha+\beta=3}^{\infty}c_{\alpha\beta}z^{\alpha}w^{\beta}\\&\quad+\sum_{\alpha+\beta=2}^{\infty}\sum_{k+j=3}^{\infty}(\alpha a_{k,j-1}-\beta b_{j,k-1})c_{\alpha\beta}z^{\alpha+k-1}w^{\beta+j-1}\\&=\sum_{\alpha+\beta=3}^{\infty}[(\alpha-\beta)c_{\alpha\beta}+\Delta_{\alpha\beta}]z^{\alpha}w^{\beta},\end{aligned}\tag{2.3.9}$$

where

$$\Delta_{\alpha\beta}=\sum_{k+j=3}^{\alpha+\beta+2}[(\alpha-k+1)a_{k,j-1}-(\beta-j+1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1}.\tag{2.3.10}$$

Thus, (2.3.9) and (2.3.10) follow the formulas (2.3.6) and (2.3.7). □

From Theorem 2.2.5, we have

Theorem 2.3.4. *Let*

$$H_{2m+2}(z,w)=(zw)^{m+1}+h.o.t.,\quad m=1,2,\cdots,\tag{2.3.11}$$

be given formal series of z and w, which coefficients be all polynomials with respect to $a_{\alpha\beta}'s$ and $b_{\alpha\beta}'s$. For system (2.3.2), one can construct successively a formal power series $F=zw+h.o.t.$, such that

$$\frac{dF}{dt}=\sum_{k=1}^{\infty}\lambda'_m H_{2m+2}(z,w).\tag{2.3.12}$$

Furthermore, we have

$$\{\lambda'_m\}\ \sim\{\mu_m\}.\tag{2.3.13}$$

Theorem 2.3.5. *For system (2.3.2), one can derived successively the formal series*

$$M(z,w)=\sum_{\alpha+\beta=0}^{\infty}c_{\alpha\beta}z^{\alpha}w^{\beta},\tag{2.3.14}$$

with $c_{00}=1$, such that

$$\frac{\partial(MZ)}{\partial z}-\frac{\partial(MW)}{\partial w}=\sum_{m=1}^{\infty}(m+1)\tilde{\lambda}_m(zw)^m.\tag{2.3.15}$$

To calculate $c_{\alpha\beta}$ and $\tilde{\lambda}_m$, we have

Theorem 2.3.6. *In* (2.3.14) *and* (2.3.15), *let* $c_{00} = 1$, $c_{kk} = 0$, $k = 1, 2, \cdots$, *then when* $\alpha + \beta \geqslant 1$ *and* $\alpha \neq \beta$, $c_{\alpha\beta}$ *and* $\tilde{\lambda}_m$ *have the recursive formulas given by*

$$c_{\alpha,\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha+1)a_{k,j-1} - (\beta+1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1} \tag{2.3.16}$$

and

$$\tilde{\lambda}_m = \sum_{k+j=3}^{2m+2} (a_{k,j-1} - b_{j,k-1})c_{m-k+1,m-j+1}, \tag{2.3.17}$$

where $c_{\alpha\beta} = 0$ *when* $\alpha < 0$ *or* $\beta < 0$.

Proof. We see from (2.3.8) that

$$\begin{aligned}
&\frac{\partial(MZ)}{\partial z} - \frac{\partial(MW)}{\partial w} \\
=& \frac{\partial M}{\partial z}Z - \frac{\partial M}{\partial w}W + \left(\frac{\partial M}{\partial z} - \frac{\partial M}{\partial w}\right)M \\
=& \sum_{\alpha+\beta=1}^{\infty} (\alpha-\beta)c_{\alpha\beta}z^\alpha w^\beta \\
&+ \sum_{\alpha+\beta=0}^{\infty} \sum_{k+j=3}^{\infty} [(\alpha+k)a_{k,j-1} - (\beta+j)b_{j,k-1}]c_{\alpha\beta}z^{\alpha+k-1}w^{\beta+j-1} \\
=& \sum_{\alpha+\beta=1}^{\infty} (\alpha-\beta)c_{\alpha\beta}z^\alpha w^\beta \\
&+ \sum_{\alpha+\beta=1}^{\infty} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha+1)a_{k,j-1} - (\beta+1)b_{j,k-1}]c_{\alpha-k+1,\beta-j+1}z^\alpha w^\beta. \qquad (2.3.18)
\end{aligned}$$

Hence, (2.3.15) and (2.3.18) implies the conclusion of this theorem. □

What is the relationship between $\{\tilde{\lambda}_m\}$ and $\{\mu_m\}$? The following theorem answer this problem.

Theorem 2.3.7. *For* $\{\tilde{\lambda}_m\}$ *defined by Theorem* 2.3.5, *we have*

$$\{\tilde{\lambda}_m\} \sim \{\mu_m\}. \tag{2.3.19}$$

Proof. Let

$$\begin{aligned}
Z^* &= MZ - \frac{1}{2}\sum_{m=1}^{\infty} \tilde{\lambda}_m z^{m+1}w^m, \\
W^* &= MW + \frac{1}{2}\sum_{m=1}^{\infty} \tilde{\lambda}_m z^m w^{m+1}.
\end{aligned} \tag{2.3.20}$$

From (2.3.15) and (2.3.20), we have

$$\frac{\partial Z^*}{\partial z} - \frac{\partial W^*}{\partial w} = 0. \tag{2.3.21}$$

From (2.3.21), there exist a formal power series $F(z,w) = zw + h.o.t.$, such that

$$\begin{aligned}\frac{\partial F}{\partial z} &= W^* = MW + \frac{1}{2}\sum_{m=1}^{\infty} \tilde{\lambda}_m z^m w^{m+1},\\ \frac{\partial F}{\partial w} &= Z^* = MZ - \frac{1}{2}\sum_{m=1}^{\infty} \tilde{\lambda}_m z^{m+1} w^m.\end{aligned} \tag{2.3.22}$$

From (2.3.22), For system (2.3.2) we have

$$\frac{dF}{dT} = \frac{\partial F}{\partial z}Z - \frac{\partial F}{\partial w}W = \sum_{m=1}^{\infty} \tilde{\lambda}_m H_{2m+2}(z,w), \tag{2.3.23}$$

where

$$H_{2m+2}(z,w) = \frac{1}{2}(zw)^m(wZ + zW) = (zw)^{m+1} + h.o.t.. \tag{2.3.24}$$

(2.3.24) and Theorem 2.2.5 implies the conclusion of this theorem. □

We see from the above discussion that when we use the recursive formulas given by Theorem 2.3.3 and Theorem 2.3.6 to compute singular point values of the origin of system (2.3.2) in a computer, we only need to perform finite many arithmetic operations, this is, plus, minus, multiply and division, to the coefficients of the system. Such calculation is symbolic and it has no error.

In principle, according to the recursive formulas given by Theorem 2.3.3 and Theorem 2.3.6 and using computer algebra systems such as Mathematica, Maple we can compute singular point values of the origin of system (2.3.2). Unfortunately, the simplification of singular values is a very difficult problem.

For an example, we use the recursive formulas of Theorem 2.3.6 and computer algebra system such as Mathematica to calculate the singular point values of the origin for the cubic system

$$\begin{aligned}\frac{dz}{dT} &= z + a_{20}z^2 + a_{11}zw + a_{02}w^2 + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3,\\ \frac{dw}{dT} &= -w - b_{20}w^2 - b_{11}wz - b_{02}z^2 - b_{30}w^3 - b_{21}w^2z - b_{12}wz^2 - b_{03}z^3.\end{aligned} \tag{2.3.25}$$

We found that the first eight singular point values have the terms shown in the following table.

μ_k	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
terms	4	42	306	1482	5694	18658	54256	143770	overflow

This table tell us that for the computation of the singular point values, to find a method for the simplification of μ_k under the conditions $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$, it is a key step.

2.4 The Algebraic Construction of Singular Values

We now introduce the work of algebraic construction of singular values (see [Liu Y.R. etc, 1989]).

Consider the two parameter transformation

$$z = \rho e^{i\phi}\tilde{z}, \quad w = \rho e^{-i\phi}\tilde{w}. \tag{2.4.1}$$

where $\tilde{z},\ \tilde{w}$ are new space variables, $\rho,\ \phi$ are complex parameters, $\rho \neq 0$.

Denote that $z = x+iy$, $w = x-iy$, $\tilde{z} = \tilde{x}+i\tilde{y}$, $\tilde{w} = \tilde{x}-i\tilde{y}$. Then, (2.4.1) becomes

$$\begin{aligned} x &= \rho(\tilde{x}\cos\phi - \tilde{y}\sin\phi), \\ y &= \rho(\tilde{x}\sin\phi + \tilde{y}\cos\phi). \end{aligned} \tag{2.4.2}$$

In the case of real variables and real parameters, (2.4.2) just is a composite transformation of a similarity and a rotation.

Definition 2.4.1. *We say that the transformation* (2.4.1) *is a generalized rotation and similar transformation.*

Under the transformation (2.4.1), system (2.3.2) is changed to

$$\begin{aligned} \frac{d\tilde{z}}{dT} &= \tilde{z} + \sum_{\alpha+\beta=2}^{\infty} \tilde{a}_{\alpha\beta}\tilde{z}^{\alpha}\tilde{w}^{\beta}, \\ \frac{d\tilde{w}}{dT} &= -\tilde{w} - \sum_{\alpha+\beta=2}^{\infty} \tilde{b}_{\alpha\beta}\tilde{w}^{\alpha}\tilde{z}^{\beta}, \end{aligned} \tag{2.4.3}$$

where

$$\begin{aligned} \tilde{a}_{\alpha\beta} &= a_{\alpha\beta}\rho^{\alpha+\beta-1}e^{i(\alpha-\beta-1)\phi}, \\ \tilde{b}_{\alpha\beta} &= b_{\alpha\beta}\rho^{\alpha+\beta-1}e^{-i(\alpha-\beta-1)\phi}. \end{aligned} \tag{2.4.4}$$

For convenience, let $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ be a polynomial with respect to finite many coefficients of system (2.3.2) and $\tilde{f} = f(\tilde{a}_{\alpha\beta}, \tilde{b}_{\alpha\beta}),\ f^* = f(a^*_{\alpha\beta}, b^*_{\alpha\beta})$, where $a^*_{\alpha\beta} = b_{\alpha\beta},\ b^*_{\alpha\beta} = a_{\alpha\beta},\ \alpha \geqslant 0,\ \beta \geqslant 0,\ \alpha+\beta \geqslant 2$.

Definition 2.4.2. *Suppose that there exist constants* λ, σ, *such that* $\tilde{f} = \rho^\lambda e^{i\sigma\vartheta} f$, *we say that* λ *is a similar exponent and* σ *a rotation exponent of system* (2.3.2) *under the transformation* (2.4.2), *which are denoted by* $I_s(f) = \lambda$ *and* $I_r(f) = \sigma$.

From (2.4.4) and Definition 2.4.2, we have

$$\begin{aligned} &I_s(a_{\alpha\beta}) = \alpha + \beta - 1, \quad I_s(b_{\alpha\beta}) = \alpha + \beta - 1, \\ &I_r(a_{\alpha\beta}) = \alpha - \beta - 1, \quad I_r(b_{\alpha\beta}) = -(\alpha - \beta - 1). \end{aligned} \tag{2.4.5}$$

Definition 2.4.3. *For system* (2.3.2), *suppose that* $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ *is a polynomial of* $a_{\alpha\beta}'s$, $b_{\alpha\beta}'s$,

(1) *If* $\tilde{f} = \rho^{2k} f$, *then* f *is called a* k*-order generalized rotation invariant under* (2.4.2);

(2) *If* f *is a generalized rotation invariant,and* f *is a monomial of* $a_{\alpha\beta}'s$, $b_{\alpha\beta}'s$, *then* f *is called a monomial generalized rotation invariant*;

(3) *If* f *is a monomial generalized rotation invariant, and it can not be expressed as a product of two monomial generalized rotation invariant, then* f *is called a elementary generalized rotation invariant.*

Definition 2.4.4. *A polynomial* $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ *of system* (2.3.2) *is called self-symmetry, if* $f^* = f$. *It is called self-antisymmetry, if* $f^* = -f$.

Theorem 2.4.1. *Let* $f_1 = f_1(a_{\alpha\beta}, b_{\alpha\beta})$ *and* $f_2 = f_2(a_{\alpha\beta}, b_{\alpha\beta})$ *be two polynomial with respect to finite many coefficients of system* (2.3.2). *If* $\tilde{f}_1 = \rho_1^\lambda e^{i\sigma_1\vartheta} f_1$, $\tilde{f}_2 = \rho_2^\lambda e^{i\sigma_2\vartheta} f_2$, *then*

$$I_s(f_1 f_2) = I_s(f_1) + I_s(f_2), \quad I_r(f_1 f_2) = I_r(f_1) + I_r(f_2). \tag{2.4.6}$$

Corollary 2.4.1. *If* f_1 *and* f_2 *are two generalized rotation invariants, then their product* $f_1 f_2$ *is also an generalized rotation invariant and* $I_s(f_1 f_2) = I_s(f_1) + I_s(f_2)$.

We see from (2.4.3) that

Theorem 2.4.2. *A monomial of the coefficients of* (2.3.2) *given by*

$$g = \prod_{j=1}^{n} a_{\alpha_j,\beta_j} \prod_{k=1}^{m} b_{\gamma_k,\delta_k} \tag{2.4.7}$$

is a N*-order generalized rotation invariant under* (2.4.2) *if and only if*

$$\begin{aligned} &I_s(g) = \sum_{j=1}^{n}(\alpha_j + \beta_j - 1) + \sum_{k=1}^{m}(\gamma_k + \delta_k - 1) = 2N, \\ &I_r(g) = \sum_{j=1}^{n}(\alpha_j - \beta_j - 1) - \sum_{k=1}^{m}(\gamma_k - \delta_k - 1) = 0. \end{aligned} \tag{2.4.8}$$

It is easy to see that N is a positive integer when (2.4.8) holds. From Theorem 2.4.2, we have

Theorem 2.4.3. *If a monomial* $g = g(a_{\alpha\beta}, b_{\alpha\beta})$ *is a* k*-order generalized rotation invariant* (*or elementary generalized rotation invariant*) *of* (2.3.2), *then so is* g^*.

We see from §1.8 that by the standard normal transformation

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j = \xi(z,w), \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j = \eta(z,w), \tag{2.4.9}$$

system (2.3.2) can become the standard normal form

$$\begin{aligned} \frac{d\xi}{dT} &= \xi + \xi \sum_{k=1}^{\infty} p_k (\xi\eta)^k, \\ \frac{d\eta}{dT} &= -\eta - \eta \sum_{k=1}^{\infty} q_k (\xi\eta)^k. \end{aligned} \tag{2.4.10}$$

Theorem 2.4.4. *All* p_k *and* q_k *given by* (2.4.10) *are* k*-order generalized rotation invariants of* (2.3.2), $k = 1,\ 2,\ \cdots$.

Proof. For the variables $\xi,\ \eta$ of the formal series given by (2.4.9), we denote that

$$\begin{aligned} \tilde{\xi} &= \frac{1}{\rho} e^{-i\phi} \xi(\rho e^{i\phi} \tilde{z}, \rho e^{-i\phi} \tilde{w}), \\ \tilde{\eta} &= \frac{1}{\rho} e^{i\phi} \eta(\rho e^{i\phi} \tilde{z}, \rho e^{-i\phi} \tilde{w}). \end{aligned} \tag{2.4.11}$$

By means of transformation (2.4.11), system (2.4.3) can be changed into the standard normal form

$$\begin{aligned} \frac{d\tilde{\xi}}{dT} &= \tilde{\xi} + \sum_{k=1}^{\infty} \rho^{2k} p_k \tilde{\xi}^{k+1} \tilde{\eta}^k, \\ \frac{d\tilde{\eta}}{dT} &= -\tilde{\eta} - \sum_{k=1}^{\infty} \rho^{2k} q_k \tilde{\eta}^{k+1} \tilde{\xi}^k. \end{aligned} \tag{2.4.12}$$

We have from (2.4.12) that

$$\tilde{p}_k = \rho^{2k} p_k, \quad \tilde{q}_k = \rho^{2k} q_k, \quad k = 1,\ 2,\ \cdots. \tag{2.4.13}$$

It follows the conclusion of this theorem. □

Theorem 2.4.5. *For system* (2.3.2), *we have*

$$p_k^* = q_k, \quad q_k^* = p_k, \quad k = 1,\ 2,\ \cdots. \tag{2.4.14}$$

Proof. By using the transformation

$$z = w^*, \quad w = z^*, \quad T = -T^*, \tag{2.4.15}$$

system (2.3.2) becomes

$$\begin{aligned} \frac{dz^*}{dT^*} &= z^* + \sum_{\alpha+\beta=2}^{\infty} a^*_{\alpha\beta}(z^*)^\alpha (w^*)^\beta, \\ \frac{dw^*}{dT^*} &= -w^* - \sum_{\alpha+\beta=2}^{\infty} b^*_{\alpha\beta}(w^*)^\alpha (z^*)^\beta. \end{aligned} \tag{2.4.16}$$

The transformation

$$\xi^* = \eta(w^*, z^*), \quad \eta^* = \xi(w^*, z^*) \tag{2.4.17}$$

makes system (2.4.16) have the following standard normal form

$$\begin{aligned} \frac{d\xi^*}{dT^*} &= \xi^* + \sum_{k=1}^{\infty} q_k (\xi^*)^{k+1}(\eta^*)^k, \\ \frac{d\eta^*}{dT^*} &= -\eta^* - \sum_{k=1}^{\infty} p_k (\eta^*)^{k+1}(\xi^*)^k. \end{aligned} \tag{2.4.18}$$

Clearly, (2.4.14) holds. □

We see from Theorem 2.4.4 and Theorem 2.4.5 that

Theorem 2.4.6. *For any positive integer k, μ_k is a k-order generalized rotation invariant and the anti-symmetry relation $\mu_k^* = -\mu_k^*$ holds. τ_k is also a k-order generalized rotation invariant having self-symmetry relation $\tau_k^* = \tau_k$.*

Theorem 2.4.6 implies the following conclusions.

Theorem 2.4.7 (The construction theorem of singular point values). *The k-order singular point value μ_k of (2.3.2) at the origin can be represented as a linear combination of k-order monomial generalized rotation invariants and their antisymmetry forms, i.e.,*

$$\mu_k = \sum_{j=1}^{N} \gamma_{kj}(g_{kj} - g^*_{kj}), \quad k = 1,\ 2,\ \cdots, \tag{2.4.19}$$

where N is a positive integer, and γ_{kj} is a rational number, g_{kj} and g^*_{kj} are k-order monomial generalized rotation invariants of (2.3.2).

Theorem 2.4.8 (The construction theorem of period constant). *The k-order period constant τ_k of (2.3.2) at the origin can be represented as a linear combination of k-order monomial generalized rotation invariants and their self-symmetry forms, i.e.,*

$$\tau_k = \sum_{j=1}^{N} \gamma'_{kj}(g_{kj} + g^*_{kj}), \quad k = 1,\ 2,\ \cdots, \tag{2.4.20}$$

where N is a positive integer and γ'_{kj} is a rational number, g_{kj} and g^*_{kj} are k-order monomial generalized rotation invariants of (2.3.2).

Theorem 2.4.9 (The extended symmetric principle). *Let g be an elementary generalized rotation invariant of (2.3.2). If for all g the symmetric condition $g = g^*$ is satisfied, then the origin of (2.3.2) is an complex center. Namely, all singular point values of the origin are zero.*

Remark 2.4.1. *The symmetry principle of a real differential system is a special case of this extended symmetric principle. In fact, suppose that real vector field (2.2.1) has a symmetry axis passing through the origin (without loss the generality, we assume that it is the x-axis). Then, (2.2.1) satisfies*

$$X(x, -y) = -X(x, y), \quad Y(x, -y) = Y(x, y). \tag{2.4.21}$$

It implies that for the associated system of (2.2.1), the relationship $Z(w, z) = W(z, w)$ holds. Hence, for all pairs (α, β) , we have $a_{\alpha\beta} = b_{\alpha\beta}$. It gives rise to the condition of the extended symmetric principle.

For example, we consider the following complex analytic system

$$\begin{aligned} \frac{dz}{dT} &= z + a_{11}zw + \sum_{k=2}^{\infty} f_k(z)w^k, \\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} h_k(z)w^k \end{aligned} \tag{2.4.22}$$

and its symmetry system

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k=2}^{\infty} h_k(w)z^k, \\ \frac{dw}{dT} &= -w - b_{11}wz - \sum_{k=2}^{\infty} f_k(w)z^k, \end{aligned} \tag{2.4.23}$$

where $f_k,\ h_k$ are two polynomials and

$$\deg(f_k) \leqslant k, \quad \deg(g_k) \leqslant k - 2, \quad k = 2,\ 3,\ \cdots. \tag{2.4.24}$$

It is easy to show that the origin of (2.4.22) and (2.4.23) is a complex center (see [Liu Y.R. etc, 1989]). The following is a new result.

Theorem 2.4.10. *There exists an analytic change of the form* (2.4.9), *such that system* (2.4.22) *and* (2.4.23) *becomes a linear system*

$$\frac{d\xi}{dT} = \xi, \quad \frac{d\eta}{dT} = -\eta. \tag{2.4.25}$$

Proof. If (2.4.24) holds, the rotation exponent of any coefficient of a nonlinear term of (2.4.22) is negative, while for (2.4.23) it is positive. We have from Theorem 1.5.2 that any generalized rotation invariant of (2.4.22) and (2.4.23) is zero. Hence, for systems (2.4.22) and (2.4.23), Theorem 2.4.2 implies that $p_k = q_k = 0,\ k = 1,\ 2,\ \cdots$, which follows the conclusion of this theorem. □

Remark 2.4.2. *By using Theorem* 2.4.1 *and the mathematical induction method, we can easily prove the following conclusion: for* $c_{\alpha\beta}$ *and* λ_m, $\tilde{\lambda}_m$ *in Theorem* 2.3.3 *and Theorem* 2.3.6, *we have*

$$\begin{aligned} &I_s(c_{\alpha\beta}) = \alpha + \beta, \quad I_r(c_{\alpha\beta}) = \alpha - \beta, \quad c_{\beta\alpha} = c^*_{\alpha\beta}, \\ &I_s(\lambda_m) = 2m, \quad I_r(\lambda_m) = 0, \\ &I_s(\tilde{\lambda}_m) = 2m, \quad I_r(\tilde{\lambda}_m) = 0. \end{aligned} \tag{2.4.26}$$

2.5 Elementary Generalized Rotation Invariants of the Cubic Systems

In this section, we consider the complex second-order cubic polynomial differential system

$$\begin{aligned} \frac{dz}{dT} =& z + a_{20}z^2 + a_{11}zw + a_{02}w^2 \\ &+ a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3, \\ \frac{dw}{dT} =& -w - b_{20}w^2 - b_{11}wz - b_{02}z^2 \\ &- b_{30}w^3 - b_{21}w^2z - b_{12}wz^2 - b_{03}z^3. \end{aligned} \tag{2.5.1}$$

By using the theory mentioned in Section 2.4, we obtained (see [Liu Y.R. etc, 1989])

Theorem 2.5.1. *System* (2.5.1) *has exactly* 120 *elementary generalized rotation invariants, which are listed as follows*:

Self-symmetry

order 1	$a_{20}b_{20}$,	$a_{11}b_{11}$,	$a_{02}b_{02}$.
order 2	$a_{30}b_{30}$,	$a_{12}b_{12}$,	$a_{03}b_{03}$.

(2.5.2)

Asymmetry

order 1	$a_{20}a_{11}$,	$b_{20}b_{11}$;	a_{21},	b_{21}.
order 2	$a_{30}a_{12}$,	$b_{30}b_{12}$;	$a_{20}^2b_{30}$,	$b_{20}^2a_{30}$;
	$a_{20}b_{11}b_{30}$,	$b_{20}a_{11}a_{30}$;	$b_{11}^2b_{30}$,	$a_{11}^2a_{30}$;
	$a_{20}^2a_{12}$,	$b_{20}^2b_{12}$;	$a_{20}b_{11}a_{12}$,	$b_{20}a_{11}b_{12}$;
	$b_{11}^2a_{12}$,	$a_{11}^2b_{12}$;	$a_{20}^3a_{02}$,	$b_{20}^3b_{02}$;
	$a_{20}^2b_{11}a_{02}$,	$b_{20}^2a_{11}b_{02}$;	$a_{20}b_{11}^2a_{02}$,	$b_{20}a_{11}^2b_{02}$;
	$b_{11}^3a_{02}$,	$a_{11}^3b_{02}$;	$a_{20}a_{30}a_{02}$,	$b_{20}b_{30}b_{02}$;
	$a_{20}b_{12}a_{02}$,	$b_{20}a_{12}b_{02}$;	$b_{11}a_{30}a_{02}$,	$a_{11}b_{30}b_{02}$;
	$b_{11}b_{12}a_{02}$,	$a_{11}a_{12}b_{02}$;	$a_{20}b_{02}a_{03}$,	$b_{20}a_{02}b_{03}$;
	$b_{11}b_{02}a_{03}$,	$a_{11}a_{02}b_{03}$.		
order 3	$a_{30}^2a_{03}$,	$b_{30}^2b_{03}$;	$a_{30}b_{12}a_{03}$,	$b_{30}a_{12}b_{03}$;
	$b_{12}^2a_{03}$,	$a_{12}^2b_{03}$;	$a_{30}^2b_{20}a_{02}$,	$b_{30}^2a_{20}b_{02}$;
	$a_{30}b_{12}b_{20}a_{02}$,	$b_{30}a_{12}a_{20}b_{02}$;	$b_{12}^2b_{20}a_{02}$,	$a_{12}^2a_{20}b_{02}$;
	$a_{30}^2a_{11}a_{02}$,	$b_{30}^2b_{11}b_{02}$;	$a_{30}b_{12}a_{11}a_{02}$,	$b_{30}a_{12}b_{11}b_{02}$;
	$b_{12}^2a_{11}a_{02}$,	$a_{12}^2b_{11}b_{02}$;	$a_{20}^4a_{03}$,	$b_{20}^4b_{03}$;
	$a_{20}^3b_{11}a_{03}$,	$b_{20}^3a_{11}b_{03}$;	$a_{20}^2b_{11}^2a_{03}$,	$b_{20}^2a_{11}^2b_{03}$;
	$a_{20}b_{11}^3a_{03}$,	$b_{20}a_{11}^3b_{03}$;	$b_{11}^4a_{03}$,	$a_{11}^4b_{03}$;
	$a_{20}b_{30}b_{03}a_{02}$,	$b_{20}a_{30}a_{03}b_{02}$;	$a_{20}a_{12}b_{03}a_{02}$,	$b_{20}b_{12}a_{03}b_{02}$;
	$b_{11}b_{30}b_{03}a_{02}$,	$a_{11}a_{30}a_{03}b_{02}$;	$b_{11}a_{12}b_{03}a_{02}$,	$a_{11}b_{12}a_{03}b_{02}$;
	$b_{30}a_{03}b_{02}^2$,	$a_{30}b_{03}a_{02}^2$;	$a_{12}a_{03}b_{02}^2$,	$b_{12}b_{03}a_{02}^2$;
	$b_{20}^2a_{03}b_{02}^2$,	$a_{20}^2b_{03}a_{02}^2$;	$b_{20}a_{11}a_{03}b_{02}^2$,	$a_{20}b_{11}b_{03}a_{02}^2$;
	$a_{11}^2a_{03}b_{02}^2$,	$b_{11}^2b_{03}a_{02}^2$;	$b_{20}^2b_{30}b_{03}$,	$a_{20}^2a_{30}a_{03}$;
	$b_{20}a_{11}b_{30}b_{03}$,	$a_{20}b_{11}a_{30}a_{03}$;	$a_{11}^2b_{30}b_{03}$,	$b_{11}^2a_{30}a_{03}$;
	$b_{20}^2a_{12}b_{03}$,	$a_{20}^2b_{12}a_{03}$;	$b_{20}a_{11}a_{12}b_{03}$,	$a_{20}b_{11}b_{12}a_{03}$;
	$a_{11}^2b_{12}a_{03}$,	$b_{11}^2a_{12}b_{03}$.		
order 4	$b_{20}b_{02}^3a_{03}^2$,	$a_{20}a_{02}^3b_{03}^2$;	$a_{11}b_{02}^3a_{03}^2$,	$b_{11}a_{02}^3b_{03}^2$;
	$a_{30}b_{02}^2a_{03}^2$,	$b_{30}a_{02}^2b_{03}^2$;	$b_{12}b_{02}^2a_{03}^2$,	$a_{12}a_{02}^2b_{03}^2$;
	$a_{30}^3a_{02}^2$,	$b_{30}^3b_{02}^2$;	$a_{30}^2b_{12}a_{02}^2$,	$b_{30}^2a_{12}b_{02}^2$;
	$a_{30}b_{12}^2a_{02}^2$,	$b_{30}a_{12}^2b_{02}^2$;	$b_{12}^3a_{02}^2$,	$a_{12}^3b_{02}^2$.
order 5	$b_{02}^4a_{03}^3$,	$a_{02}^4b_{03}^3$.		

(2.5.3)

Corollary 2.5.1. *The quadratic system*

$$\frac{dz}{dT} = z + a_{20}z^2 + a_{11}zw + a_{02}w^2,$$
$$\frac{dw}{dT} = -w - b_{20}w^2 - b_{11}wz - b_{02}z^2 \tag{2.5.4}$$

has exactly 13 *elementary generalized rotation invariants as follows :*

$$\begin{gathered} a_{20}b_{20}, \quad a_{11}b_{11}, \quad a_{02}b_{02}, \qquad a_{20}a_{11}, \quad b_{20}b_{11}, \\ a_{20}^3a_{02}, \quad b_{20}^3b_{02}; \quad a_{20}^2b_{11}a_{02}, \quad b_{20}^2a_{11}b_{02}; \\ a_{20}b_{11}^2a_{02}, \quad b_{20}a_{11}^2b_{02}; \quad b_{11}^3a_{02}, \quad a_{11}^3b_{02}. \end{gathered} \tag{2.5.5}$$

Corollary 2.5.2. *The Z_2-equivariant cubic system*

$$\begin{aligned}\frac{dz}{dT} &= z + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3,\\ \frac{dw}{dT} &= -w - b_{30}w^3 - b_{21}w^2z - b_{12}wz^2 - b_{03}z^3\end{aligned} \tag{2.5.6}$$

has exactly 13 *elementary generalized rotation invariants as follows*

$$\begin{aligned}&a_{30}b_{30},\quad a_{12}b_{12},\quad a_{03}b_{03},\quad a_{30}a_{12},\qquad b_{30}b_{12},\quad a_{21}\quad b_{21},\\ &a_{30}^2a_{03},\quad b_{30}^2b_{03},\quad a_{30}b_{12}a_{03},\quad b_{30}a_{12}b_{03},\quad b_{12}^2a_{03},\quad a_{12}^2b_{03}.\end{aligned} \tag{2.5.7}$$

2.6 Singular Point Values and Integrability Condition of the Quadratic Systems

For some concrete families of differential equations, the characterization of center and finding and simplifying focal values (or saddle values) has extensively been studied during the last decades.

[Liu Y.R. etc, 1989] studied the computation and simplification of singular point va-lues for systems (2.5.4) and (2.5.6). We now introduce their study results.

Applying recursive formulas of Theorem 2.3.6, we can compute the singular point values of the origin of system (2.5.4) and simplify them by using computer algebra system. Mathematica and Maple are very good computation software.

Theorem 2.6.1. *The first three singular point values at the origin of system* (2.5.4) *are given as follows*:

$$\begin{aligned}\mu_1 &= b_{20}b_{11} - a_{20}a_{11},\\ \mu_2 &\sim \frac{-1}{3}(2I_1 + 3I_2 - 2I_3),\\ \mu_3 &\sim \frac{5}{8}(a_{11}b_{11} - a_{02}b_{02})(2I_2 - I_3),\end{aligned} \tag{2.6.1}$$

where

$$\begin{aligned}I_0 &= a_{20}^3a_{02} - b_{20}^3b_{02},\quad I_1 = a_{20}^2b_{11}a_{02} - b_{20}^2a_{11}b_{02},\\ I_2 &= a_{20}b_{11}^2a_{02} - b_{20}a_{11}^2b_{02},\quad I_3 = b_{11}^3a_{02} - a_{11}^3b_{02}.\end{aligned} \tag{2.6.2}$$

Theorem 2.6.2. *For quadratic system* (2.5.4), *the first three singular point va-lues are all zero if and only if one of the following four conditions holds*:

$$\begin{aligned}
&C_1: 2a_{20} - b_{11} = 0, \quad 2b_{20} - a_{11} = 0;\\
&C_2: a_{20}a_{11} - b_{20}b_{11} = 0, \quad I_0 = I_1 = I_2 = I_3 = 0;\\
&C_3: a_{11} = b_{11} = 0;\\
&C_4: \begin{cases} b_{20} + 2a_{11} = a_{20} + 2b_{11} = 0, \quad a_{02}b_{02} = a_{11}b_{11},\\ |b_{11}| + |b_{02}| \neq 0, \quad |a_{11}| + |a_{02}| \neq 0. \end{cases}
\end{aligned} \tag{2.6.3}$$

Theorem 2.6.3. *For system* (2.5.4), *we write that*

$$\begin{aligned}
f_1 = &1 + 2(a_{20}z + b_{20}w) + [(a_{20}^2 + b_{20}b_{02})z^2\\
&+3(a_{20}b_{20} - a_{02}b_{02})zw + (b_{20}^2 + a_{20}a_{02})w^2]\\
&+(a_{20}b_{20} - a_{02}b_{02})(b_{02}z^3 + a_{20}z^2w + b_{20}zw^2 + a_{02}w^3),\\
f_2 = &2a_{11}b_{11}[1 - 3(b_{11}z + a_{11}w)]\\
&+[3(b_{11}z + a_{11}w) - (a_{11}z - a_{02}w)(b_{11}w - b_{02}z)]\\
&\times[a_{11}(b_{11}^2 - a_{11}b_{02})z + b_{11}(a_{11}^2 - a_{02}b_{11})w],\\
f_3 = &1 - 2(b_{11}z + a_{11}w) - (a_{11}b_{02}z^2 - 2a_{11}b_{11}zw + a_{02}b_{11}w^2).
\end{aligned} \tag{2.6.4}$$

Then,

(1) *If Condition C_1 in* (2.6.3) *holds, then system* (2.5.4) *is Hamiltonian.*

(2) *If Condition C_2 in* (2.6.3) *holds, then the conditions of the extended symmetric principle are satisfied.*

(3) *If Condition C_3 in* (2.6.3) *holds, then there exists a integrating factor f_1^{-1}.*

(4) *If Condition C_4 in* (2.6.3) *holds, then there exists a first integral $f_2^2 f_3^{-3}$.*

From Theorem 2.6.1 ∼ Theorem 2.6.3, we have

Theorem 2.6.4. *The origin of quadratic system* (2.5.4) *is a complex center if and only if the first three singular point values are all zero.*

Appendix

The computational course of the singular point values in Theorem 2.6.1 is given as follows:

Let $c_{kk} = 0,\ k = 1,\ 2,\ \cdots$. We use the recursive formulas to do the computation. The computational course of above singular point value is given as follows:

$$\mu_1 = b_{20}b_{11} - a_{20}a_{11},$$

$$\begin{aligned}
\mu_2 = &\frac{1}{3}(-24a_{11}^2a_{20}^2 - 2a_{11}^3b_{02} - 4a_{02}a_{11}a_{20}b_{02} + 15a_{11}^2a_{20}b_{11} - 2a_{02}a_{20}^2b_{11} - 3a_{02}a_{20}b_{11}^2\\
&+2a_{02}b_{11}^3 + 30a_{11}a_{20}^2b_{20} + 3a_{11}^2b_{02}b_{20} + 4a_{02}b_{02}b_{11}b_{20} - 15a_{11}b_{11}^2b_{20} + 2a_{11}b_{02}b_{20}^2\\
&-30a_{20}b_{11}b_{20}^2 + 24b_{11}^2b_{20}^2).
\end{aligned}$$

Let

$$k_{21} = \frac{1}{3}(24a_{11}a_{20} + 4a_{02}b_{02} - 15a_{11}b_{11} - 30a_{20}b_{20} + 24b_{11}b_{20}),$$

$$\mu_2 \to \mu_2 - k_{21}\mu_1,$$

then

$$\mu_2 = \frac{1}{3}(-2a_{11}^3b_{02} - 2a_{02}a_{20}^2b_{11} - 3a_{02}a_{20}b_{11}^2 + 2a_{02}b_{11}^3 + 3a_{11}^2b_{02}b_{20} + 2a_{11}b_{02}b_{20}^2),$$

i.e.,

$$\mu_2 = \frac{1}{3}(-2I_1 - 3I_2 + 2I_3),$$

$$\begin{aligned}\mu_3 = \frac{1}{72}(&-7236a_{11}^3a_{20}^3 - 696a_{02}a_{11}a_{20}^4 - 778a_{11}^4a_{20}b_{02} - 2250a_{02}a_{11}^2a_{20}^2b_{02}\\ &-153a_{02}a_{11}^3b_{02}^2 - 120a_{02}^2a_{11}a_{20}b_{02}^2 + 8424a_{11}^3a_{20}^2b_{11} - 614a_{02}a_{11}a_{20}^3b_{11}\\ &+381a_{11}^4b_{02}b_{11} + 1204a_{02}a_{11}^2a_{20}b_{02}b_{11} - 108a_{02}^2a_{20}^2b_{02}b_{11} - 2628a_{11}^3a_{20}b_{11}^2\\ &-159a_{02}a_{11}a_{20}^2b_{11}^2 - 252a_{02}^2a_{20}b_{02}b_{11}^2 + 1082a_{02}a_{11}a_{20}b_{11}^3 + 153a_{02}^2b_{02}b_{11}^3\\ &-381a_{02}a_{11}b_{11}^4 + 15876a_{11}^2a_{20}^3b_{20} + 2160a_{11}^3a_{20}b_{02}b_{20} + 2504a_{02}a_{11}a_{20}^2b_{02}b_{20}\\ &+252a_{02}a_{11}^2b_{02}^2b_{20} - 15120a_{11}^2a_{20}^2b_{11}b_{20} + 1452a_{02}a_{20}^3b_{11}b_{20} - 1082a_{11}^3b_{02}b_{11}b_{20}\\ &+120a_{02}^2b_{02}^2b_{11}b_{20} + 482a_{02}a_{20}^2b_{11}^2b_{20} - 1204a_{02}a_{11}b_{02}b_{11}^2b_{20} + 2628a_{11}^2b_{11}^3b_{20}\\ &-2160a_{02}a_{20}b_{11}^3b_{20} + 778a_{02}b_{11}^4b_{20} - 9288a_{11}a_{20}^3b_{20}^2 - 482a_{11}^2a_{20}b_{02}b_{20}^2\\ &+108a_{02}a_{11}b_{02}^2b_{20}^2 + 159a_{11}^2b_{02}b_{11}b_{20}^2 - 2504a_{02}a_{20}b_{02}b_{11}b_{20}^2 + 15120a_{11}a_{20}b_{11}^2b_{20}^2\\ &+2250a_{02}b_{02}b_{11}^2b_{20}^2 - 8424a_{11}b_{11}^3b_{20}^2 - 1452a_{11}a_{20}b_{02}b_{20}^3 + 9288a_{20}^2b_{11}b_{20}^3\\ &+614a_{11}b_{02}b_{11}b_{20}^3 - 15876a_{20}b_{11}^2b_{20}^3 + 7236b_{11}^3b_{20}^3 + 696b_{02}b_{11}b_{20}^4).\end{aligned}$$

Let

$$\begin{aligned}k_{31} = \frac{1}{144}(&14472a_{11}^2a_{20}^2 + 1392a_{02}a_{20}^3 + 290a_{11}^3b_{02} + 4500a_{02}a_{11}a_{20}b_{02} + 240a_{02}^2b_{02}^2\\ &-16848a_{11}^2a_{20}b_{11} - 38a_{02}a_{20}^2b_{11} - 2408a_{02}a_{11}b_{02}b_{11} + 5256a_{11}^2b_{11}^2 - 909a_{02}a_{20}b_{11}^2\\ &+290a_{02}b_{11}^3 - 31752a_{11}a_{20}^2b_{20} - 909a_{11}^2b_{02}b_{20} - 5008a_{02}a_{20}b_{02}b_{20}\\ &+44712a_{11}a_{20}b_{11}b_{20} + 4500a_{02}b_{02}b_{11}b_{20} - 16848a_{11}b_{11}^2b_{20} + 18576a_{20}^2b_{20}^2\\ &-38a_{11}b_{02}b_{20}^2 - 31752a_{20}b_{11}b_{20}^2 + 14472b_{11}^2b_{20}^2 + 1392b_{02}b_{20}^3),\end{aligned}$$

$$k_{32} = \frac{1}{6}(211a_{11}a_{20} + 36a_{02}b_{02} - 112a_{11}b_{11} - 252a_{20}b_{20} + 211b_{11}b_{20}),$$

$$\mu_3 \to \mu_3 - k_{31}\mu_1 - k_{32}\mu_2,$$

then

$$\mu_3 = \frac{5}{8}(-a_{02}b_{02} + a_{11}b_{11})(a_{11}^3b_{02} + 2a_{02}a_{20}b_{11}^2 - a_{02}b_{11}^3 - 2a_{11}^2b_{02}b_{20}),$$

i.e.

$$\mu_3 = \frac{5}{8}(a_{11}b_{11} - a_{02}b_{02})(2I_2 - I_3)).$$

2.7 Singular Point Values and Integrability Condition of the Cubic Systems Having Homogeneous Nonlinearities

Theorem 2.7.1. *The first five singular point values at the origin of system* (2.5.6) *are given as follows*:

$$\begin{aligned}
\mu_1 &= a_{21} - b_{21},\\
\mu_2 &\sim b_{30}b_{12} - a_{30}a_{12},\\
\mu_3 &\sim \frac{1}{8}(3I_4 + 8I_5 - 3I_6),\\
\mu_4 &\sim \frac{1}{40}(a_{21}+b_{21})(9I_4 - 6I_5 + I_6),\\
\mu_5 &\sim \frac{1}{60}(4a_{12}b_{12} - a_{03}b_{03})(9I_4 - 6I_5 + I_6),
\end{aligned} \tag{2.7.1}$$

where

$$\begin{aligned}
I_4 &= a_{30}^2 a_{03} - b_{30}^2 b_{03},\\
I_5 &= a_{30}b_{12}a_{03} - b_{30}a_{12}b_{03},\\
I_6 &= b_{12}^2 a_{03} - a_{12}^2 b_{03}.
\end{aligned} \tag{2.7.2}$$

Theorem 2.7.2. *For system* (2.5.6), *the first five singular point values are all zero if and only if one of the following four conditions holds*:

$$\begin{aligned}
&C_1 : a_{21} = b_{21}, \quad 3a_{30} - b_{12} = 3b_{30} - a_{12} = 0;\\
&C_2 : a_{21} = b_{21}, \quad a_{30}a_{12} = b_{30}b_{12}, \quad I_4 = I_5 = I_6 = 0;\\
&C_3 : \begin{cases} a_{21} = b_{21} = 0, \quad a_{03}b_{03} = 4a_{12}b_{12},\\ a_{30} + 3b_{12} = b_{30} + 3a_{12} = 0. \end{cases}
\end{aligned} \tag{2.7.3}$$

Theorem 2.7.3. *For system* (2.5.6), *we write that*

$$\begin{aligned}
f_5 = {}& 1 - 6(b_{12}z^2 + a_{12}w^2)\\
&+3(3b_{12}^2 z^4 - 2a_{12}b_{03}z^3 w + 2a_{12}b_{12}z^2w^2 - 2b_{12}a_{03}w^3 z + 3a_{12}^2 w^4)\\
&+\frac{1}{2}(2a_{12}z - a_{03}w)(2b_{12}w - b_{03}z)\\
&\times(b_{03}z^4 - 2b_{12}z^3 w - 2a_{12}w^3 z + a_{03}w^4).
\end{aligned} \tag{2.7.4}$$

Then,

(1) *If Condition C_1 in* (2.7.3) *holds, then system* (2.5.6) *is Hamiltonian.*

(2) *If Condition C_2 in* (2.7.3) *holds, then the conditions of the extended symmetric principle are satisfied.*

(3) *If Condition C_3 in* (2.7.3) *holds, then there exists a integrating factor $f_5^{-5/6}$.*

From Theorem 2.7.1 $\sim$ Theorem 2.7.3, we have

Theorem 2.7.4. *The origin of* (2.5.6) *is a complex center if and only if* $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0$.

Appendix

The computational course of the singular point values in Theorem 2.7.1 is given as follows:

$$\mu_1 = a_{21} - b_{21},$$

$$\mu_2 = -a_{12}a_{30} + b_{12}b_{30},$$

$$\begin{aligned}\mu_3 = \frac{1}{8}(&28a_{12}a_{21}a_{30} + 3a_{03}a_{30}^2 + 3a_{12}^2b_{03} - 12a_{12}a_{21}b_{12} + 8a_{03}a_{30}b_{12} - 3a_{03}b_{12}^2\\ &-20a_{12}a_{30}b_{21} + 12a_{12}b_{12}b_{21} - 36a_{21}a_{30}b_{30} - 8a_{12}b_{03}b_{30} + 20a_{21}b_{12}b_{30}\\ &+36a_{30}b_{21}b_{30} - 28b_{12}b_{21}b_{30} - 3b_{03}b_{30}^2).\end{aligned}$$

Let

$$k_{31} = \frac{3}{2}(2a_{12}a_{30} - a_{12}b_{12} - 3a_{30}b_{30} + 2b_{12}b_{30}),$$

$$k_{32} = \frac{-1}{2}(a_{21} + b_{21}),$$

$$\mu_3 \to \mu_3 - k_{31}\mu_1 - k_{32}\mu_2,$$

then

$$\mu_3 = \frac{1}{8}(3a_{03}a_{30}^2 + 3a_{12}^2b_{03} + 8a_{03}a_{30}b_{12} - 3a_{03}b_{12}^2 - 8a_{12}b_{03}b_{30} - 3b_{03}b_{30}^2),$$

i.e.

$$\mu_3 = \frac{1}{8}(3I_4 + 8I_5 - 3I_6),$$

$$\begin{aligned}\mu_4 = \frac{1}{16}(&-108a_{12}a_{21}^2a_{30} - 72a_{12}^2a_{30}^2 - 81a_{03}a_{21}a_{30}^2 - 13a_{12}^2a_{21}b_{03} - 24a_{03}a_{12}a_{30}b_{03}\\ &+28a_{12}a_{21}^2b_{12} + 32a_{12}^2a_{30}b_{12} + 24a_{03}a_{21}a_{30}b_{12} + a_{03}a_{21}b_{12}^2 + 88a_{12}a_{21}a_{30}b_{21}\\ &+75a_{03}a_{30}^2b_{21} - a_{12}^2b_{03}b_{21} - 64a_{03}a_{30}b_{12}b_{21} + 13a_{03}b_{12}^2b_{21} + 4a_{12}a_{30}b_{21}^2\\ &-28a_{12}b_{12}b_{21}^2 + 84a_{21}^2a_{30}b_{30} + 96a_{12}a_{30}^2b_{30} + 64a_{12}a_{21}b_{03}b_{30} - 4a_{21}^2b_{12}b_{30}\\ &+24a_{03}b_{03}b_{12}b_{30} - 32a_{12}b_{12}^2b_{30} - 24a_{12}b_{03}b_{21}b_{30} - 88a_{21}b_{12}b_{21}b_{30} - 84a_{30}b_{21}^2b_{30}\\ &+108b_{12}b_{21}^2b_{30} - 75a_{21}b_{03}b_{30}^2 - 96a_{30}b_{12}b_{30}^2 + 72b_{12}^2b_{30}^2 + 81b_{03}b_{21}b_{30}^2).\end{aligned}$$

Let

$$\begin{aligned}k_{41} = \frac{1}{8}(&-52a_{12}a_{21}a_{30} - 39a_{03}a_{30}^2 - 3a_{12}^2b_{03} + 14a_{12}a_{21}b_{12} + 22a_{03}a_{30}b_{12} - 3a_{03}b_{12}^2\\ &-4a_{12}a_{30}b_{21} + 14a_{12}b_{12}b_{21} + 42a_{21}a_{30}b_{30} + 22a_{12}b_{03}b_{30} - 4a_{21}b_{12}b_{30}\\ &+42a_{30}b_{21}b_{30} - 52b_{12}b_{21}b_{30} - 39b_{03}b_{30}^2),\end{aligned}$$

$$k_{42} = \frac{1}{4}(a_{21}^2 + 18a_{12}a_{30} + 6a_{03}b_{03} - 8a_{12}b_{12} + 2a_{21}b_{21} + b_{21}^2 - 24a_{30}b_{30} + 18b_{12}b_{30}),$$

$$k_{43} = \frac{-11}{10}(a_{21} + b_{21}),$$

$$\mu_4 \to \mu_4 - k_{41}\mu_1 - k_{42}\mu_2 - k_{43}\mu_3,$$

then

$$\mu_4 = \frac{-1}{40}(a_{21}+b_{21})(-9a_{03}a_{30}^2+a_{12}^2b_{03}+6a_{03}a_{30}b_{12}-a_{03}b_{12}^2-6a_{12}b_{03}b_{30}+9b_{03}b_{30}^2),$$

i.e.

$$\mu_4 = \frac{1}{40}(a_{21}+b_{21})(9I_4 - 6I_5 + I_6),$$

$$\begin{aligned}\mu_5 = \frac{1}{192}(&4608a_{12}a_{21}^3a_{30} + 7104a_{12}^2a_{21}a_{30}^2 + 2961a_{03}a_{21}^2a_{30}^2 + 1482a_{03}a_{12}a_{30}^3\\
&+423a_{12}^2a_{21}^2b_{03} + 462a_{12}^3a_{30}b_{03} + 1944a_{03}a_{12}a_{21}a_{30}b_{03} + 72a_{03}^2a_{30}^2b_{03}\\
&+104a_{03}a_{12}^2b_{03}^2 - 1728a_{12}a_{21}^3b_{12} - 4524a_{12}^2a_{21}a_{30}b_{12} - 744a_{03}a_{21}^2a_{30}b_{12}\\
&+712a_{03}a_{12}a_{30}^2b_{12} - 200a_{12}^3b_{03}b_{12} - 672a_{03}a_{12}a_{21}b_{03}b_{12} + 288a_{03}^2a_{30}b_{03}b_{12}\\
&+876a_{12}^2a_{21}b_{12}^2 - 81a_{03}a_{21}^2b_{12}^2 - 1002a_{03}a_{12}a_{30}b_{12}^2 - 104a_{03}^2b_{03}b_{12}^2 + 200a_{03}a_{12}b_{12}^3\\
&-9120a_{12}a_{21}^2a_{30}b_{21} - 4008a_{12}^2a_{30}^2b_{21} - 2448a_{03}a_{21}a_{30}^2b_{21} - 216a_{12}^2a_{21}b_{03}b_{21}\\
&-984a_{03}a_{12}a_{30}b_{03}b_{21} + 4032a_{12}a_{21}^2b_{12}b_{21} + 3060a_{12}^2a_{30}b_{12}b_{21}\\
&+168a_{03}a_{21}a_{30}b_{12}b_{21} + 672a_{03}a_{12}b_{03}b_{12}b_{21} - 876a_{12}^2b_{12}^2b_{21} + 216a_{03}a_{21}b_{12}^2b_{21}\\
&+7008a_{12}a_{21}a_{30}b_{21}^2 - 513a_{03}a_{30}^2b_{21}^2 + 81a_{12}^2b_{03}b_{21}^2 - 4032a_{12}a_{21}b_{12}b_{21}^2\\
&+1440a_{03}a_{30}b_{12}b_{21}^2 - 423a_{03}b_{12}^2b_{21}^2 - 2304a_{12}a_{30}b_{21}^3 + 1728a_{12}b_{12}b_{21}^3\\
&-5184a_{21}^3a_{30}b_{30} - 12684a_{12}a_{21}a_{30}^2b_{30} - 576a_{03}a_{30}^3b_{30} - 1440a_{12}a_{21}^2b_{03}b_{30}\\
&-2160a_{12}^2a_{30}b_{03}b_{30} - 2736a_{03}a_{21}a_{30}b_{03}b_{30} - 288a_{03}a_{12}b_{03}^2b_{30} + 2304a_{21}^3b_{12}b_{30}\\
&+10080a_{12}a_{21}a_{30}b_{12}b_{30} - 2130a_{03}a_{30}^2b_{12}b_{30} + 1002a_{12}^2b_{03}b_{12}b_{30}\\
&+984a_{03}a_{21}b_{03}b_{12}b_{30} - 3060a_{12}a_{21}b_{12}^2b_{30} + 2160a_{03}a_{30}b_{12}^2b_{30} - 462a_{03}b_{12}^3b_{30}\\
&+12096a_{21}^2a_{30}b_{21}b_{30} + 8532a_{12}a_{30}^2b_{21}b_{30} - 168a_{12}a_{21}b_{03}b_{21}b_{30}\\
&+2736a_{03}a_{30}b_{03}b_{21}b_{30} - 7008a_{21}^2b_{12}b_{21}b_{30} - 10080a_{12}a_{30}b_{12}b_{21}b_{30}\\
&-1944a_{03}b_{03}b_{12}b_{21}b_{30} + 4524a_{12}b_{12}^2b_{21}b_{30} - 12096a_{21}a_{30}b_{21}^2b_{30}\\
&+744a_{12}b_{03}b_{21}^2b_{30} + 9120a_{21}b_{12}b_{21}^2b_{30} + 5184a_{30}b_{21}^3b_{30} - 4608b_{12}b_{21}^3b_{30}\\
&+6732a_{21}a_{30}^2b_{30}^2 + 513a_{21}^2b_{03}b_{30}^2 + 2130a_{12}a_{30}b_{03}b_{30}^2 - 72a_{03}b_{03}^2b_{30}^2\\
&-8532a_{21}a_{30}b_{12}b_{30}^2 - 712a_{12}b_{03}b_{12}b_{30}^2 + 4008a_{21}b_{12}^2b_{30}^2 - 6732a_{30}^2b_{21}b_{30}^2\\
&+2448a_{21}b_{03}b_{21}b_{30}^2 + 12684a_{30}b_{12}b_{21}b_{30}^2 - 7104b_{12}^2b_{21}b_{30}^2 - 2961b_{03}b_{21}^2b_{30}^2\\
&+576a_{30}b_{03}b_{30}^3 - 1482b_{03}b_{12}b_{30}^3).\end{aligned}$$

Let

$$\begin{aligned}k_{51} = \frac{1}{64}(&1528a_{12}a_{21}^2a_{30} + 1852a_{12}^2a_{30}^2 + 987a_{03}a_{21}a_{30}^2 + 117a_{12}^2a_{21}b_{03}\\
&+488a_{03}a_{12}a_{30}b_{03} - 576a_{12}a_{21}^2b_{12} - 1264a_{12}^2a_{30}b_{12} - 320a_{03}a_{21}a_{30}b_{12}\\
&-224a_{03}a_{12}b_{03}b_{12} + 292a_{12}^2b_{12}^2 - 3a_{03}a_{21}b_{12}^2 - 1536a_{12}a_{21}a_{30}b_{21} + 171a_{03}a_{30}^2b_{21}\\
&-3a_{12}^2b_{03}b_{21} + 768a_{12}a_{21}b_{12}b_{21} - 408a_{03}a_{30}b_{12}b_{21} + 117a_{03}b_{12}^2b_{21} + 776a_{12}a_{30}b_{21}^2\\
&-576a_{12}b_{12}b_{21}^2 - 1728a_{21}^2a_{30}b_{30} - 3536a_{12}a_{30}^2b_{30} - 408a_{12}a_{21}b_{03}b_{30}\\
&-912a_{03}a_{30}b_{03}b_{30} + 776a_{21}^2b_{12}b_{30} + 3360a_{12}a_{30}b_{12}b_{30} + 488a_{03}b_{03}b_{12}b_{30}\\
&-1264a_{12}b_{12}^2b_{30} + 2304a_{21}a_{30}b_{21}b_{30} - 320a_{12}b_{03}b_{21}b_{30} - 1536a_{21}b_{12}b_{21}b_{30}\\
&-1728a_{30}b_{21}^2b_{30} + 1528b_{12}b_{21}^2b_{30} + 2244a_{30}^2b_{30}^2 + 171a_{21}b_{03}b_{30}^2 - 3536a_{30}b_{12}b_{30}^2\end{aligned}$$

$$+1852b_{12}^2b_{30}^2+987b_{03}b_{21}b_{30}^2),$$

$$k_{52}=\frac{1}{96}(-12a_{21}^3-774a_{12}a_{21}a_{30}-519a_{03}a_{30}^2-9a_{12}^2b_{03}-240a_{03}a_{21}b_{03}+366a_{12}a_{21}b_{12}$$
$$+200a_{03}a_{30}b_{12}-9a_{03}b_{12}^2-36a_{21}^2b_{21}-774a_{12}a_{30}b_{21}-240a_{03}b_{03}b_{21}$$
$$+366a_{12}b_{12}b_{21}-36a_{21}b_{21}^2-12b_{21}^3+1038a_{21}a_{30}b_{30}+200a_{12}b_{03}b_{30}-774a_{21}b_{12}b_{30}$$
$$+1038a_{30}b_{21}b_{30}-774b_{12}b_{21}b_{30}-519b_{03}b_{30}^2),$$

$$k_{53}=\frac{1}{30}(27a_{21}^2+185a_{12}a_{30}+42a_{03}b_{03}-78a_{12}b_{12}+54a_{21}b_{21}+27b_{21}^2-240a_{30}b_{30}$$
$$+185b_{12}b_{30}),$$

$$k_{54}=\frac{-3}{2}(a_{21}+b_{21}),$$

$$\mu_5\to\mu_5-k_{51}\mu_1-k_{52}\mu_2-k_{53}\mu_3-k_{54}\mu_4,$$

then

$$\mu_5=\frac{-1}{60}(-a_{03}b_{03}+4a_{12}b_{12})(-9a_{03}a_{30}^2+a_{12}^2b_{03}+6a_{03}a_{30}b_{12}-a_{03}b_{12}^2-6a_{12}b_{03}b_{30}$$
$$+9b_{03}b_{30}^2),$$

i.e.

$$\mu_5=\frac{1}{60}(4a_{12}b_{12}-a_{03}b_{03})(9I_4-6I_5+I_6).$$

Bibliographical Notes

For the study of the topic of this chapter, there exist a great number of papers. For example, the reader can consult: the quadratic systems [Bautin, 1952-1954; Qin Y.X. etc, 1981; Li C.Z., 1983] and [Zhu D.M., 1987]. Systems with homogeneous nonlinearities of degree 3,4 and 5, [Sibirskii, 1965; Liu Y.R., 1988; Chavarriga etc, 2000; Liu Y.R. etc, 2011b]; Kukles system [Christopher etc, 1995]. Quadratic-like cubic systems [Lloyd etc, 1996]. Lienard systems [Gasull etc, 1998] et al.

Chapter 3

Multiple Hopf Bifurcations

In this chapter, we discuss the bifurcations of limit cycles created from the origin for perturbed systems of (2.1.1).

3.1 The Zeros of Successor Functions in the Polar Coordinates

We consider the perturbed systems of (2.1.1) as follows:

$$\begin{aligned}\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x, y, \boldsymbol{\varepsilon}, \delta) = X(x, y, \boldsymbol{\varepsilon}, \delta),\\ \frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y, \boldsymbol{\varepsilon}, \delta) = Y(x, y, \boldsymbol{\varepsilon}, \delta),\end{aligned} \tag{3.1.1}$$

where $x,\ y$ and t are real variables, $\boldsymbol{\varepsilon}=(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ and δ are real small parameters,

$$\begin{aligned}X_k(x, y, \boldsymbol{\varepsilon}, \delta) &= \sum_{\alpha+\beta=k} A_{\alpha\beta}(\boldsymbol{\varepsilon}, \delta) x^{\alpha} y^{\beta},\\ Y_k(x, y, \boldsymbol{\varepsilon}, \delta) &= \sum_{\alpha+\beta=k} B_{\alpha\beta}(\boldsymbol{\varepsilon}, \delta) x^{\alpha} y^{\beta}\end{aligned} \tag{3.1.2}$$

are homogeneous polynomial in x, y. We assume that $A_{\alpha\beta}(\boldsymbol{\varepsilon}, \delta)'s$, $B_{\alpha\beta}(\boldsymbol{\varepsilon}, \delta)'s$ are power series of $\boldsymbol{\varepsilon}, \delta$ which have real coefficients and nonzero convergent radius. And there are $x_0,\ y_0,\ \boldsymbol{\varepsilon}_0,\ \delta_0$, such that for $|x| < x_0,\ |y| < y_0,\ |\boldsymbol{\varepsilon}| < v_0,\ |\delta| < \delta_0$, the power series $X(x, y, \boldsymbol{\varepsilon}, \delta)$ and $Y(x, y, \boldsymbol{\varepsilon}, \delta)$ of $x, y, \boldsymbol{\varepsilon}, \delta$ are convergent.

By using the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{3.1.3}$$

system (3.1.1) becomes

$$\frac{dr}{dt} = r\left[\delta + \sum_{k=1}^{\infty} \varphi_{k+2}(\theta, \boldsymbol{\varepsilon}, \delta) r^k\right],$$
$$\frac{d\theta}{dt} = 1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta, \boldsymbol{\varepsilon}, \delta) r^k, \tag{3.1.4}$$

where

$$\varphi_k(\theta, \boldsymbol{\varepsilon}, \delta) = \cos\theta X_{k-1}(\cos\theta, \sin\theta, \boldsymbol{\varepsilon}, \delta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta, \boldsymbol{\varepsilon}, \delta),$$
$$\psi_k(\theta, \boldsymbol{\varepsilon}, \delta) = \cos\theta Y_{k-1}(\cos\theta, \sin\theta, \boldsymbol{\varepsilon}, \delta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta, \boldsymbol{\varepsilon}, \delta). \tag{3.1.5}$$

From (3.1.4), we have

$$\frac{dr}{d\theta} = r\frac{\delta + \sum_{k=1}^{\infty} \varphi_{k+2}(\theta, \boldsymbol{\varepsilon}, \delta) r^k}{1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta, \boldsymbol{\varepsilon}, \delta) r^k} = r[\delta + o(r)] . \tag{3.1.6}$$

We consider the following equation

$$\frac{dr}{d\theta} = r\sum_{k=0}^{\infty} R_k(\theta, \boldsymbol{\varepsilon}, \delta) r^k = R(r, \theta, \boldsymbol{\varepsilon}, \delta), \tag{3.1.7}$$

where we assume that there exist positive real numbers $r_0, \varepsilon_0, \delta_0$, such that $R(r, \theta, \boldsymbol{\varepsilon}, \delta)$ is analytic with respect to $r, \boldsymbol{\varepsilon}, \delta$ in the region $\{|r| < r_0,\ |\boldsymbol{\varepsilon}| < \varepsilon_0,\ |\delta| < \delta_0, |\theta| < 4\pi\}$ and it is continuously differentiable with respect to the real variable θ. In addition,

$$\int_0^{2\pi} R_0(\theta, \mathbf{0}, 0) d\theta = 0,$$
$$R_k(\theta + \pi, \boldsymbol{\varepsilon}, \delta) = (-1)^k R_k(\theta, \boldsymbol{\varepsilon}, \delta), \quad k = 0,\ 1,\ \cdots . \tag{3.1.8}$$

For sufficiently small complex constant h (i.e. $|h| \ll 1$), the solution of (3.1.6) satisfying $r|_{\theta=0} = h$ and corresponding Poincaré successor function are given by

$$r = \tilde{r}(\theta, h, \boldsymbol{\varepsilon}, \delta) = \sum_{k=1}^{\infty} \nu_k(\theta, \boldsymbol{\varepsilon}, \delta) h^k,$$
$$\Delta(h, \boldsymbol{\varepsilon}, \delta) = \tilde{r}(2\pi, h, \boldsymbol{\varepsilon}, \delta) - h, \quad \nu_1(\theta, \mathbf{0}, \delta) = 1. \tag{3.1.9}$$

On the basis of the analytic dependence of solutions of a differential equation with respect to initial conditions and parameters, there exist positive numbers h_0, $\boldsymbol{\varepsilon}_0'$ and δ_0', such that $\tilde{r}(\theta, h, \boldsymbol{\varepsilon}, \delta)$ is analytic with respect to $h, \boldsymbol{\varepsilon}, \delta$ in the the region $\{|h| < h_0,\ |\boldsymbol{\varepsilon}| < \varepsilon_0',\ |\delta| < \delta_0', |\theta| < 4\pi\}$ and it is continuously differentiable with respect to the real variable θ.

Definition 3.1.1. *Suppose that* $h = h(\boldsymbol{\varepsilon}, \delta)$ *is a continuous function with complex value in real variables* $\boldsymbol{\varepsilon}, \delta$ *when* $|\boldsymbol{\varepsilon}| \ll 1$ *and* $|\delta| \ll 1$. *If* $h(\mathbf{0}, 0) = 0$ *and* $\Delta(h(\boldsymbol{\varepsilon}, \delta), \boldsymbol{\varepsilon}, \delta) \equiv 0$, *then* $h = h(\boldsymbol{\varepsilon}, \delta)$ *is called a zero of* $\Delta(h, \boldsymbol{\varepsilon}, \delta)$.

To study the limit cycles created from the origin of (3.1.1), when the origin is a weak focus or centers, the problem can be formulated as follows: for $|\boldsymbol{\varepsilon}| \ll 1$, $\delta \ll 1$, how many small positive real zeros of $\Delta(h, \boldsymbol{\varepsilon}, \delta)$ can have?

In order to answer the above problem, we shall deal with it in the area of analytic theory of differential equations. We have to investigate the numbers, positions of all complex zeros of $\Delta(h, \boldsymbol{\varepsilon}, \delta)$ as well as some algebraic and analytic properties for $\Delta(h, \boldsymbol{\varepsilon}, \delta)$.

First, Lemma 2.1.3 gives

Theorem 3.1.1. *For sufficiently small* h, $\boldsymbol{\varepsilon}$ *and* δ, *if* $h = h(\boldsymbol{\varepsilon}, \delta)$ *is a real or a complex zero of* $\Delta(h, \boldsymbol{\varepsilon}, \delta)$, *then* $h = -\tilde{r}(\pi, h(\boldsymbol{\varepsilon}, \delta), \boldsymbol{\varepsilon}, \delta)$ *so is. Thus, in the real domain, the positive zero and the negative zero of* $\Delta(h, \boldsymbol{\varepsilon}, \delta)$ *are paired appearance.*

Definition 3.1.2. *For sufficiently small* h, $\boldsymbol{\varepsilon}$ *and* δ, *if* $h = h(\boldsymbol{\varepsilon}, \delta)$ *is a real or complex zero of multiplicity* k *of* $\Delta(h, \boldsymbol{\varepsilon}, \delta)$, *then we say that* $r = \tilde{r}(\theta, h(\boldsymbol{\varepsilon}, \delta), \boldsymbol{\varepsilon}, \delta)$ *is a* 2π *period solution of multiplicity* k *of* (3.1.6)

Particularly, if $\Delta(h, \boldsymbol{\varepsilon}) \equiv 0$ when $0 < |\boldsymbol{\varepsilon} \ll 1$, $|\delta| \ll 1$, then, for all $(\boldsymbol{\varepsilon}, \delta) \in \{0 < |\boldsymbol{\varepsilon}| \ll 1,\ |\delta| \ll 1\}$, the origin of (3.1.1) is an center .

Theorem 3.1.2. *Suppose that the origin of system* $(3.1.1)_{\boldsymbol{\varepsilon}=\mathbf{0},\delta=0}$ *is a weak focus of order* m. *Then, when* $0 < |\boldsymbol{\varepsilon}| \ll 1$, $0 < |\delta| \ll 1$, *there exist exactly* $2m + 1$ *complex period solutions of system* (3.1.6) *with* 2π *period near the trivial solution* $r = 0$

Proof. Under the condition of Theorem 3.1.2, we have

$$\begin{aligned} &\nu_2(2\pi, \mathbf{0}, 0) = \nu_3(2\pi, \mathbf{0}, 0) = \cdots = \nu_{2m}(2\pi, \mathbf{0}, 0) = 0, \\ &\nu_{2m+1}(2\pi, \mathbf{0}, 0) \neq 0. \end{aligned} \tag{3.1.10}$$

From (3.1.9) and (3.1.10), we have

$$\Delta(h, \mathbf{0}, 0) = \nu_{2m+1}(2\pi, \mathbf{0}, 0)h^{2m+1} + o(h^{2m+1}). \tag{3.1.11}$$

From (3.1.11), Theorem 3.1.1 and Weierstrass preparation theorem, there exist positive number h_0', $\boldsymbol{\varepsilon}_0'$, δ_0', such that for $|\boldsymbol{\varepsilon}| < \boldsymbol{\varepsilon}_0'$, $|\delta| < \delta_0'$, $\Delta(h, \boldsymbol{\varepsilon}, \delta)$ has exact $2m+1$ complex zeros (in the multiplicity) $h = h_k(\boldsymbol{\varepsilon}, \delta)$, $k = 0, 1, 2, \cdots, 2m$ in the disc $|h| < h_0'$, where

$$h_0(\boldsymbol{\varepsilon}, \delta) \equiv 0, \quad h_{m+k}(\boldsymbol{\varepsilon}, \delta) = -\tilde{r}(\pi, h_k(\boldsymbol{\varepsilon}, \delta), \boldsymbol{\varepsilon}, \delta), \quad k = 1, 2, \cdots, m. \tag{3.1.12}$$

Thus, the conclusion of this theorem holds. □

We see from Theorem 3.1.1 that there are at most m positive real zeros in the $2m+1$ zeros given by Theorem3.1.2. Furthermore, we have the following conclusion.

Theorem 3.1.3. *If the origin of system* $(3.1.1)_{\boldsymbol{\varepsilon}=\mathbf{0},\delta=0}$ *is a weak focus of order* m*, then, when* $0 < \boldsymbol{\varepsilon} \ll 1$*,* $\delta| \ll 1$*,* (3.1.1) *has at most* m *limit cycles in a neighborhood of the origin .*

Theorem 3.1.4. *Suppose that* (1) *the origin of system* $(3.1.1)_{\boldsymbol{\varepsilon}=\mathbf{0},\delta=0}$ *is a weak focus of order* m; (2) $n \geqslant m-1$ *and there exist* $j_1, j_2, \cdots, j_{m-1} \in \{1, 2, \cdots, n\}$, *such that at the origin of* $\boldsymbol{\varepsilon} - \delta$ *parameter space, we have*

$$\frac{\partial(\nu_1, \nu_3, \nu_5, \cdots, \nu_{2m-1})}{\partial(\delta, \varepsilon_{j_1}, \varepsilon_{j_2}, \cdots, \varepsilon_{j_{m-1}})} \neq 0, \tag{3.1.13}$$

where

$$\nu_{2k+1} = \nu_{2k+1}(2\pi, \boldsymbol{\varepsilon}, \delta), \quad k = 1, 2, \cdots, m-1. \tag{3.1.14}$$

Then for choosing proper $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ *and* δ *in the parameter space of* $(n+1)$*-dimension, in a sufficiently small neighborhood of the origin, system* (3.1.1) *has exactly* m *limit circles .*

The proof of Theorem 3.1.4 will be given in Section 3.3.

3.2 Analytic Equivalence

Thereinafter we assume that the dimension of parameter space is one, i.e., $\boldsymbol{\varepsilon} = \varepsilon$ is an real small parameters. System (3.1.1) becomes

$$\begin{aligned}
\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x, y, \varepsilon, \delta) = X(x, y, \varepsilon, \delta), \\
\frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x, y, \varepsilon, \delta) = Y(x, y, \varepsilon, \delta).
\end{aligned} \tag{3.2.1}$$

Usually, when we consider the problem of multiple Hopf bifurcation of limit cycles for a concrete planar dynamical system, we are always going to find the focal value ν_{2k+1} under the conditions $\delta = 0$ and $\nu_3 = \nu_5 = \cdots = \nu_{2k-1} = 0$. We do not like to compute ν_2, ν_4, $\cdots$, ν_{2k}. Generally, a successor function has infinitely many terms, one can only find the first finitely many terms. Therefore, it is difficult to determine exactly all zeros of a successor function. We shall show in Section 3.3 that if Condition 3.3.1 is satisfied, one can find a quasi successor function, by which the first terms of all zeros of a successor function can be determined.

In this section, we study the relation of analytic equivalence of focal values.

Definition 3.2.1. *Suppose that for some positive integer $k > 1$, there exist power series $\xi_k^{(1)}$, $\xi_k^{(2)}$, $\cdots$, $\xi_k^{(k-1)}$ of ε and δ with a nonzero convergence radius, such that*

$$\nu_k(2\pi,\varepsilon,\delta) = \xi_k^{(1)}[\nu_1(2\pi,\varepsilon,\delta) - 1] + \sum_{j=2}^{k-1} \xi_k^{(j)} \nu_j(2\pi,\varepsilon,\delta) + \tilde{\nu}_k(\varepsilon,\delta). \tag{3.2.2}$$

We say that $\nu_k(2\pi,\varepsilon,\delta)$ and $\tilde{\nu}_k(\varepsilon,\delta)$ are analytic equivalent. They are written by $\nu_k(2\pi,\varepsilon,\delta) \simeq \tilde{\nu}_k(\varepsilon,\delta)$.

We see from $\nu_1(\pi,0,0) = \nu_1(2\pi,0,0) = 1$ and Theorem 2.1.1 that the following conclusions hold.

Theorem 3.2.1. *For every positive integer m, we have $\nu_{2m}(2\pi,\varepsilon,\delta) \simeq 0$.*

Theorem 3.2.2. *For every positive integer m, $\nu_{2m+1}(2\pi,\varepsilon,\delta) \simeq \tilde{\nu}_{2m+1}(\varepsilon,\delta)$ if and only if there exist power series $\eta_m^{(0)}$, $\eta_m^{(1)}$, $\cdots$, $\eta_m^{(m-1)}$ of ε and δ with nonzero convergence radius, such that*

$$\begin{aligned}\nu_{2m+1}(2\pi,\varepsilon,\delta) = {} & \eta_m^{(0)}[\nu_1(2\pi,\varepsilon,\delta) - 1] \\ & + \sum_{k=1}^{m-1} \eta_m^{(k)} \nu_{2k+1}(2\pi,\varepsilon,\delta) + \tilde{\nu}_{2m+1}(\varepsilon,\delta).\end{aligned} \tag{3.2.3}$$

Theorem 3.2.3. *For system* (3.2.1), *we have*

$$\nu_{2k+1}(2\pi,\varepsilon,\delta) \simeq \nu_{2k+1}(2\pi,\varepsilon,0), \quad k = 1,2,\cdots. \tag{3.2.4}$$

Proof. For system (3.2.1), denote that

$$\begin{aligned}&\nu_1(2\pi,\varepsilon,\delta) - 1 = e^{2\pi\delta} - 1 = 2\pi\delta g_0(\delta), \\ &\nu_{2k+1}(2\pi,\varepsilon,\delta) = \nu_{2k+1}(2\pi,\varepsilon,0) + \delta g_k(\varepsilon,\delta) \quad k = 1,2,\cdots,\end{aligned} \tag{3.2.5}$$

where $g_0(0) = 1$, $g_0(\delta)$ is a power series of δ and for all k, $g_k(\varepsilon,\delta)$ are power series of ε and δ with nonzero convergence radius. From (3.2.5), we have

$$\nu_{2k+1}(2\pi,\varepsilon,\delta) = \frac{g_k(\varepsilon,\delta)}{2\pi g_0(\delta)}[\nu_1(2\pi,\varepsilon,\delta) - 1] + \nu_{2k+1}(2\pi,\varepsilon,0). \tag{3.2.6}$$

It follows the conclusion of Theorem 3.2.3. □

Considering the algebraic equivalence of focal values, the following conclusion holds.

Theorem 3.2.4. *For systems $(3.2.1)_{\delta=0}$, if there exists a positive integer k, such that $\nu_{2k+1}(2\pi,\varepsilon,0) \sim \tilde{\nu}_{2k+1}(\varepsilon)$, then $\nu_{2k+1}(2\pi,\varepsilon,0) \simeq \tilde{\nu}_{2k+1}(\varepsilon)$.*

The associated system of $(3.2.1)_{\delta=0}$ has the form:

$$\frac{dz}{dT} = z + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta}(\varepsilon) z^{\alpha} w^{\beta},$$
$$\frac{dw}{dT} = -w - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta}(\varepsilon) w^{\alpha} z^{\beta}, \tag{3.2.7}$$

where all $a_{\alpha\beta}(\varepsilon),\ b_{\alpha\beta}(\varepsilon)$ are power series of ε with nonzero convergence radius.

Definition 3.2.2. *For systems* (3.2.7), *if there is a positive integer* $m > 1$ *and power series* $\xi_m^{(1)}(\varepsilon),\ \xi_m^{(2)}(\varepsilon),\ \cdots,\ \xi_m^{(m-1)}(\varepsilon)$ *of* ε *with nonzero convergence radius, such that*

$$\mu_m(\varepsilon) = \sum_{k=1}^{m-1} \xi_m^{(k)}(\varepsilon)\mu_k(\varepsilon) + \tilde{\mu}_m(\varepsilon). \tag{3.2.8}$$

Then, we say that $\mu_m(\varepsilon)$ *and* $\tilde{\mu}_m(\varepsilon)$ *are analytic equivalence. They are written by* $\mu_m(\varepsilon) \simeq \tilde{\mu}_m(\varepsilon)$.

Obviously, if $\mu_m(\varepsilon) \simeq \tilde{\mu}_m(\varepsilon)$, then, $\tilde{\mu}_m(\varepsilon)$ is a power series of ε with nonzero convergent radius. Similar to theorem 3.2.4, we have

Theorem 3.2.5. *For systems* (3.2.7), *if there exists a positive integer* $m > 1$, *such that* $\mu_m(\varepsilon) \sim \tilde{\mu}_m(\varepsilon)$, *then* $\mu_m(\varepsilon) \simeq \tilde{\mu}_m(\varepsilon)$.

From the relationship of focal values and singular point values, we have the following result.

Theorem 3.2.6. *For the associated system* (3.2.7) *of system* (3.2.1), *if there exists a positive integer* $m > 1$, *such that* $\mu_m(\varepsilon) \simeq \tilde{\mu}_m(\varepsilon)$, *i.e., they are analytic equivalent, then*

$$\nu_{2m+1}(2\pi, \varepsilon, 0) \simeq i\pi\tilde{\mu}_m(\varepsilon). \tag{3.2.9}$$

3.3 Quasi Successor Function

Let $\delta = \delta(\varepsilon)$ be a power series of real coefficients with respect to ε, which has a nonzero convergence radius and $\delta(0) = 0$. System (3.2.1) becomes

$$\frac{dx}{dt} = \delta(\varepsilon)x - y + \sum_{k=2}^{\infty} X_k(x, y, \varepsilon, \delta(\varepsilon)) = X(x, y, \varepsilon, \delta(\varepsilon)),$$
$$\frac{dy}{dt} = x + \delta(\varepsilon)y + \sum_{k=2}^{\infty} Y_k(x, y, \varepsilon, \delta(\varepsilon)) = Y(x, y, \varepsilon, \delta(\varepsilon)). \tag{3.3.1}$$

The *Poincaré* successor function of system (3.3.1) is given by

$$\Delta(h, \varepsilon, \delta(\varepsilon)) = \tilde{r}(2\pi, h, \varepsilon, \delta(\varepsilon)) - h = \sum_{k=1}^{\infty} \nu_k(\theta, \varepsilon, \delta(\varepsilon))h^k - h, \tag{3.3.2}$$

where h is a sufficiently small complex constant.

In this section, we suppose that $\Delta(h, \varepsilon, \delta(\varepsilon))$ is not identically zero when $0 < |\varepsilon| \ll 1$. Thus, the following condition holds

Condition 3.3.1. *There exist natural numbers N, m and $\lambda_0, \lambda_1, \cdots, \lambda_m$, which are independent of ε, such that*

$$\begin{aligned} \nu_1(2\pi, \varepsilon, \delta(\varepsilon)) - 1 &= \lambda_0 \varepsilon^{l_0+N} + o(\varepsilon^{l_0+N}), \\ \nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) &\simeq \lambda_k \varepsilon^{l_k+N} + o(\varepsilon^{l_k+N}), \quad k = 1,\ 2,\ \cdots,\ m, \end{aligned} \tag{3.3.3}$$

where $l_0,\ l_1,\ \cdots,\ l_{m-1}$ are positive integers,

$$l_m = 0, \quad \lambda_m \neq 0, \tag{3.3.4}$$

and

$$\nu_{2m+k+1}(2\pi, \varepsilon, \delta(\varepsilon)) = O(\varepsilon^N), \quad k = 1,\ 2,\ \cdots. \tag{3.3.5}$$

Remark 3.3.1. (1) *In Condition* 3.3.1, *if $N = 0$, then the origin of* $(3.3.1)_{\varepsilon=0}$ *is a m-th weak focus. In this case, we suppose that $\varepsilon^N \equiv 1$. Furthermore, if the origin of* $(3.3.1)_{\varepsilon=0}$ *is a m-th weak focus, then $N = 0$, and* (3.3.5) *hods.* (2) *If $N > 0$, then the origin of* $(3.3.1)_{\varepsilon=0}$ *is an center. Furthermore, if the origin of* $(3.3.1)_{\varepsilon=0}$ *is an center, and $N = 1$, then* (3.3.5) *hods.*

Remark 3.3.2. (1) *In Condition* 3.3.1, *if $\nu_1(2\pi, \varepsilon, \delta(\varepsilon)) \equiv 1$, we suppose that $\lambda_0 = 0,\ l_0 = \infty$.* (2) *If for some positive integer $k \in \{1, 2, \cdots, m-1\}$, we have $\nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) \sim 0$, then we suppose that $\lambda_k = 0,\ l_k = \infty$.*

Lemma 3.3.1. *If Condition* 3.3.1 *is satisfied, then $\Delta(h, \varepsilon, \delta(\varepsilon))$ can be represented as*

$$\Delta(h, \varepsilon, \delta(\varepsilon)) = \varepsilon^N h \tilde{\Delta}(h, \varepsilon), \tag{3.3.6}$$

where

$$\tilde{\Delta}(h, \varepsilon) = \sum_{k=0}^{m} \lambda_k \varepsilon^{l_k} h^{2k} g_k(h, \varepsilon) \tag{3.3.7}$$

and $g_k(h, \varepsilon)$ is a power series of h and ε with nonzero convergent radius, $g_k(0, 0) = 1,\ k = 0,\ 1,\ \cdots,\ m$.

Proof. The proof is divided into the following three steps.

(1) First, Condition 3.3.1 and Theorem 2.1.1 follow that for every positive integer k, $\varepsilon^{-N} \nu_k(2\pi, \varepsilon, \delta(\varepsilon))$ is a power series of ε with nonzero convergent radius.

(2) We rewrite (3.3.3) as

$$\begin{aligned} &\nu_1(2\pi, \varepsilon, \delta(\varepsilon)) - 1 = \lambda_0 \varepsilon^{l_0+N} \xi_0(\varepsilon), \\ &\nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) \simeq \lambda_k \varepsilon^{l_k+N} \xi_k(\varepsilon), \quad k = 1,\ 2,\ \cdots,\ m, \end{aligned} \tag{3.3.8}$$

where $\xi_k(\varepsilon)$ is a power series of ε with real coefficients, which has nonzero convergent radius and $\xi_k(0) = 1,\ k = 0,\ 1,\ \cdots,\ m$. From Theorem 2.1.1 and Definition 3.2.2, we know that when $2 \leqslant k \leqslant 2m+1$, $\nu_k(2\pi, \varepsilon, \delta(\varepsilon))$ is a linear combination of $\lambda_0\varepsilon^{l_0+N},\ \lambda_1\varepsilon^{l_1+N},\ \cdots,\ \lambda_m\varepsilon^{l_m+N}$, for which the coefficients are power series of ε with real coefficients and nonzero convergent radius. It leads to

$$\begin{aligned}&\nu_1(2\pi,\varepsilon,\delta(\varepsilon)) - 1 + \sum_{k=2}^{2m+1}\nu_k(2\pi,\varepsilon,\delta(\varepsilon))h^{k-1}\\ &= \varepsilon^N\sum_{k=0}^{m-1}\lambda_k\varepsilon^{l_k}h^{2k}g_k(h,\varepsilon) + \lambda_m\varepsilon^{l_m+N}\xi_m(\varepsilon)h^{2m},\end{aligned} \tag{3.3.9}$$

where $g_k(h, \varepsilon)$ is not only a polynomial of h, but also a power series of ε with real coefficients and nonzero convergent radius, $g_k(0, \varepsilon) = \xi_k(\varepsilon),\ k = 0,\ 1,\ \cdots,\ m-1$.

(3) From (3.3.4) and (3.3.5),

$$g_m(h,\varepsilon) = \xi_m(\varepsilon) + \frac{1}{\lambda_m\varepsilon^N}\sum_{k=2m+2}^{\infty}\nu_k(2\pi,\varepsilon,\delta(\varepsilon))h^{k-2m-1} \tag{3.3.10}$$

is a power series of h, ε with real coefficients and nonzero convergent radius. From (3.3.9) and (3.3.10), we have

$$\Delta(h,\varepsilon,\delta(\varepsilon)) = \varepsilon^N h\sum_{k=0}^{m}\lambda_k\varepsilon^{l_k}h^{2k}g_k(h,\varepsilon). \tag{3.3.11}$$

This completes the proof of this lemma. □

Definition 3.3.1. *Suppose that Condition* 3.3.1 *is satisfied. We say that*

$$L(h,\varepsilon) = \sum_{k=0}^{m}\lambda_k\varepsilon^{l_k}h^{2k} \tag{3.3.12}$$

is a quasi successor function of system (3.3.1).

Obviously, the quasi successor function of (3.3.1) can be computed by finitely many steps under Condition 3.3.1.

Clearly, we have

Lemma 3.3.2. *Under Condition* 3.3.1, *if $h = 0$ is a zero of multiplicity k of $\tilde{\Delta}(h, \varepsilon)$, then k must be an even number. Moreover, $h = 0$ is a zero of multiplicity k of $L(h, \varepsilon)$.*

Remark 3.3.3. *Suppose that $h = h(\varepsilon, \delta(\varepsilon))$ is a zero of $\Delta(h, \varepsilon, \delta(\varepsilon))$ and $h(\varepsilon, \delta(\varepsilon))$ is not identically zero when $|\varepsilon| \ll 1$. If Condition* 3.3.1 *holds, then, by using Weierstrass preparation theorem, it follows that $\varepsilon = 0$ is an algebraic zero of $h(\varepsilon, \delta(\varepsilon))$.*

Namely, there are two irredusible positive numbers p and q and a nonzero constant η, such that

$$h(\varepsilon, \delta(\varepsilon)) = \eta\varepsilon^{\frac{p}{q}} + \varepsilon^{\frac{p}{q}}\zeta(\varepsilon^{\frac{1}{q}}), \tag{3.3.13}$$

where $\zeta(\sigma)$ is a power series of σ with nonzero convergence radius and $\zeta(0) = 0$. $\eta\varepsilon^{\frac{p}{q}}$ is called the first term of $h(\varepsilon, \delta(\varepsilon))$.

From Lemma 3.3.1, we obtain

Lemma 3.3.3. *If Condition* 3.3.1 *is satisfied, then, for every positive integers p and q, we have*

$$\begin{aligned} \tilde{\Delta}(\sigma^p\eta, \sigma^q) &= \sigma^D[G(\eta) + \sigma\Phi(\eta, \sigma)], \\ L(\sigma^p\eta, \sigma^q) &= \sigma^D[G(\eta) + \sigma\Psi(\eta, \sigma)], \end{aligned} \tag{3.3.14}$$

where $\Phi(\eta, \sigma)$, $\Psi(\eta, \sigma)$ are analytic for sufficiently small σ and a bounded $|\eta|$. In addition,

$$D = \min_{0 \leqslant k \leqslant m} \{l_k q + 2kp\}, \tag{3.3.15}$$

$$G(\eta) = \sum_{k=0}^{m} \lambda'_k \eta^{2k}, \tag{3.3.16}$$

$$\lambda'_k = \begin{cases} \lambda_k, & if\ l_k q + 2kp = D, \\ 0, & if\ l_k q + 2kp > D. \end{cases} \tag{3.3.17}$$

(3.3.14) implies the following lemma.

Lemma 3.3.4. *If Condition* 3.3.1 *is satisfied, $h = h(\varepsilon, \delta(\varepsilon))$ is a zero of $\Delta(h, \varepsilon, \delta(\varepsilon))$, for which the first term is $\eta_0\varepsilon^{\frac{p}{q}}$, $\eta_0 \neq 0$, then $\eta = \eta_0$ is a zero of $G(\eta)$. Moreover, replacing $\Delta(h, \varepsilon, \delta(\varepsilon))$ with $L(h, \varepsilon)$, above conclusion also holds.*

(3.3.14) and implicit function theorem follows that

Lemma 3.3.5. *If Condition* 3.3.1 *is satisfied and $\eta = \eta_0 \neq 0$ is a simple zero of $G(\eta)$, then, $\tilde{\Delta}(\sigma^p\eta, \sigma^q)$ has also a zero $\eta = \eta_0 + f(\sigma)$correspondingly, $\Delta(h, \varepsilon, \delta(\varepsilon))$ has a zero $h = \eta_0\varepsilon^{\frac{p}{q}} + \varepsilon^{\frac{p}{q}} f(\varepsilon^{\frac{1}{q}})$, where $f(\sigma)$ is a power series of σ with nonzero radius of convergence and real coefficients, $f(0) = 0$ and η_0 is a real number.*

Moreover, replacing $\Delta(h, \varepsilon, \delta(\varepsilon))$ with $L(h, \varepsilon)$, above conclusion also holds.

Again (3.3.14) and the Weierstrass preparation theorem imply that

Lemma 3.3.6. *If Condition* 3.3.1 *holds and $\eta = \eta_0 \neq 0$ is a zero of multiplicity k of $G(\eta)$, then $\tilde{\Delta}(\sigma^p\eta, \sigma^q)$ has k zeros $\eta = \eta_0 + f_j(\sigma)$. Correspondingly, $\Delta(h, \varepsilon, \delta(\varepsilon))$ also has k zeros $h = \eta_0\varepsilon^{\frac{p}{q}} + \varepsilon^{\frac{p}{q}} f_j(\varepsilon^{\frac{1}{q}})$, where $f_j(\sigma)$ are power series of σ with nonzero convergent radius, $f_j(0) = 0$, $j = 1, 2, \cdots, k$.*

For $L(h, \varepsilon)$, we have the same conclusion.

Thus, from Lemma 3.3.1 ∼ Lemma 3.3.6, we obtain

Theorem 3.3.1. *If Condition* 3.3.1 *holds, then, the zeros of* $\tilde{\Delta}(h,\varepsilon)$ *and* $L(h,\varepsilon)$ *have the same first terms by rearranging the orders of these zeros.*

From Theorem 3.3.1 and Lemma 3.3.5, we have

Theorem 3.3.2. *If Condition* 3.3.1 *holds and for* $0<\varepsilon\ll 1$, *there exist exactly* s *zeros having positive first term in all* $2m$ *zeros of* $L(h,\varepsilon)$). *In addition, these* s *positive first terms are different each other. Then,* $\tilde{\Delta}(h,\varepsilon)$ *has exactly* s *positive zeros.*

For the case $0<-\varepsilon\ll 1$, replacing ε by $-\varepsilon$ in the quasi successor function, we obtain the corresponding result.

Theorem 3.3.3. *Suppose that Condition* 3.3.1 *holds. If* (1) *There is a positive integer* d, *such that*

$$l_k=(m-k)d,\quad k=0,\ 1,\ \cdots,\ m. \tag{3.3.18}$$

(2) $G(\eta)=\sum\limits_{k=0}^{\infty}\lambda_k\eta^{2k}$ *has exactly* m *different positive zeros* $\eta_1,\ \eta_2,\ \cdots,\ \eta_m$. *Then, for* $0<\varepsilon\ll 1$, $\tilde{\Delta}(h,\varepsilon)$ *has exactly* m *positive zeros*

$$h=h_k(\varepsilon)=\eta_k\varepsilon^{\frac{d}{2}}+o(\varepsilon^{\frac{d}{2}}),\quad k=1,\ 2,\ \cdots,m. \tag{3.3.19}$$

Correspondingly, in a sufficiently small neighborhood of the origin, system (3.3.1) *has exactly* m *limit cycles, which are close to the circles* $x^2+y^2=\eta_k^2\varepsilon^d$, $k=1,\ 2,\ \cdots,\ m$.

Proof. If the Conditions of Theorem 3.3.3 hold, then

$$L(h,\varepsilon)=\sum_{k=0}^{m}\lambda_k\varepsilon^{(m-k)d}h^{2k}.$$

By Theorem 3.3.2, we know the conclusions of this theorem. □

Theorem 3.3.4. *Suppose that Condition* 3.3.1 *holds. In addition,*

$$\begin{aligned}&\lambda_{k-1}\lambda_k<0,\quad k=1,\ 2,\ \cdots,\ m,\\&l_{k-1}-l_k>l_k-l_{k+1},\quad k=1,\ 2,\ \cdots,\ m-1.\end{aligned} \tag{3.3.20}$$

Then, for $0<\varepsilon\ll 1$, $\tilde{\Delta}(h,\varepsilon)$ *has exactly* m *positive zeros*

$$h=h_k(\varepsilon)=\sqrt{\frac{-\lambda_{k-1}}{\lambda_k}}\ \varepsilon^{\frac{l_{k-1}-l_k}{2}}+o(\varepsilon^{\frac{l_{k-1}-l_k}{2}}),\quad k=1,\ 2,\ \cdots,m. \tag{3.3.21}$$

Correspondingly, system (3.3.1) *has exactly* m *limit cycles in a sufficiently small neighborhood of the origin, which are close to the circles* $x^2+y^2=\dfrac{-\lambda_{k-1}}{\lambda_k}\ \varepsilon^{l_{k-1}-l_k}$, $k=1,\ 2,\ \cdots,\ m$.

Proof. Let $k \in \{1,\ 2,\ \cdots,\ m\}$, (3.3.20) implies that

$$l_j - l_k \geqslant (k-j)(l_{k-1} - l_k), \quad j = 0,\ 1,\ \cdots,\ m. \tag{3.3.22}$$

(3.3.22) becomes an equality if and only if $j = k-1$ or $j = k$. Let $2p = q(l_{k-1} - l_k)$, where $p,\ q$ are relatively prime. Lemma 3.3.3 follows that $D = l_k q + k(l_{k-1} - l_k)q$, $G(\eta) = \lambda_{k-1}\eta^{2k-2} + \lambda_k \eta^{2k}$. Hence, Lemma 3.3.5 follows the conclusion of Theorem 3.3.4 □

Example 3.3.1. *Suppose that for system* (3.3.1), *we have*

$$\begin{aligned} &\nu_1(2\pi, \varepsilon, \delta(\varepsilon)) - 1 = \lambda_0 \varepsilon + o(\varepsilon), \\ &\nu_{2k+1}(2\pi, 0, 0) = 0, \quad k = 1,\ 2,\ \cdots,\ m-1, \\ &\nu_{2m+1}(2\pi, 0, 0) = \lambda_m, \quad \lambda_m \neq 0. \end{aligned} \tag{3.3.23}$$

Then

$$\nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) \simeq 0, \quad k = 1,\ 2,\ \cdots,\ m-1 \tag{3.3.24}$$

and the quasi successor of system (3.3.1) *is* $L(h, \varepsilon) = \lambda_0 \varepsilon + \lambda_m h^{2m}$. *It follows the conclusion of Hopf bifurcation theorem.*

Proof of Theorem 3.1.4 Based on the Implicit function theorem, under the conditions of Theorem 3.1.4, solving

$$\begin{aligned} &\nu_1(2\pi, \boldsymbol{\varepsilon}, \delta) - 1 = c_0 \nu_{2m+1}(2\pi, \mathbf{0}, 0)\sigma^{2m}, \\ &\nu_{2k+1}(2\pi, \boldsymbol{\varepsilon}, \delta) = c_k \nu_{2m+1}(2\pi, \mathbf{0}, 0)\sigma^{2m-2k}, \quad k = 1, 2, \cdots, m-1, \end{aligned} \tag{3.3.25}$$

we obtain the unique solution

$$\delta = \delta(\sigma), \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\sigma), \tag{3.3.26}$$

where $c_0,\ c_1,\ \cdots,\ c_{m-1}$ are determined by

$$\prod_{k=1}^{m} (\eta^2 - k^2) = \sum_{k=0}^{m} c_k \eta^{2k}. \tag{3.3.27}$$

$\delta(\sigma),\ \boldsymbol{\varepsilon}(\sigma)$ is analytic at $\sigma = 0$ and $\delta(0) = 0$, $\boldsymbol{\varepsilon}(0) = \mathbf{0}$.

From (3.3.25) and (3.3.27), the quasi successor function of system $(3.1.1)_{\delta=\delta(\sigma), \boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}(\sigma)}$ is

$$\begin{aligned} L(h, \sigma) &= \nu_{2m+1}(2\pi, \mathbf{0}, 0) \sum_{k=0}^{m} c_k \sigma^{2m-2k} h^{2k} \\ &= \nu_{2m+1}(2\pi, \mathbf{0}, 0)\sigma^{2m} \prod_{k=1}^{m} \left(\frac{h^2}{\sigma^2} - k^2 \right). \end{aligned} \tag{3.3.28}$$

Thus, from Theorem 3.3.3, when $0 < \sigma \ll 1$, in a sufficiently small neighborhood of the origin, system $(3.1.1)_{\delta=\delta(\sigma),\ \varepsilon=\varepsilon(\sigma)}$ has exactly m limit cycles, which are close to the circles $x^2 + y^2 = k^2\sigma^2$, $k = 1,\ 2,\ \cdots,\ m$. It follows the conclusion of Theorem 3.1.4.

3.4 Bifurcations of Limit Circle of a Class of Quadratic Systems

In order to obtain more limit circles, as an example, in this section, we consider a planar quadratic system to show how to compute quasi successor function and find focal values in a higher order weak focus (or a center).

We investigate the planar quadratic system

$$\frac{dx}{dt} = \delta x - y + X_2(x, y), \quad \frac{dy}{dt} = x + \delta y + Y_2(x, y), \tag{3.4.1}$$

where $X_2(x, y)$, $Y_2(x, y)$ are homogenous quadratic polynomials of x, y. By transformations $z = x + iy$, $w = x - iy$, $T = it$, $i = \sqrt{-1}$, system $(3.4.1)_{\delta=0}$ becomes

$$\begin{aligned} \frac{dz}{dT} &= z + a_{20}z^2 + a_{11}zw + a_{02}w^2, \\ \frac{dw}{dT} &= -w - b_{20}w^2 - b_{11}wz - b_{02}z^2, \end{aligned} \tag{3.4.2}$$

where

$$a_{\alpha\beta} = A_{\alpha\beta} + iB_{\alpha\beta}, \quad b_{\alpha\beta} = A_{\alpha\beta} - iB_{\alpha\beta}. \tag{3.4.3}$$

Now

$$\begin{aligned} X_2(x, y) &= -(B_{20} + B_{11} + B_{02})x^2 - 2(A_{20} - A_{02})xy + (B_{20} - B_{11} + B_{02})y^2, \\ Y_2(x, y) &= (A_{20} + A_{11} + A_{02})x^2 - 2(B_{20} - B_{02})xy - (A_{20} - A_{11} + A_{02})y^2. \end{aligned} \tag{3.4.4}$$

Theorem 2.6.1 follows that the first 3 focal values of system (3.4.2) are

$$\begin{aligned} \mu_1 &= b_{20}b_{11} - a_{20}a_{11}, \\ \mu_2 &\sim -\tfrac{1}{3}(2I_1 + 3I_2 - 2I_3), \\ \mu_3 &\sim \tfrac{5}{8}(a_{11}b_{11} - a_{02}b_{02})(2I_2 - I_3), \end{aligned} \tag{3.4.5}$$

where

$$\begin{aligned} I_0 &= a_{20}^3 a_{02} - b_{20}^3 b_{02} & I_1 &= a_{20}^2 b_{11} a_{02} - b_{20}^2 a_{11} b_{02}, \\ I_2 &= a_{20} b_{11}^2 a_{02} - b_{20} a_{11}^2 b_{02}, & I_3 &= b_{11}^3 a_{02} - a_{11}^3 b_{02}. \end{aligned} \tag{3.4.6}$$

Suppose that N is a natural number, taking

$$\begin{aligned}
&\delta = \frac{25}{8}(3+\varepsilon^{2N})\varepsilon^{12+N}, \quad a_{11} = b_{11} = 1,\\
&a_{02} = 2+\varepsilon^N i, \quad b_{02} = 2-\varepsilon^N i,\\
&a_{20} = -2-\frac{15}{8}(3+\varepsilon^{2N})\varepsilon^2 - \frac{25}{8}\varepsilon^{6+N}(3+\varepsilon^{2N})i,\\
&b_{20} = -2-\frac{15}{8}(3+\varepsilon^{2N})\varepsilon^2 + \frac{25}{8}\varepsilon^{6+N}(3+\varepsilon^{2N})i.
\end{aligned} \tag{3.4.7}$$

Then Theorem 2.3.1 implies that

$$\begin{aligned}
&\nu_1(2\pi)-1 = \frac{25}{4}\pi(3+\varepsilon^{2N})\varepsilon^{12+N} + o(\varepsilon^{12+N}),\\
&\nu_3(2\pi) \simeq -\frac{25}{4}\pi(3+\varepsilon^{2N})\varepsilon^{6+N} + o(\varepsilon^{6+N}),\\
&\nu_5(2\pi) \simeq \frac{25}{4}\pi(3+\varepsilon^{2N})\varepsilon^{2+N} + o(\varepsilon^{2+N}),\\
&\nu_7(2\pi) \simeq -\frac{25}{4}\pi(3+\varepsilon^{2N})\varepsilon^{N} + o(\varepsilon^{N}).
\end{aligned} \tag{3.4.8}$$

Furthermore, when (3.4.7) holds, we have

$$\begin{aligned}
a_{20}a_{11} - b_{20}b_{11} &= o(\varepsilon^N),\\
I_k = O(\varepsilon^N), &\quad k = 0,1,2,3.
\end{aligned} \tag{3.4.9}$$

Therefore, Theorem 2.4.7 and the elementary invariants of the quadratic system given by Corollary 2.5.1 follows that

$$\nu_{2k+1} = O(\varepsilon^N), \quad k = 1,2,\cdots. \tag{3.4.10}$$

(3.4.8) and (3.4.10) give rise to that under condition (3.4.7), the quasi successor function of (3.4.1) is

$$L(h,\varepsilon) = \frac{25}{4}\pi\lambda(\varepsilon^{12} - \varepsilon^6 h^2 + \varepsilon^2 h^4 - h^6), \tag{3.4.11}$$

where

$$\lambda = \begin{cases} 4, & \text{if } N = 0,\\ 3, & \text{if } N > 0. \end{cases} \tag{3.4.12}$$

For $0 < \varepsilon \ll 1$ there exist three zeros of $L(h,\varepsilon)$

$$h = h_k(\varepsilon) = \varepsilon^k + o(\varepsilon^k), \quad k = 1,2,3. \tag{3.4.13}$$

Thus, Theorem 3.3.4 implies that

Theorem 3.4.1. *Suppose the coefficients of right hand of system* (3.4.1) *are given by* (3.4.7)*. Then,* (1) *For* $N = 0$*, the origin of system* $(3.4.1)_{\varepsilon=0}$ *is a* 3 *order weak focus, while when* $N > 0$*, it is a center.* (2) *For* $0 < |\varepsilon| \ll 1$*, system* (3.4.1) *has exactly* 3 *limit circles in a sufficiently small neighborhood of the origin, which are close to the circles* $x^2 + y^2 = \varepsilon^{2k}$*,* $k = 1, 2, 3$*.*

Remark 3.4.1. *When* $N > 0$ *in* (3.4.8)*, we must show that* (3.4.10) *holds. It was proved by using the construction theorem of singular point values.*

Bibliographical Notes

Concerning with the multiple Hopf bifurcations, a great number of papers and books had been published. For instance, see [Takens, 1973; Gobber etc, 1979; Hassard etc, 1981; Bonin etc, 1988; Li C.Z. etc, 1989; Christopher etc, 1999; Chan H.S.Y. etc, 2001; Chan H.S.Y. etc, 2002; Yu P. etc 2005a; Han M.A. etc, 2009; Liu Y.R. etc, 2009d; Liu Y.R. etc, 2009c; Han M.A. etc, 2010a] et al.

The materials of this chapter are taken from [Liu Y.R., 2001].

Chapter 4

Isochronous Center In Complex Domain

For a given planar dynamical system, when we have characterized its center, it is also interesting to know whether the center is isochronous or not. In this chapter, we extended the concepts of the period constant and the isochronous center in the real systems to the complex systems. The results and methods mentioned in this chapter are more interesting.

4.1 Isochronous Centers and Period Constants

We consider the following two-dimension complex autonomous differential system

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{k=2}^{\infty} X_k(x, y) = X(x, y), \\ \frac{dy}{dt} &= x + \sum_{k=2}^{\infty} Y_k(x, y) = Y(x, y), \end{aligned} \tag{4.1.1}$$

where x, y, t are complex variables, $X(x, y)$ and $Y(x, y)$ are power series of x, y with non-zero convergent radius. $X_k(x, y)$ and $Y_k(x, y)$ are homogeneous polynomials of degree k.

By using the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{4.1.2}$$

system (4.1.1) becomes

$$\frac{dr}{dt} = r\sum_{k=1}^{\infty} \varphi_{k+2}(\theta) r^k, \quad \frac{d\theta}{dt} = 1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) r^k, \tag{4.1.3}$$

where

$$\begin{aligned} \varphi_k(\theta) &= \cos\theta X_{k-1}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta), \\ \psi_k(\theta) &= \cos\theta Y_{k-1}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta). \end{aligned} \tag{4.1.4}$$

From (4.1.3), we have

$$\frac{dr}{d\theta} = \frac{r\sum_{k=1}^{\infty}\varphi_{k+2}(\theta)r^k}{1+\sum_{k=1}^{\infty}\psi_{k+2}(\theta)r^k}. \tag{4.1.5}$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty}\nu_k(\theta)h^k \tag{4.1.6}$$

be the solution of (4.1.5) satisfying the initial condition $r|_{\theta=0} = h$, where

$$\nu_1(\theta) \equiv 1, \quad \nu_k(0) = 0, \quad k = 2,\ 3, \cdots. \tag{4.1.7}$$

From (4.1.3), we have

$$t = \mathcal{T}(\theta, h) = \int_0^{\theta} \frac{d\vartheta}{1+\sum_{k=1}^{\infty}\psi_{k+2}(\vartheta)\tilde{r}^k(\vartheta, h)}. \tag{4.1.8}$$

Definition 4.1.1. *Suppose that for sufficiently small complex constant h (i.e., $|h| \ll 1$), we have*

$$\tilde{r}(2\pi, h) = h, \quad \mathcal{T}(2\pi, h) \equiv 2\pi. \tag{4.1.9}$$

We say that the origin of system (4.1.1) *is a complex isochronous center.*

Obviously, if system (4.1.1) is a real planar autonomous differential system and the origin of system (4.1.1) is a complex isochronous center, then the origin of system (4.1.1) is an isochronous center in the real field.

Denote that

$$\mathcal{T}(2\pi, h) = \int_0^{2\pi} \frac{d\theta}{1+\sum_{k=1}^{\infty}\psi_{k+2}(\theta)\tilde{r}^k(\theta, h)} = \pi\left(2 - \sum_{k=1}^{\infty}\mathcal{T}_k h^k\right). \tag{4.1.10}$$

It is proved in [Chicone etc, 1989] that

Lemma 4.1.1. *For system* (4.1.1)*, if $\tilde{r}(2\pi, h) = h$, then $\mathcal{T}_1 = 0$. Furthermore, for any positive integer k, if $\mathcal{T}_1 = \mathcal{T}_2 = \cdots = \mathcal{T}_{2k} = 0$, then $\mathcal{T}_{2k+1} = 0$.*

From (4.1.10) , Definition 4.1.1 and Lemma 4.1.1, we have

Theorem 4.1.1. *Suppose that the origin is a complex center of system* (4.1.1)*. Then, the origin is a complex isochronous center if and only if for any positive integer k, $\mathcal{T}_{2k} = 0$.*

By the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{4.1.11}$$

system (4.1.1) becomes its associated system

$$\begin{aligned}\frac{dz}{dT} &= \ z + \sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{k=2}^{\infty} W_k(z,w) = -W(z,w),\end{aligned} \tag{4.1.12}$$

where

$$\begin{aligned}Z_k = Y_k - iX_k &= \sum_{\alpha+\beta=k} a_{\alpha\beta} z^{\alpha} w^{\beta},\\ W_k = Y_k + iX_k &= \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}\end{aligned} \tag{4.1.13}$$

are homogeneous polynomials of degree k of z, w.

Definition 4.1.2. *Suppose that the origin of system* (4.1.1) *is a complex isochronous center. Then, we say that the origin of system* (4.1.12) *is also a complex isochronous center.*

We see from Section 1.8 that system (4.1.1) can be reduced to the following standard normal form

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \xi \sum_{k=1}^{\infty} p_k (\xi\eta)^k,\\ \frac{d\eta}{dT} &= -\eta - \eta \sum_{k=1}^{\infty} q_k (\xi\eta)^k,\end{aligned} \tag{4.1.14}$$

by means of the standard formal transformation

$$\begin{aligned}\xi &= z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j = \xi + \sum_{k=2}^{\infty} \xi_k(z,w) = \xi(z,w),\\ \eta &= w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j = \eta + \sum_{k=2}^{\infty} \eta_k(z,w) = \eta(z,w),\end{aligned} \tag{4.1.15}$$

where for all k, $\xi_k(z,w)$, $\eta_k(z,w)$ are homogeneous polynomials of degree k of z, w.

Let

$$\begin{aligned}u &= \frac{1}{2}(\xi + \eta) = x + \sum_{k=2}^{\infty} u_k(x,y) = u(x,y),\\ v &= \frac{1}{2i}(\xi - \eta) = y + \sum_{k=2}^{\infty} v_k(x,y) = v(x,y),\end{aligned} \tag{4.1.16}$$

where for all k, $u_k(x, y)$, $v_k(x, y)$ are homogeneous polynomials of degree k of x, y. By transformation (4.1.16), system (4.1.1) becomes

$$\frac{du}{dt} = -v + \frac{1}{2}\sum_{k=1}^{\infty}(\sigma_k u - \tau_k v)(u^2 + v^2)^k,$$
$$\frac{dv}{dt} = u + \frac{1}{2}\sum_{k=1}^{\infty}(\tau_k u + \sigma_k v)(u^2 + v^2)^k, \tag{4.1.17}$$

where

$$\sigma_k = i\mu_k = i(p_k - q_k), \quad \tau_k = p_k + q_k. \tag{4.1.18}$$

For the origin of system (4.1.12), $\mu_k = p_k - q_k$ is the k-th singular point value, $\tau_k = p_k + q_k$ is the k-th period constant.

We next consider the relation between $\{\mathcal{T}_{2k}\}$ and $\{\tau_k\}$.

By the transformation

$$u = \rho\cos\omega, \quad v = \rho\sin\omega, \tag{4.1.19}$$

system (4.1.17) can be reduced to

$$\frac{d\rho}{dt} = \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k\rho^{2k}, \quad \frac{d\omega}{dt} = 1 + \frac{1}{2}\sum_{k=1}^{\infty}\tau_k\rho^{2k}. \tag{4.1.20}$$

Obviously, by means of the transformation

$$\xi = \rho e^{i\omega}, \quad \eta = \rho e^{-i\omega}, \quad T = it, \tag{4.1.21}$$

system (4.1.14) can also be reduced to (4.1.20).

Let

$$\rho(\theta, h) = \sqrt{u^2(\tilde{x}, \tilde{y}) + v^2(\tilde{x}, \tilde{y})}, \tag{4.1.22}$$

where

$$\tilde{x} = \tilde{r}(\theta, h)\cos\theta, \quad \tilde{y} = \tilde{r}(\theta, h)\sin\theta. \tag{4.1.23}$$

From (4.1.22), we have

$$\rho(0, h) = \sqrt{u^2(h, 0) + v^2(h, 0)} = hA(h), \tag{4.1.24}$$

where $A(h)$ is a power series of h with non-zero convergent radius, and $A(0) = 1$.

By (4.1.3) and (4.1.20), we obtain

Lemma 4.1.2. *If the origin of system* (4.1.1) *is a complex center, then* $\rho(\theta, h)$ *is independent of* θ, *namely*

$$\rho(\theta, h) \equiv \rho(0, h) = hA(h). \tag{4.1.25}$$

Thus, (4.1.3), (4.1.20) and Lemma 4.1.2 follow that

Lemma 4.1.3. *Suppose that the origin of system* (4.1.1) *is a complex center, then when* $x = \tilde{r}(\theta, h)\cos\theta$, $y = \tilde{r}(\theta, h)\sin\theta$, *we have*

$$\omega = \omega(\theta, h), \tag{4.1.26}$$

where

$$\omega(\theta, h) = \omega(0, h) + \left[1 + \frac{1}{2}\sum_{k=1}^{\infty} \tau_k \rho^{2k}(0, h)\right] \mathcal{T}(\theta, h) \tag{4.1.27}$$

and

$$\omega(0, h) = \arctan\frac{v(h, 0)}{u(h, 0)} = \arctan\left(\frac{\sum\limits_{k=2}^{\infty} v_k(h, 0)h^{k-1}}{1 + \sum\limits_{k=2}^{\infty} u_k(h, 0)h^{k-1}}\right) \tag{4.1.28}$$

is a power series of h *with non-zero convergent radius.*

From (4.1.2) and (4.1.19), we know that

$$\begin{aligned}\omega - \theta &= \arctan\frac{xv(x, y) - yu(x, y)}{xu(x, y) + yv(x, y)} \\ &= \arctan\frac{\sum\limits_{k=2}^{\infty}\left[v_k(x, y)x - u_k(x, y)y\right]}{(u^2 + v^2) + \sum\limits_{k=2}^{\infty}\left[u_k(x, y)x + v_k(x, y)y\right]}.\end{aligned} \tag{4.1.29}$$

Thus, we have

Lemma 4.1.4. *For sufficiently small* h, *when* $x = \tilde{r}(\theta, h)\cos\theta$, $y = \tilde{r}(\theta, h)\sin\theta$, *we have*

$$\begin{aligned}&\omega(\theta, h) - \theta \\ &= \arctan\frac{\sum\limits_{k=2}^{\infty}\left[v_k(\cos\theta, \sin\theta)\cos\theta - u_k(\cos\theta, \sin\theta)\sin\theta\right]\tilde{r}^{k-1}(\theta, h)}{1 + \sum\limits_{k=2}^{\infty}\left[u_k(\cos\theta, \sin\theta)\cos\theta + v_k(\cos\theta, \sin\theta)\sin\theta\right]\tilde{r}^{k-1}(\theta, h)}.\end{aligned} \tag{4.1.30}$$

Theorem 4.1.2. *If the origin of system* (4.1.1) *is a complex center, then, in the sequences* $\{\mathcal{T}_k\}$, *we have*

$$\{\mathcal{T}_{2k-1}\} \sim \{0\}, \quad \{\mathcal{T}_{2k}\} \sim \{\tau_k\}. \tag{4.1.31}$$

Proof. Suppose that the origin of system (4.1.1) is a complex center. We see from (4.1.30) that $\omega(\theta,h)-\theta$ is a 2π periodic function of θ and

$$\omega(2\pi,h)-\omega(0,h)=2\pi. \tag{4.1.32}$$

On the other hand, from (4.1.27), we have

$$\begin{aligned}\omega(2\pi,h)-\omega(0,h)&=\left[1+\frac{1}{2}\sum_{k=1}^{\infty}\tau_k\rho^{2k}(0,h)\right]\mathcal{T}(2\pi,h)\\&=2\pi\left[1+\frac{1}{2}\sum_{k=1}^{\infty}\tau_k\rho^{2k}(0,h)\right]\left(1-\frac{1}{2}\sum_{k=1}^{\infty}\mathcal{T}_k h^k\right).\end{aligned} \tag{4.1.33}$$

(4.1.25), (4.1.32) and (4.1.33) imply that

$$\sum_{k=1}^{\infty}\mathcal{T}_k h^k=\sum_{k=1}^{\infty}\tau_k h^{2k}B_{2k}(h), \tag{4.1.34}$$

where for any positive integer k,

$$B_{2k}(h)=\frac{A^{2k}(h)}{1+\frac{1}{2}\sum_{k=1}^{\infty}\tau_k h^{2k}A^{2k}(h)} \tag{4.1.35}$$

is a power series of h with non-zero convergent radius and $B_{2k}(0)=1$. From (4.1.34) and Theorem 2.2.1, we obtain (4.1.31). Thus, the conclusion of Theorem 4.1.2 holds. □

From Theorem 4.1.1 and Theorem 4.1.2, we obtain

Theorem 4.1.3. *The origin of system* (4.1.12) *is a complex isochronous center if and only if*

$$\{\mu_k\}=\{0\},\quad \{\tau_k\}=\{0\}. \tag{4.1.36}$$

We see from Theorem 4.1.3 and Theorem 1.8.18 that

Theorem 4.1.4. *The origin of system* (4.1.12) *is a complex isochronous center if and only if system* (4.1.12) *is linearizable in a neighborhood of the origin.*

4.2 Linear Recursive Formulas to Compute Period Constants

In this section, we discuss a method to compute τ_k, which was developed by [Liu Y.R. etc, 2003a].

Theorem 4.2.1. *For system* (4.1.12), *one can derive uniquely the formal series*

$$f(z,w) = z + \sum_{k+j=2}^{\infty} c'_{kj} z^k w^j = \sum_{m=1}^{\infty} f_m(z,w),$$
$$g(z,w) = w + \sum_{k+j=2}^{\infty} d'_{kj} w^k z^j = \sum_{m=1}^{\infty} g_m(z,w), \tag{4.2.1}$$

where $f_m(z,w)$, $g_m(z,w)$ *are homogeneous polynomials of degree* m, *and*

$$c'_{j+1,j} = d'_{j+1,j} = 0, \quad j = 1, 2, \cdots, \tag{4.2.2}$$

such that system (4.1.12) *reduces to*

$$\frac{df}{dT} = f + \sum_{k=1}^{\infty} p'_k z^{k+1} w^k,$$
$$\frac{dg}{dT} = -g - \sum_{k=1}^{\infty} q'_k w^{k+1} z^k. \tag{4.2.3}$$

Theorem 4.2.2. *In* (4.2.1) *and* (4.2.3), *when* $k - j - 1 \neq 0$, c'_{kj} *and* d'_{kj} *are determined by the recursive formulas*

$$c'_{kj} = \frac{C_{kj}}{j+1-k}, \quad d'_{kj} = \frac{D_{kj}}{j+1-k}, \tag{4.2.4}$$

and for any positive integer m, p'_m *and* q'_m *are determined by the recursive formulas*

$$p'_m = C_{m+1,m}, \quad q'_m = D_{m+1,m}, \tag{4.2.5}$$

where

$$C_{kj} = \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}]c'_{k-\alpha+1,j-\beta+1},$$
$$D_{kj} = \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}]d'_{k-\alpha+1,j-\beta+1}. \tag{4.2.6}$$

In (4.2.6), *we have taken* $c'_{10} = d'_{10} = 1, c'_{01} = d'_{01} = 0$ *and if* $\alpha < 0$ *or* $\beta < 0$, *we take* $a_{\alpha\beta} = b_{\alpha\beta} = c'_{\alpha\beta} = d'_{\alpha\beta} = 0$.

Proof. From (4.2.3), we have

$$\frac{df}{dT} - f = \sum_{m=2}^{\infty} \left(\frac{\partial f_m}{\partial z} z - \frac{\partial f_m}{\partial w} w - f_m \right) + \sum_{m=2}^{\infty} \Phi_m(z,w),$$
$$\frac{dg}{dT} + g = -\sum_{m=2}^{\infty} \left(\frac{\partial g_m}{\partial w} w - \frac{\partial g_m}{\partial z} z - g_m \right) - \sum_{m=2}^{\infty} \Psi_m(z,w), \tag{4.2.7}$$

where

$$\frac{\partial f_m}{\partial z}z - \frac{\partial f_m}{\partial w}w - f_m = \sum_{k+j=m}(k-j-1)c'_{kj}z^k w^j,$$
$$\frac{\partial g_m}{\partial w}w - \frac{\partial g_m}{\partial z}z - g_m = \sum_{k+j=m}(k-j-1)d'_{kj}w^k z^j \qquad (4.2.8)$$

and

$$\Phi_m(z,w) = \sum_{k=1}^{m-1}\left(\frac{\partial f_k}{\partial z}Z_{m-k+1} - \frac{\partial f_k}{\partial w}W_{m-k+1}\right) = \sum_{k+j=m}C_{kj}z^k w^j,$$
$$\Psi_m(z,w) = \sum_{k=1}^{m-1}\left(\frac{\partial g_k}{\partial w}W_{m-k+1} - \frac{\partial g_k}{\partial z}Z_{m-k+1}\right) = \sum_{k+j=m}D_{kj}w^k z^j. \quad (4.2.9)$$

From (4.2.3) and (4.2.7) $\sim$ (4.2.9), we have

$$\sum_{k+j=2}^{\infty}\left[(k-j-1)c'_{kj} + C_{kj}\right]z^k w^j = \sum_{m=1}^{\infty}p'_m z^{m+1}w^m,$$
$$\sum_{k+j=2}^{\infty}\left[(k-j-1)d'_{kj} + D_{kj}\right]w^k z^j = \sum_{m=1}^{\infty}q'_m w^{m+1}z^m. \qquad (4.2.10)$$

(4.2.10) follows the conclusion of this theorem. □

We next consider the relations between (p_j, q_j) and (p'_j, q'_j).

Theorem 4.2.3. *Let $(p_0, q_0) = (p'_0, q'_0) = (0,0)$. For any positive integer m, if*

$$(p_0, q_0) = (p_1, q_1) = \cdots = (p_{m-1}, q_{m-1}) = (0,0),$$
$$(p'_0, q'_0) = (p'_1, q'_1) = \cdots = (p'_{m-1}, q'_{m-1}) = (0,0), \qquad (4.2.11)$$

then

$$(p'_m, q'_m) = (p_m, q_m). \qquad (4.2.12)$$

Proof. Suppose that there exists a positive integer m, such that (4.2.11) holds. Let

$$\xi^* = z + \sum_{k+j=2}^{2m+1}c_{kj}z^k w^j, \quad \eta^* = w + \sum_{k+j=2}^{2m+1}d_{kj}w^k z^j,$$
$$f^* = z + \sum_{k+j=2}^{2m+1}c'_{kj}z^k w^j, \quad g^* = w + \sum_{k+j=2}^{2m+1}d'_{kj}w^k z^j. \qquad (4.2.13)$$

Then, from (4.1.15), we have

$$\frac{d\xi^*}{dT} = \xi^* + p_m z^{m+1}w^m + h.o.t., \quad \frac{d\eta^*}{dT} = -\eta^* - q_m w^{m+1}z^m + h.o.t. \qquad (4.2.14)$$

and from (4.2.3), we obtain

$$\frac{df^*}{dT} = f^* + p'_m z^{m+1} w^m + h.o.t., \quad \frac{dg^*}{dT} = -g^* - q'_m w^{m+1} z^m + h.o.t. \tag{4.2.15}$$

Because of ξ, η, f and g are all unique, from (4.2.14) and (4.2.15), we have that $\xi^* = f^*$, $\eta^* = g^*$. It follows the conclusion of this theorem. □

Remark 4.2.1. *For system* (4.1.12)*, if* (4.2.11) *holds, then, we have* $p'_m + q'_m = p_m + q_m = \tau_m$, $p'_m - q'_m = p_m - q_m = \mu_m$.

The above three theorems give rise to an algorithm of τ_m. The algorithm is recursive. It can be easily realized by computer algebra systems.

Similar to Theorem 4.2.1, we have

Theorem 4.2.4. *For system* (4.1.12)*, we can derive uniquely the formal series*

$$\tilde{f}(z,w) = z + \sum_{k+j=2}^{\infty} \tilde{c}_{kj} z^k w^j = \sum_{m=1}^{\infty} \tilde{f}_m(z,w),$$
$$\tilde{g}(z,w) = w + \sum_{k+j=2}^{\infty} \tilde{d}_{kj} w^k z^j = \sum_{m=1}^{\infty} \tilde{g}_m(z,w), \tag{4.2.16}$$

where $\tilde{f}_m(z,w)$ *and* $\tilde{g}_m(z,w)$ *are homogeneous polynomials of order* m*,*

$$\tilde{c}_{j+1,j} = \tilde{d}_{j+1,j} = 0, \quad j = 1, 2, \cdots, \tag{4.2.17}$$

such that

$$\frac{d\tilde{f}}{dT} = \tilde{f}\left[1 + \sum_{k=1}^{\infty} \tilde{p}_k (zw)^k\right],$$
$$\frac{d\tilde{g}}{dT} = -\tilde{g}\left[1 + \sum_{k=1}^{\infty} \tilde{q}_k (zw)^k\right]. \tag{4.2.18}$$

Thus, we obtain the following result as Theorem 4.2.2.

Theorem 4.2.5. *In* (4.2.1) *and* (4.2.3)*, when* $k - j - 1 \neq 0$, $\tilde{c}_{kj}$ *and* $\tilde{d}_{kj}$ *are determined by the recursive formulas*

$$\tilde{c}_{kj} = \frac{1}{j+1-k}\left(\tilde{C}_{kj} - \sum_{s=1}^{[(k+j)/2]-1} \tilde{p}_s \tilde{c}_{k-s,j-s}\right),$$
$$7\tilde{d}_{kj} = \frac{1}{j+1-k}\left(\tilde{D}_{kj} - \sum_{s=1}^{(k+j)/2]-1} \tilde{q}_s \tilde{d}_{k-s,j-s}\right), \tag{4.2.19}$$

and for any positive integer m, $\tilde{p}_m$ *and* $\tilde{q}_m$ *are determined by the recursive formulas*

$$\tilde{p}_m = \tilde{C}_{m+1,m} - \sum_{s=1}^{m-1} \tilde{p}_s \tilde{c}_{k-s,j-s},$$
$$\tilde{q}_m = \tilde{D}_{m+1,m} - \sum_{s=1}^{m-1} \tilde{q}_s \tilde{d}_{k-s,j-s}, \tag{4.2.20}$$

where

$$\tilde{C}_{kj} = \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)a_{\alpha,\beta-1} - (j-\beta+1)b_{\beta,\alpha-1}]\tilde{c}_{k-\alpha+1,j-\beta+1},$$
$$\tilde{D}_{kj} = \sum_{\alpha+\beta=3}^{k+j+1} [(k-\alpha+1)b_{\alpha,\beta-1} - (j-\beta+1)a_{\beta,\alpha-1}]\tilde{d}_{k-\alpha+1,j-\beta+1}. \tag{4.2.21}$$

In (4.2.19) $\sim$ (4.2.21)*, we have taken* $\tilde{c}_{10} = \tilde{d}_{10} = 1$, $\tilde{c}_{01} = \tilde{d}_{01} = 0$*, and if* $\alpha < 0$ *or* $\beta < 0$*, we take* $a_{\alpha\beta} = b_{\alpha\beta} = \tilde{c}_{\alpha\beta} = \tilde{d}_{\alpha\beta} = 0$.

For (4.2.18), we write that

$$\tilde{\mu}_k = \tilde{p}_k - \tilde{q}_k, \quad \tilde{\tau}_k = \tilde{p}_k + \tilde{q}_k, \quad k = 1, 2, \cdots. \tag{4.2.22}$$

Theorem 4.2.6. *In* (4.2.22)*, we have*

$$\{\mu_k\} \sim \{\tilde{\mu}_k\}. \tag{4.2.23}$$

Proof. Let $F(z,w) = \tilde{f}(z,w)\tilde{g}(z,w) = zw + h.o.t.$. From (4.2.18), we have

$$\frac{dF}{dT} = \tilde{f}\tilde{g}\sum_{k=1}^{\infty} \tilde{\mu}_k (zw)^k. \tag{4.2.24}$$

(4.2.24) and Theorem 2.3.4 implies the conclusion of this theorem. □

Theorem 4.2.7. *If the origin of system* (4.1.1) *is a complex center, then in* (4.2.22)*, we have*

$$\{\tau_k\} \sim \{\tilde{\tau}_k\}. \tag{4.2.25}$$

Proof. Let

$$\tilde{f} = \tilde{\rho}\cos\varpi, \quad \tilde{g} = \tilde{\rho}\sin\varpi, \tag{4.2.26}$$

then

$$\frac{d\varpi}{dt} = 1 + \frac{1}{2}\sum_{k=1}^{\infty} \tilde{\tau}_k (zw)^k. \tag{4.2.27}$$

Suppose that the origin of system (4.1.1) is a complex center, then when $x = \tilde{r}(\theta, h)\cos\theta, \quad y = \tilde{r}(\theta, h)\sin\theta$, we have $\varpi = \varpi(\theta, h)$ and

$$\varpi(\theta, h) - \theta = \frac{1}{2i} \ln \left(\frac{1 + \sum_{k=2}^{\infty} e^{-i\theta} \tilde{f}_k(e^{i\theta}, e^{-i\theta}) \tilde{r}^{k-1}(\theta, h)}{1 + \sum_{k=2}^{\infty} e^{i\theta} \tilde{g}_k(e^{i\theta}, e^{-i\theta}) \tilde{r}^{k-1}(\theta, h)} \right) \tag{4.2.28}$$

is a 2π periodic function in θ. Thus, we get

$$\varpi(2\pi, h) - \varpi(0, h) = 2\pi. \tag{4.2.29}$$

(4.2.29) and (4.1.27) imply that

$$\begin{aligned} 2\pi = & \int_0^{2\pi} \frac{d\varpi}{d\theta} d\theta = \int_0^{2\pi} \frac{1 + \frac{1}{2}\sum_{k=1}^{\infty} \tilde{\tau}_k \tilde{r}^{2k}(\theta, h)}{1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) \tilde{r}^k(\theta, h)} d\theta \\ = & \mathcal{T}(2\pi, h) + \frac{1}{2} \sum_{k=1}^{\infty} \tilde{\tau}_k \int_0^{2\pi} \frac{\tilde{r}^{2k}(\theta, h) d\theta}{1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) \tilde{r}^k(\theta, h)} \\ = & \mathcal{T}(2\pi, h) + \pi \sum_{k=1}^{\infty} \tilde{\tau}_k h^{2k} \tilde{B}_{2k}(h), \end{aligned} \tag{4.2.30}$$

where for any positive integer k,

$$\tilde{B}_{2k}(h) = \frac{1}{2\pi h^{2k}} \int_0^{2\pi} \frac{\tilde{r}^{2k}(\theta, h) d\theta}{1 + \frac{1}{2}\sum_{k=1}^{\infty} \tilde{\tau}_k \tilde{r}^{2k}(\theta, h)} \tag{4.2.31}$$

is an unit formal power series of h. From (4.1.10), (4.1.34) and (4.2.30), we obtain

$$\sum_{k=1}^{\infty} \tilde{\tau}_k h^{2k} \tilde{B}_{2k}(h) = \sum_{k=1}^{\infty} \tau_k h^{2k} B_{2k}(h). \tag{4.2.32}$$

Hence, by (4.2.32) and Theorem 2.2.1 we complete the proof of this theorem. $\square$

4.3 Isochronous Center for a Class of Quintic System in the Complex Domain

In this section, we investigate an example to show how to apply the method mentioned in Section 4.2. Consider the following complex quintic system:

$$\begin{aligned}\frac{dz}{dT} &= z + a_{30}z^3 + a_{21}z^2w + (a_{12}+3b_{30})zw^2 + a_{03}w^3 + \lambda z^3w^2,\\ \frac{dw}{dT} &= -w - b_{30}w^3 - b_{21}w^2z - (b_{12}+3a_{30})wz^2 - b_{03}z^3 - \lambda w^3z^2. \quad (4.3.1)\end{aligned}$$

First we have the following conclusion, which is similar to Corollary 2.5.2.

Lemma 4.3.1. *System* (4.3.1) *has exactly* 14 *elementary invariants as follows*

$$\begin{aligned}&a_{30}b_{30},\quad a_{12}b_{12},\quad a_{03}b_{03},\quad a_{30}a_{12},\quad b_{30}b_{12},\quad a_{21}\quad b_{21},\quad \lambda.\\ &a_{30}^2a_{03},\quad b_{30}^2b_{03},\quad a_{30}b_{12}a_{03},\quad b_{30}a_{12}b_{03},\quad b_{12}^2a_{03},\quad a_{12}^2b_{03}. \quad (4.3.2)\end{aligned}$$

We next discuss the conditions that the origin of system (4.3.1) is a center. Applying the recursive formulas given by Theorem 2.3.5, we obtain

Theorem 4.3.1. *The first seven singular point values at the origin of* (4.3.1) *are as follows:*

$$\begin{aligned}\mu_1 &= a_{21} - b_{21},\\ \mu_2 &\sim b_{12}b_{30} - a_{12}a_{30},\\ \mu_3 &\sim \frac{1}{8}[3(a_{12}^2b_{03} - b_{12}^2a_{03}) - 10(b_{12}a_{30}a_{03} - a_{12}b_{30}b_{03})],\\ \mu_4 &\sim -\frac{1}{40}(a_{21}+b_{21})(a_{12}^2b_{03} - b_{12}^2a_{03}),\\ \mu_5 &\sim \frac{1}{1500}(25a_{03}b_{03} - a_{12}b_{12} - 150\lambda)(a_{12}^2b_{03} - b_{12}^2a_{03}),\\ \mu_6 &\sim 0,\\ \mu_7 &\sim -\frac{1}{200}a_{12}b_{12}\lambda(a_{12}^2b_{03} - b_{12}^2a_{03}). \quad (4.3.3)\end{aligned}$$

Theorem 4.3.1 follows that

Theorem 4.3.2. *The first seven singular point values of system* (4.3.1) *are all zero if and only if one of the following four conditions are satisfied*:

$$\begin{aligned}&C_1 : a_{21} = b_{21},\ a_{12} = b_{12} = 0;\\ &C_2 : a_{21} = b_{21},\ b_{12}b_{30} = a_{12}a_{30},\ a_{12}^2b_{03} = b_{12}^2a_{03},\ |a_{12}| + |b_{12}| \neq 0;\\ &C_3 : \begin{cases}\lambda = a_{21} = b_{21} = 0,\ 10a_{30} + 3b_{12} = 10b_{30} + 3a_{12} = 0,\\ a_{12}b_{12} = 25a_{03}b_{03}\ |a_{12}| + |b_{12}| \neq 0,\quad b_{12}^2a_{03} - a_{12}^2b_{03} \neq 0;\end{cases}\\ &C_4 : a_{21} = b_{21} = a_{12} = b_{30} = 0,\ 10a_{30} + 3b_{12} = 0,\ a_{03}b_{03} = 6\lambda,\ b_{12}a_{03}b_{03} \neq 0;\\ &C_4^* : a_{21} = b_{21} = b_{12} = a_{30} = 0, 10b_{30} + 3a_{12} = 0, a_{03}b_{03} = 6\lambda,\ a_{12}a_{03}b_{03} \neq 0. \quad (4.3.4)\end{aligned}$$

4.3.1 The Conditions of Isochronous Center Under Condition C_1

Suppose that Condition C_1 in Theorem 4.3.2 is satisfied. Then, there exists a constant s, such that $a_{21} = b_{21} = s$. Thus, system (4.3.1) becomes

$$\begin{aligned}\frac{dz}{dT} &= z + a_{30}z^3 + sz^2w + 3b_{30}zw^2 + a_{03}w^3 + \lambda z^3w^2,\\ \frac{dw}{dT} &= -w - b_{30}w^3 - sw^2z - 3a_{30}wz^2 - b_{03}z^3 - \lambda w^3z^2.\end{aligned} \tag{4.3.5}$$

Applying recursive formulas given by Theorem 4.2.2 to do computations, we obtain

Theorem 4.3.3. *For the origin of system* (4.3.5), *the first six complex period constants are as follows:*

$$\begin{aligned}\tau_1 &= 2s,\\ \tau_2 &\sim \frac{1}{2}(-3a_{03}b_{03} - 48a_{30}b_{30} + 4\lambda),\\ \tau_3 &\sim 30(a_{03}b_{30}^2 + b_{03}b_{30}^2),\\ \tau_4 &\sim \frac{15}{32}(3a_{03}^2b_{03}^2 - 128a_{03}b_{03}a_{30}b_{30} + 768a_{30}^2b_{30}^2),\\ \tau_5 &\sim 0,\\ \tau_6 &\sim 7a_{30}^2b_{30}^2(1111a_{03}b_{03} - 2688a_{30}b_{30}).\end{aligned} \tag{4.3.6}$$

Moreover, the first six complex period constants are all zero if and only if one of the following two conditions holds:

$$\begin{aligned}C_{11}:&\ \ s = \lambda = a_{30} = b_{03} = 0;\\ C_{11}^*:&\ \ s = \lambda = b_{30} = a_{03} = 0.\end{aligned} \tag{4.3.7}$$

Theorem 4.3.4. *If Condition C_{11} or C_{11}^* holds, then the origin of system* (4.3.5) *is a complex isochronous center.*

Proof. If Condition C_{11} is satisfied, then system (4.3.5) becomes

$$\frac{dz}{dT} = z + 3b_{30}zw^2 + a_{03}w^3, \quad \frac{dw}{dT} = -w(1 + b_{30}w^2). \tag{4.3.8}$$

System (4.3.8) is linearizable by using the transformation

$$\xi = (z + b_{30}zw^2 + \frac{1}{4}a_{03}w^3)\sqrt{1 + b_{30}w^2}, \quad \eta = \frac{w}{\sqrt{1 + b_{30}w^2}}\ . \tag{4.3.9}$$

Thus, the origin of system (4.3.9) is a complex isochronous center.

If Condition C_{11}^* is satisfied, then by using the same method as the above, we know that the origin of system (4.3.5) is also a complex isochronous center. □

4.3.2 The Conditions of Isochronous Center Under Condition C_2

If Condition C_2 in Theorem 4.3.2 holds, then there exist constants p, q, s, such that

$$a_{21} = b_{21} = s, \quad a_{30} = p\, b_{12}, \quad b_{30} = p\, a_{12}, \quad a_{03} = q\, a_{12}^2, \quad b_{03} = q\, b_{12}^2. \tag{4.3.10}$$

Thus, system (4.3.1) becomes

$$\begin{aligned} \frac{dz}{dT} &= z + pb_{12}z^3 + sz^2w + (1+3p)a_{12}zw^2 + q\, a_{12}^2 w^3 + \lambda z^3 w^2, \\ \frac{dw}{dT} &= -w - pa_{12}w^3 - sw^2 z - (1+3p)b_{12}wz^2 - q\, b_{12}^2 z^3 - \lambda w^3 z^2. \end{aligned} \tag{4.3.11}$$

Theorem 4.3.5. *The first six complex period constants of the origin of system* (4.3.11) *are as follows*:

$$\begin{aligned} \tau_1 &= 2s, \\ \tau_2 &\sim \frac{1}{2}[4\lambda - a_{12}b_{12}(4 + 28p + 48p^2 + 3a_{12}b_{12}q^2))], \\ \tau_3 &\sim \frac{1}{4}a_{12}^2 b_{12}^2 (3+10p)(7+24p)q, \\ \tau_4 &\sim \frac{1}{96}a_{12}^2 b_{12}^2 [192(1+3p)^2(1+4p)(2+5p) \\ &\quad + 8a_{12}b_{12}(13+42p)q^2 + 135a_{12}^2 b_{12}^2 q^4], \\ \tau_5 &\sim -\frac{1}{50400}a_{12}^3 b_{12}^3 (1559 + 5178p)q(63 + 226p + 105a_{12}b_{12}q^2), \\ \tau_6 &\sim \frac{1}{8100}a_{12}^3 b_{12}^3 [-12(37019 + 310766p + 858060p^2 + 781200p^3) \\ &\quad + a_{12}b_{12}(1006867 + 3466140p)q^2)](1+3p)^2(1+4p). \end{aligned} \tag{4.3.12}$$

In addition, the first six complex period constants are all zero if and only if one of the following five conditions holds:

$$\begin{aligned} &C_{21}: \ s = 0, \ r = 0, \ q = 0, p = -\frac{1}{3}; \\ &C_{22}: \ s = 0, \ r = 0, \ q = 0, \ p = -\frac{1}{4}; \\ &C_{23}: \ s = 0, \ r = 0, \ p = -\frac{7}{24}, \ 36a_{12}b_{12}q^2 - 1 = 0; \\ &C_{24}: \ \ s = 0, \ r = 0, \ a_{12} = 0; \\ &C_{24}^*: \ \ s = 0, \ r = 0, \ b_{12} = 0. \end{aligned} \tag{4.3.13}$$

Proposition 4.3.1. *If Condition C_{21} holds, then the origin of system* (4.3.11) *is a complex isochronous center.*

Proof. When Condition C_{21} holds, system (4.3.11) becomes

$$\frac{dz}{dT} = z - \frac{1}{3}b_{12}z^3, \quad \frac{dw}{dT} = -w + \frac{1}{3}a_{12}w^3. \tag{4.3.14}$$

This system is linearizable by using the transformation

$$\xi = \frac{z}{\sqrt{1 - \frac{1}{3}b_{12}z^2}}, \quad \eta = \frac{w}{\sqrt{1 - \frac{1}{3}a_{12}w^2}}. \tag{4.3.15}$$

It follows the conclusion of this proposition. □

Proposition 4.3.2. *If Condition C_{22} holds, then the origin of system* (4.3.11) *is a complex isochronous center.*

Proof. When Condition C_{22} holds, system (4.3.11) becomes

$$\frac{dz}{dT} = \frac{1}{4}z(4 - b_{12}z^2 + a_{12}w^2), \quad \frac{dw}{dT} = -\frac{1}{4}w(4 - a_{12}z^2 + b_{12}w^2). \tag{4.3.16}$$

Let

$$z = re^{i\theta}, \quad w = re^{-i\theta}, \quad T = it, \tag{4.3.17}$$

then $\theta = \frac{1}{2i}(ln\, z - ln\, w)$, we have $\frac{d\theta}{dt} = i\frac{d\theta}{dT} \equiv 1$. It implies the conclusion of this proposition. □

Proposition 4.3.3. *If Condition C_{23} holds, then the origin of system* (4.3.11) *is a complex isochronous center.*

Proof. When Condition C_{23} holds, system (4.3.11) becomes

$$\begin{aligned}\frac{dz}{dT} &= z - \frac{7b_{12}z^3}{24} + \frac{zw^2}{288b_{12}q^2} + \frac{w^3}{1296b_{12}^2q^3},\\ \frac{dw}{dT} &= -w - b_{12}^2qz^3 - \frac{b_{12}wz^2}{8} + \frac{7w^3}{864b_{12}q^2}.\end{aligned} \tag{4.3.18}$$

System (4.3.18) is linearizable by using the transformation

$$\xi = \frac{f_1}{\sqrt{f_3}}, \quad \eta = \frac{f_1}{\sqrt{f_3}}, \tag{4.3.19}$$

where

$$\begin{aligned}f_1 &= z - \frac{b_{12}z^3}{24} + \frac{z^2w}{48q} - \frac{zw^2}{288b_{12}q^2} + \frac{w^3}{5184b_{12}^2q^3},\\ f_2 &= w + \frac{b_{12}^2qz^3}{4} - \frac{b_{12}z^2w}{8} + \frac{zw^2}{48q} - \frac{w^3}{864b_{12}q^2},\\ f_3 &= 1 - \frac{3b_{12}z^2}{8} - \frac{zw}{16q} - \frac{w^2}{96b_{12}q^2}.\end{aligned} \tag{4.3.20}$$

It gives the conclusion of this proposition. □

Proposition 4.3.4. *If Condition C_{24} holds, then the origin of system* (4.3.11) *is a complex isochronous center.*

Proof. When Condition C_{24} holds, system (4.3.11) becomes

$$\frac{dz}{dT}=z(1+b_{12}pz^2),\quad \frac{dw}{dT}=-w-b_{12}^2qz^3-3b_{12}pz^2w-b_{12}z^2w. \tag{4.3.21}$$

System (4.3.21) is linearizable by using the transformation

$$\xi=\frac{z}{\sqrt{1+b_{12}z^2}},\quad \eta=f_4f_5, \tag{4.3.22}$$

where

$$f_4=\begin{cases}(1+b_{12}pz^2)^{\frac{1+3p}{2p}}, & \text{if } p\neq 0,\\ e^{\frac{b_{12}z^2}{2}}, & \text{if } p=0,\end{cases}$$

$$f_5=w-\frac{qg(z)}{z} \tag{4.3.23}$$

and

$$g(z)=\begin{cases}2-b_{12}z^2-2e^{\frac{-b_{12}z^2}{2}}, & \text{if } p=0;\\ b_{12}z^2+2\ln\left(1-\dfrac{b_{12}z^2}{2}\right), & \text{if } p=-\dfrac{1}{2};\\ -2b_{12}z^2-2(4-b_{12}z^2)\ln\left(1-\dfrac{b_{12}z^2}{4}\right), & \text{if } p=-\dfrac{1}{4};\\ \dfrac{2-b_{12}(1+2p)z^2-2(1+b_{12}pz^2)^{\frac{-1+2p}{2p}}}{(1+2p)(1+4p)}, & \text{others.}\end{cases} \tag{4.3.24}$$

It follows the conclusion of this proposition. □

Remark 4.3.1. *From* (4.3.24), *we have* $g(0)=g'(0)=g''(0)=0$.

Similar to Proposition 4.3.4, we have

Proposition 4.3.5. *If Condition C_{24}^* holds, then the origin of system* (4.3.11) *is a complex isochronous center.*

Proposition 4.3.1 ∼ Proposition 4.3.5 imply that

Theorem 4.3.6. *The origin of system* (4.3.11) *is a complex isochronous center if and only if one of condition in* (4.3.13) *holds.*

4.3.3 The Conditions of Isochronous Center Under Condition C_3

If Condition C_3 in Theorem 4.3.2 holds, then system (4.3.1) becomes

$$\begin{aligned}\frac{dz}{dT} &= z - \frac{3}{10}b_{12}z^3 + \frac{1}{10}a_{12}zw^2 + a_{03}w^3,\\ \frac{dw}{dT} &= -w - b_{03}z^3 - \frac{1}{10}b_{12}z^2w + \frac{3}{10}a_{12}w^3,\end{aligned} \tag{4.3.25}$$

where

$$b_{12}^2a_{03} - a_{12}^2b_{03} = 0, \quad a_{12}b_{12} = 25a_{03}b_{03}, \quad |a_{12}| + |b_{12}| \neq 0. \tag{4.3.26}$$

We have

Theorem 4.3.7. *The first two complex period constants of the origin of system* (4.3.25) *are as follows:*

$$\tau_1 = 0, \quad \tau_2 = -\frac{a_{12}b_{12}}{50}. \tag{4.3.27}$$

In addition, the first two complex period constants are all zero if and only if one of the following two conditions are satisfied:

$$\begin{aligned}&C_{31}: \quad a_{12} = 0, \quad b_{03} = 0,\\ &C_{31}^*: \quad b_{12} = 0, \quad a_{03} = 0.\end{aligned} \tag{4.3.28}$$

Theorem 4.3.8. *If Condition C_{31} or C_{31}^* holds, then the origin of system* (4.3.25) *is a complex isochronous center.*

Proof. When Condition C_{31} holds, system (4.3.25) becomes

$$\frac{dz}{dT} = z - \frac{3}{10}b_{12}z^3 + a_{03}w^3, \quad \frac{dw}{dT} = -w\left(1 + \frac{1}{10}b_{12}z^2\right). \tag{4.3.29}$$

System (4.3.29) is linearizable by using the transformation

$$\xi = f_6 f_7^{\frac{-1}{2}}, \quad \eta = w f_7^{\frac{-1}{6}}, \tag{4.3.30}$$

where

$$\begin{aligned}f_6 &= z + \frac{1}{4}a_{03}w^3,\\ f_7 &= 1 - \frac{3}{10}b_{12}z^2 - \frac{3}{10}a_{03}b_{12}zw^3 - \frac{1}{20}a_{03}^2b_{12}w^6.\end{aligned} \tag{4.3.31}$$

Thus, the origin of system (4.3.29) is a complex isochronous center.

Similarly, if Condition C_{31}^* holds, the origin of system (4.3.25) is also a complex isochronous center. □

4.3.4 Non-Isochronous Center under Condition C_4 and C_4^*

If Condition C_4 or C_4^* in Theorem 4.3.2 holds, then for system (4.3.1),we have

$$\tau_1 = 0, \quad \tau_2 = -\frac{7}{6}a_{03}b_{03} \neq 0. \tag{4.3.32}$$

Thus,we have

Theorem 4.3.9. *If Condition C_4 or C_4^* in Theorem* 4.3.2 *holds, then the origin is not an isochronous center.*

Thus, the problem of the complex isochronous center for the system is completely solved in this section.

Remark 4.3.2. *On the basis of the conclusions in this section, for system* (4.3.1)*, if $\lambda \neq 0$, then the origin is not an isochronous center.*

Remark 4.3.3. *System* $(4.3.1)_{\lambda=0}$ *is a cubic systems having homogeneous nonlinearities. For the origin of system* $(4.3.1)_{\lambda=0}$*, the problem of the complex isochronous center is correspondencely solved in this section.*

4.4 The Method of Time-Angle Difference

In this section, we introduce another method to characterize isochronous centers of system (4.1.1) and system (4.1.12) (see[Liu, Li, 2006]).

A center of an analytic system is isochronous if and only if there exists an analytic change of coordinates such that the original system is reduced to a linear system. Clearly, such a change of variables needs to determine two functions of two variables. We now give another new method to characterize isochronous centers of polynomial systems. Unlike the above method of the linearized system, this method only needs to determine a function which is called the function of the time-angle difference. In addition, other two algorithms to compute period constants τ_k are also given.

Theorem 4.4.1. *For system* (4.1.12)*, one can derive successively the terms of the following formal series*

$$G(z,w) = \sum_{k=1}^{\infty} \frac{g_{3k}(z,w)}{(zw)^k}, \tag{4.4.1}$$

where, for any positive integer k,

$$g_{3k}(z,w) = \sum_{\alpha+\beta=3k} C_{\alpha\beta} z^\alpha w^\beta \tag{4.4.2}$$

is a homogeneous polynomial of degree $3k$ *in* z, w, C_{kk} *can take any constant, such that*

$$\frac{dG}{dT} + i + \frac{d\theta}{dT} = \frac{1}{2i}\sum_{m=1}^{\infty} \tau_m'(zw)^m, \tag{4.4.3}$$

where

$$\theta = \frac{1}{2i}\ln\frac{z}{w}, \qquad \frac{d\theta}{dT} = \frac{wZ + zW}{2izw}. \tag{4.4.4}$$

Theorem 4.4.2. *In Theorem* 4.4.1, *let* $C_{0,0} = 0$, *for any positive integer* m, *when* $\alpha + \beta = 3m$ *and* $\alpha \neq \beta$, $C_{\alpha\beta}$ *is given by the following recursive formula*

$$C_{\alpha\beta} = \frac{1}{3(\beta - \alpha)}\Delta_{\alpha\beta}. \tag{4.4.5}$$

In addition, τ_m' *is given by the following recursive formula*

$$\begin{aligned}\tau_m' =& a_{m+1,m} + b_{m+1,m}\\ &+2i\sum_{k+j=3}^{2m+1}[(m-k+1)a_{k,j-1}-(m-j+1)b_{j,k-1}]C_{3m-2k-j+3,3m-2j-k+3},\end{aligned} \tag{4.4.6}$$

where

$$\begin{aligned}\Delta_{\alpha\beta} =& \frac{-3i}{2}(a_{\alpha-m+1,\beta-m} + b_{\beta-m+1,\alpha-m})\\ &+\sum_{k+j=3}^{m+1}[(2\alpha-\beta-3k+3)a_{k,j-1}-(2\beta-\alpha-3j+3)b_{j,k-1}]\\ &\times_{\alpha-2k-j+3,\beta-2j-k+3}.\end{aligned} \tag{4.4.7}$$

In the above two formulas, if $\alpha < 0$ *or* $\beta < 0$, *then we define that* $a_{\alpha\beta} = b_{\alpha\beta} = C_{\alpha\beta} = 0$.

Proof. Notice that $\theta = \dfrac{1}{2i}\ln\dfrac{z}{w}$. We have

$$\begin{aligned}\frac{dG}{dT} + i + \frac{d\theta}{dT} &= \sum_{m=1}^{\infty}(zw)^{-m}\left[\frac{\partial g_{3m}}{\partial z}z - \frac{\partial g_{3m}}{\partial w}w + H_{3m}\right]\\ &= \sum_{m=1}^{\infty}(zw)^{-m}\left[\sum_{\alpha+\beta=3m}(\alpha-\beta)C_{\alpha\beta}z^{\alpha}w^{\beta} + H_{3m}\right],\end{aligned} \tag{4.4.8}$$

where for any positive integer m,

$$\begin{aligned}H_{3m} = &\frac{-i}{2}(zw)^{m-1}(wZ_{m+1} + zW_{m+1})\\ &+ \sum_{k=1}^{m-1}(zw)^{k-1}\left[\frac{\partial g_{3m-3k}}{\partial z}z - (m-k)g_{3m-3k}\right]wZ_{k+1}\\ &- \sum_{k=1}^{m-1}(zw)^{k-1}\left[\frac{\partial g_{3m-3k}}{\partial w}w - (m-k)g_{3m-3k}\right]zW_{k+1}\\ = &\frac{1}{3}\sum_{\alpha+\beta=3m}\Delta_{\alpha\beta}z^{\alpha}w^{\beta}. \end{aligned} \tag{4.4.9}$$

From (4.4.8), (4.4.9), and (4.4.3), the conclusion of this theorem holds. □

Corollary 4.4.1. *For system* (4.1.1), *one can derive successively the terms of the following formal series*

$$G^*(x,y) = G(x+iy, x-iy) = \sum_{k=1}^{\infty}\frac{g_{3k}(x+iy, x-iy)}{(x^2+y^2)^k}, \tag{4.4.10}$$

such that

$$\frac{dG^*}{dt} - 1 + \frac{d\theta}{dt} = \frac{1}{2}\sum_{m=1}^{\infty}\tau'_m(x^2+y^2)^m. \tag{4.4.11}$$

Corollary 4.4.2. *For system* (4.1.3), *one can derive successively the terms of the following formal series*

$$\mathcal{G}(r,\theta) = G(re^{i\theta}, re^{-i\theta}) = \sum_{k=1}^{\infty}g_{3k}(e^{i\theta}, e^{-i\theta})r^{3k}, \tag{4.4.12}$$

such that

$$\frac{d\mathcal{G}}{dt} - 1 + \frac{d\theta}{dt} = \frac{1}{2}\sum_{m=1}^{\infty}\tau'_m r^{2m}. \tag{4.4.13}$$

Theorem 4.4.3. *If the origin of* (4.1.12) *is a complex center, then*

$$\{\tau'_m\} \sim \{\tau_m\}. \tag{4.4.14}$$

Proof. From (4.4.13) and (4.1.3), we have

$$\frac{dt}{d\theta} = \frac{d\mathcal{G}(\tilde{r}(\theta,h),\theta)}{d\theta} + 1 - \frac{\frac{1}{2}\sum_{k=1}^{\infty}\tau'_k\tilde{r}^{2k}(\theta,h)}{1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta)\tilde{r}^k(\theta,h)}. \tag{4.4.15}$$

Because the origin of (4.1.12) is a complex center, so $\mathcal{G}(\tilde{r}(\theta,h),\theta)$ is a 2π periodic function in θ. Integrating the two sides of (4.4.15) from 0 to 2π, it follows

$$T(2\pi,h)=2\pi-\pi\sum_{k=1}^{\infty}\tau'_k h^{2k}\tilde{B}_{2k}(h), \tag{4.4.16}$$

where for any positive integer k, $\tilde{B}_{2k}(h)$ is given in (4.2.31). From (4.1.10), (4.1.34) and (4.1.16) we have

$$\sum_{k=1}^{\infty}\tau_k h^{2k}B_{2k}(h)=\sum_{k=1}^{\infty}\tau'_k h^{2k}\tilde{B}_{2k}(h). \tag{4.4.17}$$

(4.4.17) and Theorem 2.2.1 imply the result of Theorem 4.4.3. □

Theorem 4.4.1 and Theorem 4.4.3 give an algorithm to compute period constants τ_k.

Definition 4.4.1. (1) *For system* (4.1.12)*, if there exists a formal series* $G(z,w)$ *having the form* (4.4.1)*, such that*

$$\frac{dG}{dT}+i+\frac{d\theta}{dT}=0, \tag{4.4.18}$$

then $G(z,w)$ *is called a function of the time-angle difference in a neighborhood of the origin;*

(2) *For system* (4.1.1)*, if there exists a formal series* $G^*(x,y)$ *having the form* (4.4.10)*, such that*

$$\frac{dG^*}{dt}-1+\frac{d\theta}{dt}=0, \tag{4.4.19}$$

then $G^*(x,y)$ *is called a function of the time-angle difference in a neighborhood of the origin.*

For system (4.1.12), we notice that the function of the time-angle difference is not unique. In fact, if $G(z,w)$ is a function of the time-angle difference and $F(z,w)$ is a first integral satisfying $F(0,0)=0$, then $G(z,w)+F(z,w)$ is also a function of the time-angle difference . In addition, if $G_1(z,w)$ and $G_2(z,w)$ are two functions of the time-angle difference and $G_1(z,w)-G_2(z,w)$ is not a constant, then $G_1(z,w)-G_2(z,w)$ is a formal first integral.

From Theorem 4.4.1 and 4.4.3, we have

Theorem 4.4.4. (***Theorem of time-angle difference***) *For system* (4.1.1) *and* (4.1.12)*, if the origin is a complex center, then the origin is a complex isochronous center, if and only if there exists a function of the time-angle difference in a neighborhood of the origin.*

This theorem tell us that if the origin of (4.1.1) is a complex isochronous center, then, when $|h| \ll 1$, we have formally

$$t - \theta = G^*(\tilde{r}(\theta, h)\cos\theta,\ \tilde{r}(\theta, h)\sin\theta) - G^*(h, 0). \tag{4.4.20}$$

Now we discuss some properties of the function of the time-angle difference of system (4.1.12). Suppose that ξ and η are given by (4.1.15) satisfying (4.1.14). We consider the function

$$\tilde{G}(z, w) = \frac{1}{2i}\left[\ln\frac{\xi(z, w)}{z} - \ln\frac{\eta(z, w)}{w}\right]. \tag{4.4.21}$$

Letting $z = u^2v, \quad w = uv^2$, then, from (4.4.15) we obtain

$$\begin{aligned} \frac{\xi(z, w)}{z} &= 1 + \sum_{k=2}^{\infty} u^{k-2}v^{k-1}\xi_k(u, v), \\ \frac{\eta(z, w)}{w} &= 1 + \sum_{k=2}^{\infty} u^{k-1}v^{k-2}\eta_k(u, v), \end{aligned} \tag{4.4.22}$$

where for any positive integer $k > 1$, $u^{k-2}v^{k-1}\xi_k(u, v), \quad u^{k-1}v^{k-2}\eta_k(u, v)$ are two homogeneous polynomials of degree $3(k-1)$ in u, v. Hence, $\tilde{G}(u^2v, uv^2)$ is a power formal series having the form

$$\tilde{G}(u^2v, uv^2) = \sum_{k=1}^{\infty} \tilde{f}_{3k}(u, v), \tag{4.4.23}$$

where for any positive integer k, $\tilde{f}_{3k}(u, v)$ is homogeneous polynomials of degree $3k$ in u, v. So that, we have

$$\tilde{G}(z, w) = \sum_{k=1}^{\infty} \frac{\tilde{f}_{3k}(z, w)}{(zw)^k}. \tag{4.4.24}$$

Theorem 4.4.5. *For the function $\tilde{G}(z, w)$ defined by* (4.4.21)*, we have*

$$\frac{d\tilde{G}(z, w)}{dT} + i + \frac{d\theta}{dT} = \frac{1}{2i}\sum_{k=1}^{\infty} \tau_k(\xi\eta)^k. \tag{4.4.25}$$

Proof. We see from (4.4.21) that

$$\tilde{G}(z, w) = \frac{1}{2i}[\ln\xi - \ln\eta] - \theta. \tag{4.4.26}$$

By using (4.1.15) and (4.4.26), we obtain (4.4.25). □

Theorem 4.4.6. *Suppose that the origin is a complex isochronous center of* (4.1.12). *Then,* $\tilde{G}(z,w)$ *is a function of the time-angle difference of* (4.1.12) *in a neighborhood of the origin. As a series of u and v, the convergence radius of* $\tilde{G}(u^2v, uv^2)$ *is not zero.*

Specially, if $z = 0$ and $w = 0$ are two complex straight line solutions of system (4.1.12), then (4.1.12) has the form

$$\frac{dz}{dT} = zP(z,w), \quad \frac{dw}{dT} = -wQ(z,w), \tag{4.4.27}$$

where

$$P(z,w) = 1 + \sum_{k+j=1}^{\infty} a'_{kj} z^k w^j, \quad Q(z,w) = 1 + \sum_{k+j=1}^{\infty} b'_{kj} w^k z^j. \tag{4.4.28}$$

Theorem 4.4.7. *For system* (4.4.27), *one can derive successively the terms of the following formal series*

$$G(z,w) = \sum_{k=1}^{\infty} g_k(z,w), \tag{4.4.29}$$

where for any positive integer k,

$$g_k(z,w) = \sum_{\alpha+\beta=k} C_{\alpha\beta} z^\alpha w^\beta \tag{4.4.30}$$

is a homogeneous polynomial of degree k in z, w, C_{kk} can take any constant, such that

$$\frac{dG}{dT} + i + \frac{d\theta}{dT} = \frac{1}{2i} \sum_{m=1}^{\infty} \tau'_m (zw)^m. \tag{4.4.31}$$

Theorem 4.4.8. *In Theorem* 4.4.7, *denote $C_{00} = 0$, then for all $\alpha \neq \beta$, $C_{\alpha\beta}$ is given by the following recursive formula*

$$C_{\alpha\beta} = \frac{1}{\beta-\alpha} \left\{ \frac{a'_{\alpha\beta} + b'_{\beta\alpha}}{2i} + \sum_{k+j=1}^{\alpha+\beta-1} \left[(\alpha-k)a'_{kj} - (\beta-j)b'_{jk} \right] C_{\alpha-k,\beta-j} \right\}. \tag{4.4.32}$$

For any integer $m > 0$, τ'_m is given by the following recursive formula

$$\tau'_m = a'_{mm} + b'_{mm} + 2i \sum_{k+j=1}^{2m-1} \left[(m-k)a'_{kj} - (m-j)b'_{jk} \right] C_{m-k,m-j}. \tag{4.4.33}$$

In the above two formulas, if $\alpha < 0$ or $\beta < 0$, then we define that $a_{\alpha\beta} = b_{\alpha\beta} = C_{\alpha\beta} = 0$.

Theorem 4.4.8 gives an algorithm to calculate complex period constants of the origin of system (4.4.27). In this case, $G(z,w)$ is a formal power series of z, w.

4.5 The Conditions of Isochronous Center of the Origin for a Cubic System

In this section, as an application of the method of Time-Angle Difference, we consider the following real cubic system

$$\begin{aligned}\frac{dx}{dt} &= -y - 2a_1xy - (3a_4 + a_5 - a_6)x^2y + (a_4 - a_5 + a_6)y^3,\\ \frac{dy}{dt} &= x+(a_1 + a_2)x^2 - (a_1 - a_2)y^2 + (a_4 + a_5 + a_6)x^3 - (3a_4 - a_5 - a_6)xy^2\end{aligned} \tag{4.5.1}$$

and its associated system

$$\begin{aligned}\frac{dz}{dT} &= z + a_1z^2 + a_2zw + a_4z^3 + a_5z^2w + a_6zw^2,\\ \frac{dw}{dT} &= -w - a_1w^2 - a_2wz - a_4w^3 - a_5w^2z - a_6wz^2.\end{aligned} \tag{4.5.2}$$

Using Theorem 4.4.8 and computer algebra systems, we have the following result.

Theorem 4.5.1. *The first 5 period constants of the origin of* (4.5.2) *are as follows:*

$$\begin{aligned}\tau_1 &= 2(a_5 - a_1a_2 - a_2^2),\\ \tau_2 &\sim 2[(3a_2^2 - a_5 - a_6)(a_4 + a_5 + a_6) + 3a_5a_6],\\ \tau_3 &\sim \frac{1}{3}(a_4 + a_5 + a_6)f_3,\\ \tau_4 &\sim \frac{1}{30}(a_4 + a_5 + a_6)f_4,\\ \tau_5 &\sim \frac{1}{402969600000}a_5(a_5 - 2a_6)(a_4 + a_5 + a_6)f_5,\end{aligned} \tag{4.5.3}$$

where

$$\begin{aligned}f_3 &= -20a_2^2a_5 - 80a_2^2a_6 + 3a_4a_5 + 7a_5^2 + 6a_4a_6 + 9a_5a_6 + 2a_6^2,\\ f_4 &= 36a_4a_5^2 + 4a_5^3 - 77a_4a_5a_6 - 58a_5^2a_6 - 98a_4a_6^2 + 247a_5a_6^2 - 186a_6^3,\\ f_5 &= 1371835883790a_4^2 - 57411367448a_4a_5 - 16682041862a_5^2\\ &\quad -3793858253681a_4a_6 + 1649355227996a_5a_6 - 989885411687a_6^2.\end{aligned} \tag{4.5.4}$$

From Theorem 4.5.1, we have

Theorem 4.5.2. *The first 5 period constants of the origin of* (4.5.2) *are all zero, if and only if one of the following 5 conditions holds:*

$$\begin{aligned}&C_1 : a_2 = a_5 = a_6 = 0,\\ &C_2 : a_2 = a_5 = a_4 + a_6 = 0,\\ &C_3 : a_1 + a_2 = a_5 = a_4 + a_6 = 0,\\ &C_4 : a_1 - 3a_2 = a_4 - 2a_2^2 = a_5 - 4a_2^2 = a_6 - 2a_2^2 = 0,\\ &C_5 : a_4 + a_2(a_1 + a_2) = a_5 - a_2(a_1 + a_2) = a_6 = 0.\end{aligned} \tag{4.5.5}$$

We next consider respectively 5 conditions $C_1 \sim C_5$.

When Condition C_1 holds, system (4.5.2) becomes

$$\frac{dz}{dT} = z(1 + a_1 z + a_4 z^2), \quad \frac{dw}{dT} = -w(1 + a_1 w + a_4 w^2). \tag{4.5.6}$$

Proposition 4.5.1. *For this system, there exists a function of the time-angle difference*

$$t - \theta = \frac{i}{2}\left[\int_0^z \frac{a_1 + a_4 z}{1 + a_1 z + a_4 z^2} dz - \int_0^w \frac{a_1 + a_4 w}{1 + a_1 w + a_4 w^2} dw\right]. \tag{4.5.7}$$

When Condition C_2 holds, system (4.5.2) becomes

$$\begin{aligned} \frac{dz}{dT} &= z(1 + a_1 z + a_4 z^2 - a_4 w^2), \\ \frac{dw}{dT} &= -w(1 + a_1 w + a_4 w^2 - a_4 z^2). \end{aligned} \tag{4.5.8}$$

The associated system of (4.5.8) is that

$$\begin{aligned} \frac{dx}{dt} &= -y(1 + 2a_1 x + 4a_4 x^2), \\ \frac{dy}{dt} &= x + a_1 x^2 - a_1 y^2 - 4a_4 x y^2. \end{aligned} \tag{4.5.9}$$

We have

Proposition 4.5.2. *For system* (4.5.9), *there exists a function of the time-angle difference*

$$t - \theta = \begin{cases} \dfrac{a_1}{2\sqrt{4a_4 - a_1^2}} \ln\left(\dfrac{1 + a_1 x - \sqrt{4a_4 - a_1^2}\, y}{1 + a_1 x + \sqrt{4a_4 - a_1^2}\, y}\right), & if \ \ a_1^2 - 4a_4 < 0, \\ \dfrac{-a_1 y}{1 + a_1 x}, \quad if \ a_1^2 - 4a_4 = 0, \\ \dfrac{-a_1}{\sqrt{a_1^2 - 4a_4}} \arctan\left(\dfrac{\sqrt{a_1^2 - 4a_4}\, y}{1 + a_1 x}\right), & if \ a_1^2 - 4a_4 > 0. \end{cases} \tag{4.5.10}$$

When Condition C_3 holds, system (4.5.2) becomes

$$\begin{aligned} \frac{dz}{dT} &= z(1 + a_1 z - a_1 w + a_4 z^2 - a_4 w^2), \\ \frac{dw}{dT} &= -w(1 + a_1 w - a_1 z + a_4 w^2 - a_4 z^2). \end{aligned} \tag{4.5.11}$$

Proposition 4.5.3. *For system* (4.5.11), *there exists a function of the time-angle difference*

$$t - \theta = 0. \tag{4.5.12}$$

When Condition C_4 holds, system (4.5.2) becomes

$$\begin{aligned}\frac{dz}{dT} &= z(1+3a_2z+a_2w+2a_2^2z^2+4a_2^2zw+2a_2^2w^2),\\ \frac{dw}{dT} &= -w(1+3a_2w+a_2z+2a_2^2w^2+4a_2^2wz+2a_2z^2).\end{aligned} \tag{4.5.13}$$

The associated system of (4.5.13) is

$$\begin{aligned}\frac{dx}{dt} &= -y(1+2a_2x)(1+4a_2x),\\ \frac{dy}{dt} &= x+4a_2x^2-2a_2y^2+8a_2^2x^3.\end{aligned} \tag{4.5.14}$$

We obtain

Proposition 4.5.4. *For system* (4.5.14), *there exists a function of the time-angle difference*

$$t-\theta = -\arctan\frac{4a_2(1+2a_2x)y}{1+4a_2x+8a_2^2x^2}. \tag{4.5.15}$$

When Condition C_5 holds, system (4.5.2) becomes

$$\begin{aligned}\frac{dz}{dT} &= z[1+(a_1+a_2)z][1-a_2z+a_2w],\\ \frac{dw}{dT} &= -w[1+(a_1+a_2)w][1-a_2w+a_2z].\end{aligned} \tag{4.5.16}$$

The associated system of (4.5.16) is

$$\begin{aligned}\frac{dx}{dt} &= -y[1+2a_1x-2a_2(a_1+a_2)x^2+2a_2(a_1+a_2)y^2],\\ \frac{dy}{dt} &= x+(a_1+a_2)x^2-(a_1-a_2)y^2+4a_2(a_1+a_2)xy^2.\end{aligned} \tag{4.5.17}$$

We have

Proposition 4.5.5. *For system* (4.5.17), *there exists a function of the time-angle difference*

$$t-\theta = -\arctan\frac{(a_1+a_2)y}{1+(a_1+a_2)x}. \tag{4.5.18}$$

Theorem 4.4.4, Theorem 4.5.2 and Proposition 4.5.1 $\sim$ Proposition 4.5.5 follow that

Theorem 4.5.3. *The origin of the real system* (4.5.1) *is an isochronous center if and only if the first 5 period constants of* (4.5.1) *are all zero, i.e., one of the conditions* $C_1 \sim C_5$ *is satisfied.*

Remark 4.5.1. *If* a_1, a_2, a_4, a_5, a_6 *are all complex constants, and* x, y, t *are all complex variables, then the functions of the time-angle difference given in Proposition* 4.5.1 $\sim$ *Proposition* 4.5.5 *are all power series with non-zero convergent radius. Thus, the conclusion of Theorem* 4.5.3 *is also correct in the complex field.*

Bibliographical Notes

In 1988, the definition of period constants for complex polynomial systems and a computational method had been given in [Li J.B. etc,1991b]. The materials of this chapter are taken from [Liu Y.R. etc, 2003a] and [Liu Y.R. etc, 2006b] in which the authors extended the concepts of the period constant and the isochronous center in the real systems to the complex systems.

The method determining an isochronous center is not unique. Many mathematicians had made important contributed in this study direction (see [Loud, 1964; Pleshkan, 1969; Mardesić etc, 1995; Cima etc, 1997; Christopher ect, 1997; Gasull, 1997b; Cairó etc, 1999; Chavarriga etc, 1999; Chavarriga etc, 1999; Chavarriga etc, 2000; Li C.Z. etc, 2000b; Romanovskii ect, 2001c; Chavarriga etc, 2001; Li C.Z. etc, 2002; Liu Y.R. etc, 2003a; Liu Y.R. etc, 2004; Huang W.T. etc, 2005; Liu Y.R. etc, 2006b; Huang W.T. etc, 2007; Liu Y.R. etc, 2008a]), for instance.

The basic method is computation of period constants. A necessary and sufficient condition for isochronicity is that all period constants vanish. In the applications, we often need to obtain the necessary condition of isochronicity from the first period constants vanishing and then prove the conditions are sufficient with other methods such as the method of the change of variables. The same as the computation of Lyapunov constants, the computation of period constants is a much more difficult problem. There are a lot of authors who deal with this problem (see[Gasull, 1997b; Hassard etc, 1978; Farr etc, 1989]).

Chapter 5

Theory of Center-Focus and Bifurcation of Limit Cycles at Infinity of a Class of Systems

We have mentioned in Preface of this book that a real planar polynomial vector field V can be compactified on the sphere. The vector field $p(V)$ restricted to the upper hemisphere completed with the equator Γ_∞ is called *Poincaré compactification of a polynomial vector field*. If a real polynomial vector field has no real singular point in the equator Γ_∞ of the *Poincaré disc* and Γ_∞ can be seen a trajectory, all trajectories in a inner neighborhood of Γ_∞ are spirals or closed orbits, then Γ_∞ is called the equator cycle of the vector field. Γ_∞ can be become a point by using the Bendixson reciprocal radius transformation. This point is called infinity of the system.

In this chapter, we discuss the center-focus problem of infinity (i.e., to distinguish when the trajectories in a inner neighborhood of Γ_∞ are either closed orbits or spirals) and the bifurcation of limit cycles at infinity for a class of systems.

5.1 Definition of the Focal Values of Infinity

Consider the following real planar polynomial system of degree $(2n+1)$:

$$\frac{dx}{dt} = \sum_{k=0}^{2n+1} X_k(x,y), \quad \frac{dy}{dt} = \sum_{k=0}^{2n+1} Y_k(x,y), \tag{5.1.1}$$

where n is a positive integer and $X_k(x,y)$, $Y_k(x,y)$ are homogeneous polynomials of degree k in x, y of the form

$$\begin{aligned} k(x,y) &= \sum_{\alpha+\beta=k} A_{\alpha\beta} x^\alpha y^\beta, \\ Y_k(x,y) &= \sum_{\alpha+\beta=k} B_{\alpha\beta} x^\alpha y^\beta. \end{aligned} \tag{5.1.2}$$

Suppose that the function $xY_{2n+1}(x,y) - yX_{2n+1}(x,y)$ is not identically zero. Then, system (5.1.1) only has finite real or complex singular points in Γ_∞. It has no real singular point in Γ_∞ if and only if $xY_{2n+1}(x,y)-yX_{2n+1}(x,y)$ is a positive (or negative) definite function in the real field. This function can be expressed as a product of linear terms in the complex field as follows:

$$xY_{2n+1}(x,y)-yX_{2n+1}(x,y)=\prod_{k=1}^{2n+2}(\alpha_k x+\beta_k y). \tag{5.1.3}$$

On the Poincaré disk, all infinite singular points (real and complex) of system (5.1.1) are the intersection points of the straight line $\alpha_k x+\beta_k y=0$ and the unit circle $x^2+y^2=1$, $k=1,2,\cdots,2n+2$.

Without loss of the generality, we assume that $I(x,y)$ is positive definite (otherwise, we can take a transformation $t \rightharpoonup -t$), then, there exists a positive numbers d, such that

$$xY_{2n+1}(x,y)-yX_{2n+1}(x,y)\geqslant d(x^2+y^2)^{n+1}. \tag{5.1.4}$$

By using the transformation

$$x=\frac{cos\theta}{r},\quad y=\frac{sin\theta}{r}, \tag{5.1.5}$$

system (5.1.1) becomes

$$\begin{aligned}\frac{dr}{dt}&=\frac{-1}{r^{2n-1}}\sum_{k=0}^{2n+1}\varphi_{2n+2-k}(\theta)r^k,\\ \frac{d\theta}{dt}&=\frac{1}{r^{2n}}\sum_{k=0}^{2n+1}\psi_{2n+2-k}(\theta)r^k.\end{aligned} \tag{5.1.6}$$

Thus, we have

$$\frac{dr}{d\theta}=-r\frac{\varphi_{2n+2}(\theta)+\sum\limits_{k=1}^{2n+1}\varphi_{2n+2-k}(\theta)r^k}{\psi_{2n+2}(\theta)+\sum\limits_{k=1}^{2n+1}\psi_{2n+2-k}(\theta)r^k}, \tag{5.1.7}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are given by (2.1.5). Especially,

$$\begin{aligned}\varphi_{2n+2}(\theta)&=\cos\theta X_{2n+1}(\cos\theta,\sin\theta)+\sin\theta Y_{2n+1}(\cos\theta,\sin\theta),\\ \psi_{2n+2}(\theta)&=\cos\theta Y_{2n+1}(\cos\theta,\sin\theta)-\sin\theta X_{2n+1}(\cos\theta,\sin\theta).\end{aligned} \tag{5.1.8}$$

(5.1.4) and (5.1.8) follow that

$$\psi_{2n+2}(\theta)\geqslant d>0. \tag{5.1.9}$$

Since for all k, $\varphi_k(\theta)$ and $\psi_k(\theta)$ are homogeneous polynomials of degree k in $(\cos\theta, \sin\theta)$, we have

$$\varphi_k(\theta+\pi) = (-1)^k\varphi_k(\theta), \quad \psi_k(\theta+\pi) = (-1)^k\psi_k(\theta). \tag{5.1.10}$$

It implies that equation (5.1.7) is the specific form of the equation (2.1.7).

For a sufficiently small constant h, we write the solution of (5.1.7) with the initial condition $r|_{\theta=0} = h$ as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta)h^k. \tag{5.1.11}$$

From (5.1.7) and (5.1.11), we obtain

$$\nu_1(\theta) = \exp\int_0^\theta \frac{-\varphi_{2n+2}(\vartheta)\ d\vartheta}{\psi_{2n+2}(\vartheta)}. \tag{5.1.12}$$

By Corollary 2.1.1, if $\nu_1(2\pi) = 1$, then the first positive integer k satisfying $\nu_k(2\pi) \neq 0$ is an odd number.

Definition 5.1.1. *For any positive integer k, $\nu_{2k+1}(2\pi)$ is called the k-th focal value at infinity of system* (5.1.1)

Definition 5.1.2. *For system* (5.1.1):

(1) *If $\nu_1(2\pi) \neq 1$ and when $\nu_1(2\pi) < 1$ (> 1), infinity is called a stable* (*an unstable*) *rough focus;*

(2) *If $\nu_1(2\pi) = 1$ and there exists a positive integer k, such that $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k}(2\pi) = 0$ and $\nu_{2k+1}(2\pi) \neq 0$, then when $\nu_{2k+1}(2\pi) < 0$ (> 0), infinity is called a stable* (*an unstable*) *weak focus;*

(3) *If $\nu_1(2\pi) = 1$ and for any positive integer k, we have $\nu_{2k+1}(2\pi) = 0$, then infinity is called a center.*

From Corollary 2.1.1 and the geometric properties of the Poincaré successor function $\Delta(h) = \tilde{r}(2\pi, h) - h$, we obtain

Theorem 5.1.1. *If infinity is a stable* (*an unstable*) *focus of system* (5.1.1), *then Γ_∞ is an internal stable* (*an internal unstable*) *limit cycle.*

If infinity is a center, then there exists a family of closed orbits of system (5.1.1) *in a inner neighborhood of the equator Γ_∞.*

For a given polynomial system, to solve the center-focus problem of infinity, it depends on the computations of the focal values of infinity. In next sections, we discuss this difficult problem.

5.2 Conversion of Questions

First, we consider a special case of system (5.1.1). Letting

$$X_{2n+1}(x,y) = (\delta x - y)(x^2+y^2)^n,$$
$$Y_{2n+1}(x,y) = (x + \delta y)(x^2+y^2)^n, \tag{5.2.1}$$

then system (5.1.1) becomes

$$\frac{dx}{dt} = (\delta x - y)(x^2+y^2)^n + \sum_{\alpha+\beta=0}^{2n} A_{\alpha\beta}x^\alpha y^\beta,$$
$$\frac{dy}{dt} = (x + \delta y)(x^2+y^2)^n + \sum_{\alpha+\beta=0}^{2n} B_{\alpha\beta}x^\alpha y^\beta. \tag{5.2.2}$$

For system (5.2.2), (5.1.8) reduces to

$$\varphi_{2n+2}(\theta) \equiv \delta, \quad \psi_{2n+2}(\theta) \equiv 1. \tag{5.2.3}$$

Thus, (5.1.7) becomes

$$\frac{dr}{d\theta} = -r\frac{\delta + \sum\limits_{k=1}^{2n+1}\varphi_{2n+2-k}(\theta)r^k}{1 + \sum\limits_{k=1}^{2n+1}\psi_{2n+2-k}(\theta)r^k}. \tag{5.2.4}$$

It is easy to prove that

Proposition 5.2.1. *For system* (5.2.2)*, we have* $\nu_1(\theta) = e^{-\delta\theta}$ *and when* $\delta > 0$ (< 0)*, infinity is a stable* (*an unstable*) *focus.*

From Lemma 2.1.2, we obtain

Proposition 5.2.2. *If* $\delta = 0$*, then for system* (5.2.2)*, all* $\nu_k(\theta)$ *are polynomials in* θ*,* $\sin\theta$*,* $\cos\theta$*, and their coefficients are polynomials in* $A_{\alpha\beta}$*,* $B_{\alpha\beta}$*. Especially, for all* k*,* $\nu_k(\pi)$*,* $\nu_k(2\pi)$ *are polynomials in* $A_{\alpha\beta}$*,* $B_{\alpha\beta}$*.*

Notice that infinity of system (5.2.2) can be changed to the origin by using a suitable transformation. In fact, by the transformation

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}, \quad \frac{dt}{d\tau} = (u^2+v^2)^{2n} \tag{5.2.5}$$

system (5.2.2) becomes

$$\frac{du}{d\tau} = -(\delta u + v)(u^2+v^2)^n + \sum_{k=0}^{2n}(u^2+v^2)^{2n-k}[(u^2-v^2)X_k(u,v) + 2uvY_k(u,v],$$
$$\frac{dv}{d\tau} = (u - \delta v)(u^2+v^2)^n + \sum_{k=0}^{2n}(u^2+v^2)^{2n-k}[(u^2-v^2)Y_k(u,v) - 2uvX_k(u,v]. \tag{5.2.6}$$

In the transformation (5.2.5),

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2} \tag{5.2.7}$$

is called Bendixson reciprocal radius transformation. Making the polar coordinate transformation $u = r\cos\theta$, $v = r\sin\theta$, the transformation (5.2.7) becomes the transformation (5.1.5).

The transformation (5.2.5) makes infinity of system (5.2.2) become the origin of system (5.2.6). Thus, the studies of the center-focus problem and the bifurcation of limit cycles of infinity of system (5.2.2) can be changed to the studies of the corresponding problems for the origin of system (5.2.6). Since the origin of system (5.2.6) is a higher-order singular point (or degenerate singular point), it leads to some difficult problems. We discuss them in Section 6.

If for all $k \in \{n+1, n+2, \cdots, 2n\}$, we have $X_k(x,y) = Y_k(x,y) = 0$, then system (5.2.2) becomes

$$\frac{dx}{dt} = (\delta x - y)(x^2+y^2)^n + \sum_{k=0}^{n} X_k(x,y),$$
$$\frac{dy}{dt} = (x + \delta y)(x^2+y^2)^n + \sum_{k=0}^{n} Y_k(x,y). \tag{5.2.8}$$

Hence, we have the following conclusion.

Theorem 5.2.1. *By the transformation*

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}, \quad \frac{dt}{d\tau} = (u^2+v^2)^n, \tag{5.2.9}$$

system (5.2.8) *becomes the following polynomial system*

$$\frac{du}{d\tau} = -\delta u - v + \sum_{k=0}^{n}(u^2+v^2)^{n-k}[(u^2-v^2)X_k(u,v) + 2uvY_k(u,v],$$
$$\frac{dv}{d\tau} = u - \delta v + \sum_{k=0}^{n}(u^2+v^2)^{n-k}[(u^2-v^2)Y_k(u,v) - 2uvX_k(u,v], \tag{5.2.10}$$

for which the origin is an elementary singular point.

We can use the following transformation

$$x = \frac{u}{(u^2+v^2)^{m_1}}, \quad y = \frac{v}{(u^2+v^2)^{m_1}}, \quad \frac{dt}{d\tau} = (u^2+v^2)^{m_2}, \tag{5.2.11}$$

such that infinity of system (5.2.2) changes to the origin which is an elementary singular point.

Theorem 5.2.2. *By the transformation*

$$x = \frac{u}{(u^2+v^2)^{n+1}}, \quad y = \frac{v}{(u^2+v^2)^{n+1}}, \quad \frac{dt}{d\tau} = (u^2+v^2)^{n(2n+1)}, \tag{5.2.12}$$

system (5.2.2) *becomes the following polynomial system:*

$$\begin{aligned} \frac{du}{d\tau} &= \frac{-\delta u}{2n+1} - v + \sum_{k=1}^{2n+1} P_{2nk+k+1}(u,v), \\ \frac{dv}{d\tau} &= u - \frac{\delta v}{2n+1} + \sum_{k=1}^{2n+1} Q_{2nk+k+1}(u,v), \end{aligned} \tag{5.2.13}$$

for which the origin is an elementary singular point, where

$$\begin{aligned} P_{2nk+k+1}(u,v) = \Bigg[& \left(v^2 - \frac{1}{2n+1}\, u^2\right) X_{2n+1-k}(u,v) \\ & - \frac{2n+2}{2n+1}\, uv\, Y_{2n+1-k}(u,v)\Bigg] (u^2+v^2)^{(k-1)(n+1)} \end{aligned} \tag{5.2.14}$$

and

$$\begin{aligned} Q_{2nk+k+1}(u,v) = \Bigg[& \left(u^2 - \frac{1}{2n+1}\, v^2\right) Y_{2n+1-k}(u,v) \\ & - \frac{2n+2}{2n+1}\, uv\, X_{2n+1-k}(u,v)\Bigg] (u^2+v^2)^{(k-1)(n+1)} \end{aligned} \tag{5.2.15}$$

are homogeneous polynomials of degree $(2n+1)k+1$ *of* u *and* v, $k = 1,\ 2,\ \cdots,\ 2n+1$.

This theorem tell us that the studies of the center-focus problem and bifurcation of limit cycles at infinity of system (5.2.2) can be change to the studies of the corresponding problems at the elementary singular point $O(0,0)$ of system (5.2.13). Because system (5.2.13) is a class of particular systems of (2.1.1). Therefore, we can apply all known theory for the center-focus problem of system (2.1.1) to system (5.2.13).

Of course, system (5.2.13) have the following particular properties.

(1) The subscripts (i.e., the degree of homogeneous polynomials) of $P_{2nk+k+1}$, $Q_{2nk+k+1}$ form an arithmetic sequence having common difference $2n+1$, $k = 1, 2, \cdots, 2n+1$.

(2) $P_{2nk+k+1}$ and $Q_{2nk+k+1}$ have the common factor $(u^2+v^2)^{(k-1)(n+1)}$.

(3) System (5.2.13) has a pair of conjugated complex straight line solutions $u \pm iv = 0$.

We can use these special properties to study the theory of center-focus at infinity for system (5.2.2).

5.3 Method of Formal Series and Singular Point Value of Infinity

By the polar coordinate transformation

$$u = \rho\cos\theta, \quad v = \rho\sin\theta \tag{5.3.1}$$

system (5.2.13) becomes

$$\frac{d\rho}{d\theta} = \frac{-\rho}{2n+1} \cdot \frac{\delta + \sum\limits_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta)\rho^{k(2n+1)}}{1 + \sum\limits_{k=1}^{2n+1} \psi_{2n+2-k}(\theta)\rho^{k(2n+1)}}. \tag{5.3.2}$$

Substituting (5.3.1) and (5.1.5) into (5.2.12), we have

$$r = \rho^{2n+1}. \tag{5.3.3}$$

Obviously, equation (5.3.2) can also be obtained from equation (5.2.4) by using transformation (5.3.3).

Let $\rho = \tilde{\rho}(\theta, \rho_0)$ be the solution of (5.3.2) satisfying the initial condition $\rho|_{\theta=0} = \rho_0$. By using the particular properties of (5.2.13) mentioned in the above section, we obtain

Proposition 5.3.1. *$\tilde{\rho}(\theta, \rho_0)\rho_0^{-1}$ is a power series of ρ_0^{2n+1}, i.e., $\tilde{\rho}(\theta, \rho_0)$ has the following form:*

$$\tilde{\rho}(\theta, \rho_0) = \sum_{m=1}^{\infty} \sigma_{(m-1)(2n+1)+1}(\theta)\rho_0^{(m-1)(2n+1)+1}. \tag{5.3.4}$$

Proof. Let $r = \tilde{r}(\theta, h) = \sum\limits_{m=1}^{\infty} \nu_m(\theta)h^m$ be the solution of (5.2.4) satisfying the initial condition $r|_{\theta=0} = h$. From (5.3.3), we obtain

$$\tilde{\rho}^{2n+1}(\theta, \rho_0) = \tilde{r}(\theta, \rho_0^{2n+1}). \tag{5.3.5}$$

Let $h_0 = \rho_0^{2n+1}$. (5.3.5) follows that

$$\frac{\tilde{\rho}(\theta,\rho_0)}{\rho_0} = \left(\frac{\tilde{r}(\theta,h_0)}{h_0}\right)^{\frac{1}{2n+1}} = \left[\sum_{m=1}^{\infty} \nu_m(\theta) h_0^{m-1}\right]^{\frac{1}{2n+1}}. \tag{5.3.6}$$

Since the right hand of (5.3.6) can be expanded as a power series of h_0, it follows the conclusion of this proposition. □

Clearly,

$$\begin{aligned}
\nu_1(2\pi) - 1 &= e^{-2\pi\delta} - 1 = -2\pi\delta + o(\delta),\\
\sigma_1(2\pi) - 1 &= e^{\frac{-2\pi\delta}{2n+1}} - 1 = \frac{-2\pi\delta}{2n+1} + o(\delta).
\end{aligned} \tag{5.3.7}$$

Theorem 5.3.1. *If $\delta = 0$, for any positive integer k, we have*

$$\begin{aligned}
&\sigma_{2k(2n+1)+1}(2\pi) \sim \frac{1}{2n+1}\nu_{2k+1}(2\pi),\\
&\sigma_{(2k-1)(2n+1)+1}(2\pi) \sim 0,
\end{aligned} \tag{5.3.8}$$

and when m is not an integer multiple of $2n+1$, $\sigma_{m+1}(2\pi) = 0$, where $\nu_{2k+1}(2\pi)$ is the k-th focal value at infinity of system (5.2.2) *and $\sigma_{2k(2n+1)+1}(2\pi)$ is the $k(2n+1)$-th focal value at the origin of system* (5.2.13).

Proof. First, when $\delta = 0$, we see from (5.3.7) that $\nu_1(2\pi) = \sigma_1(2\pi) = 1$. Thus, from (5.3.4), we have

$$\begin{aligned}
&\tilde{\rho}^{2n+1}(2\pi,\rho_0) - \rho_0^{2n+1}\\
=&\sum_{j=0}^{2n} \rho_0^{2n-j}\tilde{\rho}^j(2\pi,\rho_0)[\tilde{\rho}(2\pi,\rho_0) - \rho_0]\\
=&(2n+1)\rho_0^{2n} G(\rho_0)[\tilde{\rho}(2\pi,\rho_0) - \rho_0]\\
=&(2n+1)G(\rho_0)\sum_{m=2}^{\infty} \sigma_{(m-1)(2n+1)+1}(2\pi)\rho_0^{m(2n+1)},
\end{aligned} \tag{5.3.9}$$

where $G(\rho_0)$ is an unit formal power series in ρ_0 (see Definition 1.2.3).

On the other hand, (5.3.5) follows that

$$\begin{aligned}
\tilde{\rho}^{2n+1}(2\pi,\rho_0) - \rho_0^{2n+1} &= \tilde{r}(2\pi,\rho_0^{2n+1}) - \rho_0^{2n+1}\\
&= \sum_{m=2}^{\infty} \nu_m(2\pi)\rho_0^{m(2n+1)}.
\end{aligned} \tag{5.3.10}$$

By (5.3.9) and (5.3.10), we have

$$\begin{aligned}
&\sum_{m=2}^{\infty} \sigma_{(m-1)(2n+1)+1}(2\pi)\rho_0^{m(2n+1)}\\
=&\frac{1}{(2n+1)G(\rho_0)}\sum_{m=2}^{\infty} \nu_m(2\pi)\rho_0^{m(2n+1)}.
\end{aligned} \tag{5.3.11}$$

Comparing the coefficients of the same power of ρ_0 on the two sides of (5.3.11), it gives rise to the conclusion of this theorem. □

By the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \tag{5.3.12}$$

system (5.2.2) becomes

$$\begin{aligned}\frac{dz}{dT} &= (1 - i\delta)z^{n+1}w^n + \sum_{k=0}^{2n} Z_k(z, w),\\ \frac{dw}{dT} &= -(1 + i\delta)w^{n+1}z^n - \sum_{k=0}^{2n} W_k(z, w),\end{aligned} \tag{5.3.13}$$

where

$$\begin{aligned}Z_k(z, w) &= \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta\\ &= Y_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) - iX_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right),\\ W_k(z, w) &= \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta\\ &= Y_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) + iX_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right).\end{aligned} \tag{5.3.14}$$

We say that system (5.2.2) is the associated system of (5.3.13) and vice versa.

Let

$$\xi = u + iv, \quad \eta = u - iv, \quad \mathcal{T} = i\tau. \tag{5.3.15}$$

Then, from (5.3.12), (5.3.15) and (5.2.12), we have

$$z = \frac{\xi}{(\xi\ \eta)^{n+1}}, \quad w = \frac{\eta}{(\xi\ \eta)^{n+1}}, \quad \frac{dT}{d\mathcal{T}} = (\xi\ \eta)^{n(2n+1)}. \tag{5.3.16}$$

By transformation (5.3.16), system (5.3.13) can be reduced to

$$\begin{aligned}\frac{d\xi}{d\mathcal{T}} &= \left(1 + \frac{i\delta}{2n+1}\right)\xi + \xi \sum_{k=1}^{2n+1} \Phi_{k(2n+1)}(\xi, \eta),\\ \frac{d\eta}{d\mathcal{T}} &= -\left(1 - \frac{i\delta}{2n+1}\right)\eta - \eta \sum_{k=1}^{2n+1} \Psi_{k(2n+1)}(\xi, \eta),\end{aligned} \tag{5.3.17}$$

where

$$\begin{aligned}\Phi_{k(2n+1)}(\xi, \eta) = \Bigg[&\frac{n}{2n+1}\ \eta Z_{2n+1-k}(\xi, \eta)\\ &+ \frac{n+1}{2n+1}\ \xi W_{2n+1-k}(\xi, \eta)\Bigg] (\xi\ \eta)^{(k-1)(n+1)},\end{aligned}$$

$$\Psi_{k(2n+1)}(\xi,\eta)=\left[\frac{n}{2n+1}\,\xi\,W_{2n+1-k}(\xi,\eta)\right.$$
$$\left.+\frac{n+1}{2n+1}\,\eta Z_{2n+1-k}(\xi,\eta)\right](\xi\ \eta)^{(k-1)(n+1)} \tag{5.3.18}$$

are homogeneous polynomials of degree $k(2n+1)$ in ξ,η.

Obviously, system (5.3.17) can also be obtained from system (5.2.13) by using transformation (5.3.15), thus, system (5.2.13) is the associated system of (5.3.17) and vice versa.

We next consider the case of $\delta=0$. When $\delta=0$, system (5.2.2), (5.2.13), (5.3.13) and (5.3.17) take the following forms, respectively,

$$\begin{aligned}\frac{dx}{dt}&=-y(x^2+y^2)^n+\sum_{k=0}^{2n}X_k(x,y)=X(x,y),\\ \frac{dy}{dt}&=x(x^2+y^2)^n+\sum_{k=0}^{2n}Y_k(x,y)=Y(x,y);\end{aligned} \tag{5.3.19}$$

$$\begin{aligned}\frac{du}{d\tau}&=-v+\sum_{k=1}^{2n+1}P_{2nk+k+1}(u,v)=P(u,v),\\ \frac{dv}{d\tau}&=u+\sum_{k=1}^{2n+1}Q_{2nk+k+1}(u,v)=Q(u,v);\end{aligned} \tag{5.3.20}$$

$$\begin{aligned}\frac{dz}{dT}&=z^{n+1}w^n+\sum_{k=0}^{2n}Z_k(z,w)=Z(z,w),\\ \frac{dw}{dT}&=-w^{n+1}z^n-\sum_{k=0}^{2n}W_k(z,w)=-W(z,w);\end{aligned} \tag{5.3.21}$$

$$\begin{aligned}\frac{d\xi}{d\mathcal{T}}&=xi+\xi\sum_{k=1}^{2n+1}\Phi_{k(2n+1)}(\xi,\eta)=\Phi(\xi,\eta),\\ \frac{d\eta}{d\mathcal{T}}&=-\eta-\eta\sum_{k=1}^{2n+1}\Psi_{k(2n+1)}(\xi,\eta)=-\Psi(\xi,\eta).\end{aligned} \tag{5.3.22}$$

The right hand of system (5.3.22) have the following particular properties:

(1) The subscripts (the degree of homogeneous polynomials) of $\Phi_{k(2n+1)}$, $\Psi_{k(2n+1)}$ form an arithmetic sequence with common difference $2n+1$, $k=1,2,\cdots,2n+1$.

(2) $\Phi_{k(2n+1)}$ and $\Psi_{k(2n+1)}$ have the common factor $(\xi\eta)^{(k-1)(n+1)}$.

(3) System (5.3.22) has a pair of straight line solutions $\xi=0$ and $\eta=0$.

From these properties of the right hand of system (5.3.22), we have

Theorem 5.3.2. *For system* (5.3.22), *one can derive uniquely and successively the terms of the following formal series*

$$F(\xi,\eta)=(\xi\eta)^{2n+1}\left[1+\sum_{m=1}^{\infty}f_{m(2n+1)}(\xi,\eta)\right], \tag{5.3.23}$$

such that

$$\left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)}=\sum_{m=1}^{\infty}\mu_m(\xi\eta)^{(m+1)(2n+1)}, \tag{5.3.24}$$

where

$$f_{m(2n+1)}(\xi,\eta)=\sum_{\alpha+\beta=m(2n+1)}c_{\alpha\beta}\xi^{\alpha}\eta^{\beta} \tag{5.3.25}$$

are homogeneous polynomials of degree $m(2n+1)$ *in* ξ,η $(m=1,2,\cdots)$ *and we take*

$$c_{00}=1,\quad c_{k(2n+1),k(2n+1)}=0,\quad k=1,2,\cdots. \tag{5.3.26}$$

Definition 5.3.1. *For any positive integer* m, μ_m *given by* (5.3.24) *is called the* m*-th singular point value at infinity of system* (5.3.21).

If there exists a positive integer k, *such that* $\mu_1=\mu_2=\cdots=\mu_{k-1}=0$, $\mu_k\neq 0$, *then infinity of system* (5.3.21) *is called a weak critical singular point of order* k.

If for all positive integer k, $\mu_k=0$, *then infinity of system* (5.3.21) *is called a complex center.*

Theorem 5.3.3. *In the* (5.3.25), *for all pairs* (α,β), *when* $\alpha\neq\beta$, *and* $\alpha+\beta\geqslant 1$, $c_{\alpha\beta}$ *is given by*

$$\begin{aligned}c_{\alpha\beta}=&\frac{1}{(2n+1)(\beta-\alpha)}\\&\times\sum_{k+j=1}^{2n+1}\Big\{[n\alpha-(n+1)\beta+(n-k)(2n+1)]a_{k,j-1}\\&-[n\beta-(n+1)\alpha+(n-j)(2n+1)]b_{j,k-1}\Big\}\\&\times c_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}.\end{aligned} \tag{5.3.27}$$

For any positive integer m, μ_m *is given by*

$$\begin{aligned}\mu_m=&\sum_{k+j=1}^{2n+1}[(n-k-m)a_{k,j-1}-(n-j-m)b_{j,k-1}]\\&\times c_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)},\end{aligned} \tag{5.3.28}$$

where for all pairs (α,β), *when* $\alpha<0$ *or* $\beta<0$, *we take* $a_{\alpha\beta}=b_{\alpha\beta}=c_{\alpha\beta}=0$.

Proof. From (5.3.23), we have

$$\left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)} = (\xi\eta)^{2n+1}\left\{\sum_{m=1}^{\infty}\left(\frac{\partial f_{m(2n+1)}}{\partial\xi}\,\xi - \frac{\partial f_{m(2n+1)}}{\partial\eta}\,\eta\right)\right.$$
$$+\sum_{m=1}^{\infty}\sum_{s=1}^{2n+1}\left[\xi\frac{\partial\ f_{(m-s)(2n+1)}}{\partial\xi} + (2n+1)f_{(m-s)(2n+1)}\right]\Phi_{s(2n+1)}$$
$$\left.-\sum_{m=1}^{\infty}\sum_{s=1}^{2n+1}\left[\eta\frac{\partial f_{(m-s)(2n+1)}}{\partial\eta} + (2n+1)f_{(m-s)(2n+1)}\right]\Psi_{s(2n+1)}\right\}. \tag{5.3.29}$$

By (5.3.25) and (5.3.29), we obtain

$$\left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)} = (\xi\eta)^{2n+1}\sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta=\\ m(2n+1)}}(\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}$$
$$+(\xi\eta)^{2n+1}\sum_{m=1}^{\infty}\sum_{s=1}^{2n+1}\sum_{\substack{\alpha+\beta=\\ (m-s)(2n+1)}}\left[(\alpha+2n+1)\Phi_{s(2n+1)}\right.$$
$$\left.-\ (\beta+2n+1)\Psi_{s(2n+1)}\right]c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}. \tag{5.3.30}$$

From (5.3.18) and (5.3.30), we get

$$(\xi\eta)^{-(2n+1)}\left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)} = \sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta=\\ m(2n+1)}}(\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}$$
$$+\frac{1}{2n+1}\sum_{m=1}^{\infty}\sum_{s=1}^{2n+1}\sum_{\substack{\alpha+\beta=\\ (m-s)(2n+1)}}\left[(n\alpha-n\beta-\beta-2n-1)\eta Z_{2n+1-s}\right.$$
$$\left.-(n\beta-n\alpha-\alpha-2n-1)\xi W_{2n+1-s}\right]c_{\alpha\beta}\xi^{\alpha+(s-1)(n+1)}\eta^{\beta+(s-1)(n+1)}. \tag{5.3.31}$$

(5.3.14) becomes

$$Z_{2n+1-s}(\xi,\eta) = \sum_{k+j=2n+2-s} a_{k,j-1}\xi^{k}\eta^{j-1},$$
$$W_{2n+1-s}(\xi,\eta) = \sum_{k+j=2n+2-s} b_{j,k-1}\xi^{k-1}\eta^{j}. \tag{5.3.32}$$

Thus, (5.3.31) and (5.3.32) follow that

$$(\xi\eta)^{-(2n+1)} \left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)} = \sum_{m=1}^{\infty} \sum_{\substack{\alpha+\beta= \\ m(2n+1)}} (\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}$$
$$+\frac{1}{2n+1}\sum_{m=1}^{\infty}\sum_{s=1}^{2n+1} \sum_{\substack{\alpha+\beta= \\ (m-s)(2n+1)}} \sum_{\substack{k+j= \\ 2n+2-s}} \big[(n\alpha-n\beta-\beta-2n-1)a_{k,j-1}$$
$$-(n\beta-n\alpha-\alpha-2n-1)b_{j,k-1}\big]\, c_{\alpha\beta}\xi^{\alpha+(s-1)(n+1)+k}\eta^{\beta+(s-1)(n+1)+j}. \quad (5.3.33)$$

Write that

$$\alpha_1 = \alpha + (s-1)(n+1) + k,$$
$$\beta_1 = \beta + (s-1)(n+1) + j. \quad (5.3.34)$$

Hence, when $k+j = 2n+2-s$, $\alpha+\beta = (m-s)(2n+1)$, we have

$$\alpha_1 + \beta_1 = m(2n+1),$$
$$\alpha = \alpha_1 + nk + (n+1)j - (n+1)(2n+1),$$
$$\beta = \beta_1 + nj + (n+1)k - (n+1)(2n+1),$$
$$n\alpha - n\beta - \beta - 2n - 1 = n\alpha_1 - (n+1)\beta_1 + (n-k)(2n+1),$$
$$n\beta - n\alpha - \alpha - 2n - 1 = n\beta_1 - (n+1)\alpha_1 + (n-j)(2n+1). \quad (5.3.35)$$

Substituting (5.3.34), (5.3.35) into (5.3.33), and using the symbols α, β instead of α_1, β_1, we obtain

$$\left.\frac{dF}{d\mathcal{T}}\right|_{(5.3.22)} = (\xi\eta)^{2n+1} \sum_{m=1}^{\infty} \sum_{\substack{\alpha+\beta= \\ m(2n+1)}} \left[(\alpha-\beta)c_{\alpha\beta} + H_{\alpha\beta}\right] \xi^{\alpha}\eta^{\beta}, \quad (5.3.36)$$

where

$$H_{\alpha\beta} = \frac{1}{2n+1} \sum_{k+j=1}^{2n+1} \{[n\alpha-(n+1)\beta+(n-k)(2n+1)]a_{k,j-1}$$
$$-\ [n\beta-(n+1)\alpha+(n-j)(2n+1)]b_{j,k-1}\}$$
$$\times\ c_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}. \quad (5.3.37)$$

From (5.3.24) and (5.3.36), it gives rise to the conclusion of this theorem. □

For any positive integer m, let μ'_m be the *m-th* singular point values at the origin of system (5.3.22), we have

Theorem 5.3.4. *For any positive integer k, we have*

$$\mu'_{k(2n+1)} \sim \frac{\mu_k}{2n+1} \tag{5.3.38}$$

and when m is not an integer multiple of $2n+1$, we have $\mu'_m = 0$.

Proof. For the function $F(\xi, \eta)$ given by (5.3.23), let

$$\hat{F}(\xi,\eta) = F^{\frac{1}{2n+1}}(\xi,\eta) = \xi\eta\left[1 + \sum_{m=1}^{\infty} f_{m(2n+1)}(\xi,\eta)\right]^{\frac{1}{2n+1}}. \tag{5.3.39}$$

From (5.3.24) and (5.3.39), we have

$$\left.\frac{d\hat{F}}{d\mathcal{T}}\right|_{(5.3.22)} = \frac{\sum\limits_{m=1}^{\infty} \mu_m (\xi\eta)^{m(2n+1)+1}}{(2n+1)\left[1 + \sum\limits_{m=1}^{\infty} f_{m(2n+1)}(\xi,\eta)\right]^{2n/(2n+1)}}. \tag{5.3.40}$$

(5.3.40) follows the conclusion of the theorem. □

From Theorem 5.3.1, Theorem 5.3.4 and Theorem 1.4.4, we have

Theorem 5.3.5. *For any positive integer k,*

$$\begin{aligned} \sigma_{2k(2n+1)+1}(2\pi) &\sim \frac{i\pi}{2n+1}\mu_k, \\ \nu_{2k+1}(2\pi) &\sim i\pi\mu_k, \end{aligned} \tag{5.3.41}$$

where $\sigma_{2k(2n+1)+1}(2\pi)$ is the $k(2n+1)$-th focal value at the origin of system (5.3.20), *$\nu_{2k+1}(2\pi)$ is the k-th focal value at infinity of system* (5.3.19) *and μ_k is the k-th singular point value at infinity of system* (5.3.21)

From Theorem 2.3.4 and the particular properties of the right hand of system (5.3.22), we have

Theorem 5.3.6. *For system* (5.3.22), *one can derive successively the terms of the following formal series*

$$M(\xi,\eta) = 1 + \sum_{m=1}^{\infty} g_{m(2n+1)}(\xi,\eta), \tag{5.3.42}$$

such that

$$\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta} = \sum_{m=1}^{\infty} \frac{2mn+m+1}{2n+1}\lambda_m(\xi\eta)^{m(2n+1)}, \tag{5.3.43}$$

where, for any positive integer m,

$$g_{m(2n+1)}(\xi,\eta)=\sum_{\alpha+\beta=m(2n+1)} d_{\alpha\beta}\xi^{\alpha}\eta^{\beta} \tag{5.3.44}$$

is a homogeneous polynomial of degree $m(2n+1)$ in ξ,η, and

$$\lambda_m \sim (2n+1)\mu'_{m(2n+1)} \sim \mu_m. \tag{5.3.45}$$

Similar to Theorem 5.3.3, we have

Theorem 5.3.7. *In the right hand of* (5.3.42), *letting* $d_{00}=1$ *and taking* $d_{k(2n+1),k(2n+1)}(k=1,\ 2,\ \cdots)$ *as arbitrary numbers, then for all* (α,β), *when* $\alpha\neq\beta$, *and* $\alpha+\beta\geqslant 1$, $d_{\alpha\beta}$ *is given by*

$$\begin{aligned}d_{\alpha\beta}=&\frac{1}{(2n+1)(\beta-\alpha)}\sum_{k+j=1}^{2n+1}\Big\{[n\alpha-(n+1)\beta-1]a_{k,j-1}\\&-[n\beta-(n+1)\alpha-1]b_{j,k-1}]\Big\}\\&\times d_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}\end{aligned} \tag{5.3.46}$$

and for any positive integer m, λ_m *is determined by*

$$\begin{aligned}\lambda_m=&\sum_{k+j=1}^{2n+1}(b_{j,k-1}-a_{k,j-1})\\&\times d_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)},\end{aligned} \tag{5.3.47}$$

where for all (α,β), *when* $\alpha<0$ *or* $\beta<0$, *we take* $a_{\alpha\beta}=b_{\alpha\beta}=d_{\alpha\beta}=0$.

Theorem 5.3.3 and Theorem 5.3.7 give the recursive formulas to compute singular point values at infinity of system (5.3.21).

Theorem 5.3.8. *For system* (5.3.21), *one can derive successively the terms of the following formal series*

$$\mathcal{F}(z,w)=\frac{1}{zw}\left[1+\sum_{m=1}^{\infty}\frac{f_{m(2n+1)}(z,w)}{(zw)^{m(n+1)}}\right], \tag{5.3.48}$$

such that

$$\left.\frac{d\mathcal{F}}{dT}\right|_{(5.3.21)}=(zw)^n\sum_{m=1}^{\infty}\frac{\mu_m}{(zw)^{m+1}}, \tag{5.3.49}$$

where μ_m *is the m-th singular point value at infinity of system* (5.3.21), $m=1,2,\cdots$.

Proof. The inverse transformation of (5.3.16) is

$$\xi = z(zw)^{\frac{-(n+1)}{2n+1}}, \quad \eta = w(zw)^{\frac{-(n+1)}{2n+1}}, \quad \frac{d\mathcal{T}}{dT} = (zw)^n. \tag{5.3.50}$$

By (5.3.23) and (5.3.48), we have

$$\mathcal{F}(z,w) = F\left(z(zw)^{\frac{-(n+1)}{2n+1}}, w(zw)^{\frac{-(n+1)}{2n+1}}\right). \tag{5.3.51}$$

From (5.3.50), (5.3.51) and (5.3.24), it gives rise to the conclusion of this theorem. □

Theorem 5.3.9. *For system* (5.3.21), *one can derive successively the terms of the following formal series*

$$\mathcal{M}(z,w) \;=\; (zw)^{-n-1-\frac{1}{2n+1}}\left[1+\sum_{m=1}^{\infty}\frac{g_{m(2n+1)}(z,w)}{(zw)^{m(n+1)}}\right], \tag{5.3.52}$$

such that

$$\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w} = (zw)^{-1-\frac{1}{2n+1}}\sum_{m=1}^{\infty}\frac{(2mn+m+1)\lambda_m}{(2n+1)(zw)^m}, \tag{5.3.53}$$

where $\lambda_m \sim \mu_m,\ m=1,2,\cdots$.

Proof. First, by (5.3.42), we have

$$\mathcal{M}(z,w) = (zw)^{-n-1-\frac{1}{2n+1}} M\left[z(zw)^{\frac{-(n+1)}{2n+1}}, w(zw)^{\frac{-(n+1)}{2n+1}}\right]. \tag{5.3.54}$$

We consider the system

$$\begin{aligned}\frac{dz}{d\mathcal{T}} &= \frac{MZ}{(zw)^n} = \mathcal{Z}(z,w),\\ \frac{dw}{d\mathcal{T}} &= -\frac{MW}{(zw)^n} = -\mathcal{W}(z,w).\end{aligned} \tag{5.3.55}$$

By the transformation

$$\xi = z(zw)^{\frac{-(n+1)}{2n+1}}, \quad \eta = w(zw)^{\frac{-(n+1)}{2n+1}}, \tag{5.3.56}$$

system (5.3.55) becomes

$$\frac{d\xi}{d\mathcal{T}} = M(\xi,\eta)\Phi(\xi,\eta), \quad \frac{d\eta}{d\mathcal{T}} = -M(\xi,\eta)\Psi(\xi,\eta). \tag{5.3.57}$$

The Jacobin determinant of transformation (5.3.56) is given by

$$J = \frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial w} - \frac{\partial\xi}{\partial w}\frac{\partial\eta}{\partial z} = \frac{-1}{2n+1}(zw)^{-1-\frac{1}{2n+1}}. \tag{5.3.58}$$

Then, from (5.3.54), (5.3.55) and (5.3.58), we have

$$\frac{-1}{2n+1}\mathcal{M}Z = J\mathcal{Z}, \quad \frac{-1}{2n+1}\mathcal{M}W = J\mathcal{W}. \tag{5.3.59}$$

Thus,

$$\frac{-1}{2n+1}\left[\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w}\right] = \frac{\partial(J\mathcal{Z})}{\partial z} - \frac{\partial(J\mathcal{W})}{\partial w}. \tag{5.3.60}$$

By applying Proposition (1.1.3) to systems (5.3.55) and (5.3.57), from (5.3.57) we get

$$\begin{aligned}\frac{\partial(J\mathcal{Z})}{\partial z} - \frac{\partial(J\mathcal{W})}{\partial w} &= J\left[\frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta}\right] \\ &= \frac{-1}{2n+1}(zw)^{-1-\frac{1}{2n+1}}\left[\frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta}\right]. \end{aligned} \tag{5.3.61}$$

From (5.3.60) and (5.3.61), it follows that

$$\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w} = (zw)^{-1-\frac{1}{2n+1}}\left[\frac{\partial(M\Phi)}{\partial \xi} - \frac{\partial(M\Psi)}{\partial \eta}\right]. \tag{5.3.62}$$

(5.3.62), (5.3.43) and (5.3.56) give rise to the conclusion of this theorem. □

We now consider the following formal series

$$H(z,w) \ = 1 + \sum_{m=1}^{\infty} \frac{h_{m(2n+1)}(z,w)}{(zw)^{m(n+1)}}, \tag{5.3.63}$$

where

$$h_{m(2n+1)}(\xi,\eta) = \sum_{\alpha+\beta=m(2n+1)} e_{\alpha\beta} z^{\alpha} w^{\beta} \tag{5.3.64}$$

are homogeneous polynomials of degree $m(2n+1)$ in z, w ($m = 1, 2, \cdots$), and $h_0 = e_{00} = 1$.

Reference [Liu Y.R., 2001] gave the following two theorems.

Theorem 5.3.10. *For all $s \neq 0$, $\gamma \neq 0$, one can derive successively the terms of the formal series*

$$\tilde{F}(z,w) = (zw)^s H^{\frac{1}{\gamma}}(z,w), \tag{5.3.65}$$

such that

$$\left.\frac{d\tilde{F}}{dT}\right|_{(5.3.21)} = \frac{1}{\gamma}(zw)^{n+s} H^{\frac{1}{\gamma}-1} \sum_{m=1}^{\infty} \frac{\lambda'_m}{(zw)^m} \tag{5.3.66}$$

and for any positive integer m,

$$\lambda'_m \ \sim \ -s\gamma\mu_m. \tag{5.3.67}$$

Theorem 5.3.11. *Let s, γ be two constants, if for any positive integer m, $\gamma(s+n+1-m) \neq 0$, then one can derive successively the terms of the formal series*

$$\tilde{M}(z,w) = (zw)^s H^{\frac{1}{\gamma}}(z,w), \tag{5.3.68}$$

such that

$$\frac{\partial(\tilde{M}Z)}{\partial z} - \frac{\partial(\tilde{M}W)}{\partial w} = \frac{1}{\gamma}(zw)^{n+s} H^{\frac{1}{\gamma}-1} \sum_{m=1}^{\infty} \frac{\lambda''_m}{(zw)^m} \tag{5.3.69}$$

and for any positive integer m,

$$\lambda''_m \sim -\gamma(s+n+1-m)\mu_m. \tag{5.3.70}$$

By Theorem 5.3.10 and Theorem 5.3.11, the authors of [Chen H.B. etc, 2005b] gave the recursive formulas to compute singular point values of infinity of system (5.3.21) as follows.

Theorem 5.3.12. *For the formal series $\tilde{F}$ given by Theorem 5.3.10, $e_{k(2n+1),k(2n+1)}$ can be taken arbitrarily, $k = 1, 2, \cdots$. If $\alpha \neq \beta$ and $\alpha + \beta \geqslant 1$, $e_{\alpha\beta}$ is given by*

$$\begin{aligned} e_{\alpha\beta} = & \frac{1}{(2n+1)(\beta-\alpha)} \\ & \times \sum_{k+j=1}^{2n+1} \Big\{ [n\alpha - (n+1)\beta + (\gamma s + n + 1 - k)(2n+1)] a_{k,j-1} \\ & - [n\beta - (n+1)\alpha + (\gamma s + n + 1 - j)(2n+1)] b_{j,k-1} \Big\} \\ & \times e_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)}. \end{aligned} \tag{5.3.71}$$

For any positive integer m, λ'_m is given by

$$\begin{aligned} \lambda'_m = & \sum_{k+j=1}^{2n+1} [(\gamma s + n + 1 - k - m) a_{k,j-1} \\ & - (\gamma s + n + 1 - j - m) b_{j,k-1}] \\ & \times e_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)}. \end{aligned} \tag{5.3.72}$$

In above two recursive formulas, for all (α, β), if $\alpha < 0$ or $\beta < 0$, we take $a_{\alpha\beta} = b_{\alpha\beta} = e_{\alpha\beta} = 0$.

Theorem 5.3.13. *For the formal series $\tilde{M}$ given by Theorem 5.3.11, $e_{k(2n+1),k(2n+1)}$ can be taken arbitrarily, $k = 1, 2, \cdots$. If $\alpha \neq \beta$, and $\alpha + \beta \geqslant 1$, $e_{\alpha\beta}$ is given by*

$$
\begin{aligned}
e_{\alpha\beta} = & \frac{1}{(2n+1)(\beta-\alpha)} \\
& \times \sum_{k+j=1}^{2n+1} \Big\{ [n\alpha-(n+1)\beta+(\gamma s+n+1-k)(2n+1)]a_{k,j-1} \\
& -[n\beta-(n+1)\alpha+(\gamma s+\gamma j+n+1-j)(2n+1)]b_{j,k-1} \Big\} \\
& \times e_{\alpha+nk+(n+1)j-(n+1)(2n+1),\beta+nj+(n+1)k-(n+1)(2n+1)},
\end{aligned} \tag{5.3.73}
$$

else $e_{\alpha\beta}=0$.

For any positive integer m, λ''_m *is given by*

$$
\begin{aligned}
\lambda''_m = & \sum_{k+j=1}^{2n+1} [(\gamma s+\gamma k+n+1-k-m)a_{k,j-1} \\
& -(\gamma s+\gamma j+n+1-j-m)b_{j,k-1}] \\
& \times e_{nk+(n+1)j+(m-n-1)(2n+1),nj+(n+1)k+(m-n-1)(2n+1)}.
\end{aligned} \tag{5.3.74}
$$

In above two recursive formulas, for $\forall(\alpha,\beta)$, *when* $\alpha<0$ *or* $\beta<0$, *we take* $a_{\alpha\beta}=b_{\alpha\beta}=e_{\alpha\beta}=0$.

From Theorem 5.3.10 and Theorem 5.3.11, we have

Theorem 5.3.14. *Infinity of system* (5.3.21) *is a complex center if and only if there exists a first integral* $\tilde{F}(z,w)$ *with the form* (5.3.65).

Theorem 5.3.15. *Infinity of system* (5.3.21) *is a complex center if and only if there exists an integral factor* $\tilde{M}(z,w)$ *with the form* (5.3.68).

5.4 The Algebraic Construction of Singular Point Values of Infinity

By means of the transformation

$$z=\rho e^{i\phi}\hat{z},\quad w=\rho e^{-i\phi}\hat{w},\quad T=\rho^{-2n}\hat{T}, \tag{5.4.1}$$

system (5.3.21) becomes

$$
\begin{aligned}
\frac{d\hat{z}}{d\hat{T}} &= (\hat{z})^{n+1}(\hat{w})^n + \sum_{\alpha+\beta=0}^{2n} \hat{a}_{\alpha\beta}(\hat{z})^\alpha(\hat{w})^\beta, \\
\frac{d\hat{w}}{d\hat{T}} &= -(\hat{w})^{n+1}(\hat{z})^n - \sum_{\alpha+\beta=0}^{2n} \hat{b}_{\alpha\beta}(\hat{w})^\alpha(\hat{z})^\beta,
\end{aligned} \tag{5.4.2}
$$

where $\hat{z}, \hat{w}, \hat{T}$ are new variables and ρ, ϕ are complex constants, $\rho \neq 0$ and for all (α, β),

$$\begin{aligned}\hat{a}_{\alpha\beta} &= a_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{i(\alpha-\beta-1)\phi},\\ \hat{b}_{\alpha\beta} &= b_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{-i(\alpha-\beta-1)\phi}.\end{aligned} \tag{5.4.3}$$

If

$$\begin{aligned}&z = x + iy, \hat{z} = \hat{x} + i\hat{y},\\ &w = x - iy, \hat{w} = \hat{x} - i\hat{y},\\ &T = it, \hat{T} = i\hat{t},\end{aligned} \tag{5.4.4}$$

then transformation (5.4.1) becomes

$$x = \rho(\hat{x}\cos\phi - \hat{y}\sin\phi), \quad y = \rho(\hat{x}\sin\phi + \hat{y}\cos\phi), \quad t = \rho^{-2n}\hat{t}. \tag{5.4.5}$$

Compared with transformation (2.4.1), transformation (5.4.1) has a new time scale $T = \rho^{-2n}\hat{T}$. We say that transformation (5.4.1) is a generalized rotation and similar transformation with time exponent n.

Definition 5.4.1. *For systems* (5.4.1), *assume that $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ is a polynomial in $a_{\alpha\beta}$, $b_{\alpha\beta}$. Denote that $\hat{f} = f(\hat{a}_{\alpha\beta}, \hat{b}_{\alpha\alpha})$, $f^* = f(b_{\alpha\beta}, a_{\alpha\beta})$. If there exist λ, σ, such that $\hat{f} = \rho^{\lambda}e^{i\sigma\phi}f$, then λ and σ are respectively called the similar exponent and the rotation exponent with time exponent n of f under the transformation* (5.4.1), *which are represented by $I_s^{(n)}(f) = \lambda$, $I_r^{(n)}(f) = \sigma$.*

We see from (5.4.3) and Definition 5.4.1 that

$$\begin{aligned}&I_s^{(n)}(a_{\alpha\beta}) = \alpha + \beta - 2n - 1, \quad I_r^{(n)}(a_{\alpha\beta}) = \alpha - \beta - 1,\\ &I_s^{(n)}(b_{\alpha\beta}) = \alpha + \beta - 2n - 1, \quad I_r^{(n)}(b_{\alpha\beta}) = -(\alpha - \beta - 1).\end{aligned} \tag{5.4.6}$$

Obviously, for the generalized rotation and similar transformation in Definition Section 2.4, the similar exponent, the rotation exponent and the generalized rotation invariant all have time exponent 0. From (2.4.4) and (5.4.3), we have

$$\hat{a}_{\alpha\beta} = \rho^{-2n}\tilde{a}_{\alpha\beta}, \quad \hat{b}_{\alpha\beta} = \rho^{-2n}\tilde{b}_{\alpha\beta}. \tag{5.4.7}$$

In addition, (2.4.5) and (5.4.6) imply that

$$\begin{aligned}&I_s^{(n)}(a_{\alpha\beta}) = I_s(a_{\alpha\beta}) - 2n, \quad I_r^{(n)}(a_{\alpha\beta}) = I_r(a_{\alpha\beta}),\\ &I_s^{(n)}(b_{\alpha\beta}) = I_s(b_{\alpha\beta}) - 2n, \quad I_r^{(n)}(b_{\alpha\beta}) = I_r(b_{\alpha\beta}).\end{aligned} \tag{5.4.8}$$

Proposition 5.4.1. *Suppose that* $f_1 = f_1(a_{\alpha\beta}, b_{\alpha\beta})$ *and* $f_2 = f_2(a_{\alpha\beta}, b_{\alpha\beta})$ *are polynomials in* $a_{\alpha\beta}$, $b_{\alpha\beta}$. *If there exist* $\lambda_1, \lambda_2, \sigma_1, \sigma_2$, *such that* $\hat{f}_1 = \rho^{\lambda_1} e^{i\sigma_1\phi} f_1$, $\hat{f}_2 = \rho^{\lambda_2} e^{i\sigma_2\phi} f_2$, *then*

$$I_s^{(n)}(f_1 f_2) = I_s^{(n)}(f_1) + I_s^{(n)}(f_2), \quad I_r^{(n)}(f_1 f_2) = I_r^{(n)}(f_1) + I_r^{(n)}(f_2). \tag{5.4.9}$$

We see from Proposition 5.4.1 and formula (5.4.6) that

Proposition 5.4.2. *For* $m_1 + m_2$ *order monomial*

$$g = \prod_{j=1}^{m_1} a_{\alpha_j,\beta_j} \prod_{k=1}^{m_2} b_{\gamma_k,\delta_k}, \tag{5.4.10}$$

given by the coefficients of system (5.3.21), *we have*

$$\begin{aligned} I_s^{(n)}(g) &= \sum_{j=1}^{m_1}(\alpha_j + \beta_j - 2n - 1) + \sum_{k=1}^{m_2}(\gamma_k + \delta_k - 2n - 1), \\ I_r^{(n)}(g) &= \sum_{j=1}^{m_1}(\alpha_j - \beta_j - 1) - \sum_{k=1}^{m_2}(\gamma_k - \delta_k - 1). \end{aligned} \tag{5.4.11}$$

Remark 5.4.1. *For the coefficients* $a_{\alpha\beta}$, $b_{\alpha\beta}$ *of system* (5.3.21), *we have* $0 \leqslant \alpha + \beta \leqslant 2n$, *thus, from* (5.4.11), *for* $m_1 + m_2$ *order monomial* g *of the coefficients of system* (5.3.21), *we have* $I_s^{(n)}(g) < 0$.

From Proposition 5.4.2 and Theorem (t2.4.2), we obtain

Proposition 5.4.3.

$$I_s^{(n)}(g) = I_s(g) - 2n(m_1 + m_2), \quad I_r^{(n)}(g) = I_r(g). \tag{5.4.12}$$

Definition 5.4.2. (1) *Suppose that* $f = f(a_{\alpha\beta}, b_{\alpha\beta})$ *is a polynomial in* $a_{\alpha\beta}$, $b_{\alpha\beta}$. *If* $\hat{f} = \rho^{2k} f$, *then* f *is called a* k*-order generalized rotation invariant with time exponent* n *under the transformation* (5.4.1).

(2) *A generalized rotation invariant* f *is called a monomial generalized rotation invariant, if* f *is a monomial of* $a_{\alpha\beta}, b_{\alpha\beta}$.

(3) *A monomial generalized rotation invariant* f *is called an elementary generalized rotation invariant if it can not be expressed as a product of two monomial generalized rotation invariant.*

(4) *A generalized rotation invariant* f *is called self-symmetry, if* $f^* = f$. *It is called antisymmetry, if* $f^* = -f$.

From Proposition 5.4.1 and Definition 5.4.2, we have

Proposition 5.4.4. *Suppose that f_1 and f_2 are monomial generalized rotation invariants (or elementary generalized rotation invariants), then so are f_1^* and $f_1 f_2$, moreover,*

$$I_s^{(n)}(f_1^*) = I_s^{(n)}(f_1), \quad I_s^{(n)}(f_1 f_2) = I_s^{(n)}(f_1) + I_s^{(n)}(f_2). \tag{5.4.13}$$

We see from Proposition 5.4.2 and Definition 5.4.2 that

Proposition 5.4.5. *The m_1+m_2 order monomial given by (5.4.10) is a N-order generalized rotation invariant if and only if*

$$\begin{aligned} I_s^{(n)}(g) &= \sum_{j=1}^{m_1}(\alpha_j + \beta_j - 2n - 1) + \sum_{k=1}^{m_2}(\gamma_k + \delta_k - 2n - 1) = 2N, \\ I_r^{(n)}(g) &= \sum_{j=1}^{m_1}(\alpha_j - \beta_j - 1) - \sum_{k=1}^{m_2}(\gamma_k - \delta_k - 1) = 0. \end{aligned} \tag{5.4.14}$$

Lemma 5.4.1. *For any positive integer m, the m-th singular point value μ_m at infinity of system (5.3.21) is a "$-m$" order generalized rotation invariant with time exponent n under the transformation (5.4.1), i.e.,*

$$\hat{\mu}_m = \rho^{-2m}\mu_m. \tag{5.4.15}$$

Proof. For the function $\mathcal{F}(z,w)$ given by Theorem 5.3.8, let $\hat{\mathcal{F}} = \rho^2\mathcal{F}(\rho e^{i\phi}\hat{z}, \rho e^{-i\phi}\hat{w})$, then, from (5.3.48), we have

$$\hat{\mathcal{F}} = \frac{1}{\hat{z}\hat{w}}\left[1 + \sum_{m=1}^{\infty}\frac{f_{m(2n+1)}(\hat{z}e^{i\phi}, \hat{w}e^{-i\phi})}{\rho^m(\hat{z}\hat{w})^{m(n+1)}}\right]. \tag{5.4.16}$$

And from (5.3.49), we have

$$\left.\frac{d\hat{\mathcal{F}}}{d\hat{T}}\right|_{(5.4.2)} = (\hat{z}\hat{w})^n\sum_{m=1}^{\infty}\frac{\mu_m}{\rho^{2m}(\hat{z}\hat{w})^{m+1}}. \tag{5.4.17}$$

(5.4.17) leads (5.4.15), thus, Lemma 5.4.1 holds. □

Lemma 5.4.2. *For any positive integer m, the m-th singular point value μ_m at infinity of system (5.3.21) is antisymmetry, i.e.,*

$$\hat{\mu}_m^* = -\mu_m. \tag{5.4.18}$$

Proof. By the antisymmetry transformation

$$z = w^*, \quad w = z^*, \quad T = -T^*, \tag{5.4.19}$$

system (5.3.21) becomes

$$\begin{aligned}\frac{dz^*}{dT^*} &= (z^*)^{n+1}(w^*)^n + \sum_{k=0}^{2n} W_k(w^*, z^*) = W(w^*, z^*),\\ \frac{dw^*}{dT^*} &= -(w^*)^{n+1}(z^*)^n - \sum_{k=0}^{2n} Z_k(w, z) = -Z(w^*, z^*).\end{aligned} \tag{5.4.20}$$

For the function $\mathcal{F}(z, w)$ given by Theorem 5.3.8, let $\mathcal{F}^* = \mathcal{F}(w^*, z^*)$, then from (5.3.48), we have

$$\mathcal{F}^* = \frac{1}{z^*w^*}\left[1 + \sum_{m=1}^{\infty} \frac{f_{m(2n+1)}(w^*, z^*)}{(z^*w^*)^{m(n+1)}}\right]. \tag{5.4.21}$$

From (5.3.49), we obtain

$$\left.\frac{d\mathcal{F}^*}{dT^*}\right|_{(5.4.20)} = (z^*w^*)^n \sum_{m=1}^{\infty} \frac{(-\mu_m^*)}{(z^*w^*)^{m+1}}. \tag{5.4.22}$$

(5.4.22) follows (5.4.18). Thus, the conclusion of this lemma holds. □

We see from Lemma 5.4.1 and 5.4.2 that

Theorem 5.4.1 (The construction theorem of singular point values at infinity). *For any positive integer m, the m-th singular point value μ_m at infinity of system* (5.3.21) *can be represented as a linear combination of "$-m$" order monomial generalized rotation invariants with time exponent n and their antisymmetry forms, i.e.,*

$$\mu_m = \sum_{j=1}^{N} \gamma_{kj}(g_{kj} - g_{kj}^*), \quad k = 1,\ 2,\ \cdots, \tag{5.4.23}$$

where N is a positive integer and γ_{kj} are rational numbers, g_{kj} and g_{kj}^* are $-m$ order monomial generalized rotation invariants with time exponent n of system (5.3.21).

This theorem follows that

Theorem 5.4.2 (The extended symmetric principle at infinity). *If all elementary generalized rotation invariants g of* (5.3.21) *satisfy symmetric condition $g = g^*$, then all singular point values at infinity of system* (5.3.21) *are zero.*

Under the translational transformation

$$z' = z - z_0, \quad w' = w - w_0, \tag{5.4.24}$$

system (5.3.21) becomes

$$\frac{dz'}{dT} = (z')^{n+1}(w')^n + \sum_{\alpha+\beta=0}^{2n} a'_{\alpha\beta}(z')^\alpha(w')^\beta,$$
$$\frac{dw'}{dT} = -(w')^{n+1}(z')^n - \sum_{\alpha+\beta=0}^{2n} b'_{\alpha\beta}(w')^\alpha(z')^\beta. \tag{5.4.25}$$

For any positive integer m, the m-th singular point value at infinity of system (5.4.25) is written by μ'_m. In [Liu Y.R. etc, 2006c], the authors proved that

Theorem 5.4.3. *In the sense of the algebraic equivalence, singular point value at infinity of system* (5.3.21) *have the property of translational invariance. Namely,*

$$\{\mu'_m\} \sim \{\mu_m\}. \tag{5.4.26}$$

5.5 Singular Point Values at Infinity and Integrable Conditions for a Class of Cubic System

Consider a class of real planar cubic system

$$\frac{dx}{dt} = X_1(x,y) + X_2(x,y) + (\delta x - y)(x^2+y^2),$$
$$\frac{dy}{dt} = Y_1(x,y) + Y_2(x,y) + (x + \delta y)(x^2+y^2). \tag{5.5.1}$$

where X_k, Y_k are homogeneous polynomials of degree k in x, y, $k = 1, 2$.

When $\delta = 0$, by means of transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \tag{5.5.2}$$

system (5.5.1) can be reduced to

$$\frac{dz}{dT} = a_{10}z + a_{01}w + a_{20}z^2 + a_{11}zw + a_{02}w^2 + z^2w,$$
$$\frac{dw}{dT} = -b_{10}w - a_{01}z - b_{20}w^2 - b_{11}wz - b_{02}w^2 - w^2z. \tag{5.5.3}$$

If

$$\begin{aligned}
a_{10} &= A_{10} + iB_{10}, \quad b_{10} = A_{10} - iB_{10},\\
a_{01} &= A_{01} + iB_{01}, \quad b_{01} = A_{01} - iB_{01}\\
a_{20} &= A_{20} + iB_{20}, \quad b_{20} = A_{20} - iB_{20},\\
a_{11} &= A_{11} + iB_{11}, \quad b_{11} = A_{11} - iB_{11},\\
a_{02} &= A_{02} + iB_{02}, \quad b_{02} = A_{02} - iB_{01},
\end{aligned} \tag{5.5.4}$$

system $(5.5.1)_{\delta=0}$ can be reduced to

$$\begin{aligned}\frac{dx}{dt}=&-(B_{10}+B_{01})x-(A_{10}-A_{01})y-(B_{20}+B_{11}+B_{02})x^2\\&-2(A_{20}-A_{02})xy+(B_{20}-B_{11}+B_{02})y^2-y(x^2+y^2),\\\frac{dy}{dt}=&(A_{10}+A_{01})x-(B_{10}-B_{01})y+(A_{20}+A_{11}+A_{02})x^2\\&-2(B_{20}-B_{02})xy-(A_{20}-A_{11}+A_{20})y^2+x(x^2+y^2).\end{aligned} \tag{5.5.5}$$

By using the following generalized rotation and similar transformation with time exponent 1:

$$z=\rho e^{i\phi}\hat{z},\quad w=\rho e^{-i\phi}\hat{w},\quad T=\rho^{-2}\hat{T} \tag{5.5.6}$$

and (5.4.6), we obtain the similar exponent and rotation exponent of all $a_{\alpha\beta}$, $b_{\alpha\beta}$ as follows:

$$\begin{aligned}&I_s^{(1)}(a_{10})=I_s^{(1)}(b_{10})=-3,\quad I_r^{(1)}(a_{10})=0,\quad I_r^{(1)}(b_{10})=0,\\&I_s^{(1)}(a_{01})=I_s^{(1)}(b_{01})=-3,\quad I_r^{(1)}(b_{01})=2,\quad I_r^{(1)}(a_{01})=-2,\\&I_s^{(1)}(a_{20})=I_s^{(1)}(b_{20})=-1,\quad I_r^{(1)}(a_{20})=1,\quad I_r^{(1)}(b_{20})=-1,\\&I_s^{(1)}(a_{11})=I_s^{(1)}(b_{11})=-1,\quad I_r^{(1)}(b_{11})=1,\quad I_r^{(1)}(a_{11})=-1,\\&I_s^{(1)}(a_{02})=I_s^{(1)}(b_{02})=-1,\quad I_r^{(1)}(b_{02})=3,\quad I_r^{(1)}(a_{02})=-3.\end{aligned} \tag{5.5.7}$$

From (5.5.7), we know all elementary generalized rotation invariant of system (5.5.3). For example, since

$$I_s^{(1)}(b_{01}^3a_{02}^2)=3I_s^{(1)}(b_{01})+2I_s^{(1)}(a_{02})=-8,$$

$$I_r^{(1)}(b_{01}^3a_{02}^2)=3I_r^{(1)}(b_{01})+2I_r^{(1)}(a_{02})=0,$$

$b_{01}^3a_{02}^2$ is "$-8/2=-4$" order generalized rotation invariant. It can not be expressed as a product of two monomial generalized rotation invariants. Namely, the generalized rotation invariant is "elementary". Similarly, $(b_{01}^3a_{02}^2)^*=a_{01}^3b_{02}^2$ is also a "-4" order elementary generalized rotation invariant.

Theorem 5.5.1. *System* (5.5.3) *has exactly* 32 *elementary generalized rotation invariants with time exponent* 1 *at infinity, which are listed as follows.*

By means of transformation

$$z=\frac{\xi}{\xi^2\eta^2},\quad w=\frac{\eta}{\xi^2\eta^2},\quad \frac{dT}{d\mathcal{T}}=\xi^3\eta^3, \tag{5.5.8}$$

Degree	elementary generalized rotation invariant	number
-1	$a_{20}b_{20},\ a_{11}b_{11},\ a_{02}b_{02}$ (self-symmetry) $a_{10},\ b_{10}$ $a_{20}a_{11},\ b_{20}b_{11}$	7
-2	$a_{01}b_{01}$ (self-symmetry) $b_{01}b_{20}^2,\ b_{01}b_{20}a_{11},\ b_{01}a_{11}^2,\ b_{01}a_{20}a_{02},\ b_{01}b_{11}a_{02}$ $a_{01}a_{20}^2,\ a_{01}a_{20}b_{11},\ a_{01}b_{11}^2,\ a_{01}b_{20}b_{02},\ a_{01}a_{11}b_{02}$ $a_{20}^3a_{02},\ a_{20}^2b_{11}a_{02},\ a_{20}b_{11}^2a_{02},\ b_{11}^3a_{02}$ $b_{20}^3b_{02},\ b_{20}^2a_{11}b_{02},\ b_{20}a_{11}^2b_{02},\ a_{11}^3b_{02}$	9
-3	$b_{01}^2b_{20}a_{02},\ a_{01}^2a_{20}b_{02},\ b_{01}^2a_{11}a_{02},\ a_{01}^2b_{11}b_{02}$	4
-4	$b_{01}^3a_{02}^2,\ a_{01}^3b_{02}^2$	2

system (5.5.3) becomes a 7-th differential system with the elementary singular point as follows:

$$\begin{aligned}\frac{d\xi}{d\mathcal{T}} &= \xi + \frac{1}{3}\left[2b_{02}\xi^3 + (a_{20}+2b_{11})\xi^2\eta + (a_{11}+2b_{20})\xi\eta^2\right]\xi\\ &\quad + \frac{1}{3}\left[2b_{01}\xi^2 + (a_{10}+2b_{10})\xi\eta + a_{01}\eta^2\right]\xi^3\eta^2 = \Phi(\xi,\eta),\\ \frac{d\eta}{d\mathcal{T}} &= -\eta - \frac{1}{3}\left[2a_{02}\eta^3 + (b_{20}+2a_{11})\eta^2\xi + (b_{11}+2a_{20})\eta\xi^2\right]\eta\\ &\quad - \frac{1}{3}\left[2a_{01}\eta^2 + (b_{10}+2a_{10})\eta\xi + b_{01}\xi^2\right]\eta^3\xi^2 = -\Psi(\xi,\eta).\end{aligned} \tag{5.5.9}$$

From Theorem 5.3.6, we have

Theorem 5.5.2. *For system* (5.5.9), *one can derive successively the terms of the following formal series*

$$M(\xi,\eta) \ = \ 1 + \sum_{m=1}^{\infty}\sum_{\alpha+\beta=3m} d_{\alpha\beta}\xi^\alpha\eta^\beta, \tag{5.5.10}$$

such that

$$\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta} = \sum_{m=1}^{\infty}\frac{3m+1}{3}\lambda_m(\xi\eta)^{3m}. \tag{5.5.11}$$

In addition, for any positive integer number m,

$$\lambda_m \ \sim \ 3\mu'_{3m} \ \sim \ \mu_m, \tag{5.5.12}$$

where μ'_{3m} is the $3m$-th singular point value at the origin of system (5.5.9), *and μ_m is the m-th singular point value at infinity of system* (5.5.3).

From Theorem 5.3.7, we know that

Theorem 5.5.3. *For* (5.5.10), *letting* $d_{00} = 1$ *and taking* $d_{3k,3k} = 0$, $k = 1, 2, \cdots$, *then for all* (α, β) *and* $\alpha \neq \beta$, $d_{\alpha\beta}$ *is given by the recursive formula*

$$d_{\alpha\beta} = \frac{1}{3(\beta-\alpha)} \sum_{k+j=1}^{3} [(\alpha - 2\beta - 1)a_{k,j-1} \\ - (\beta - 2\alpha - 1)b_{j,k-1}]\, d_{\alpha+k+2j-6,\beta+j+2k-6}. \tag{5.5.13}$$

For any positive integer m, λ_m *is given by the recursive formula*

$$\lambda_m = \sum_{k+j=1}^{3} (b_{j,k-1} - a_{k,j-1}) d_{k+2j+3m-6,j+2k+3m-6}. \tag{5.5.14}$$

where for all (α, β), *when* $\alpha < 0$ *or* $\beta < 0$, *we take* $a_{\alpha\beta} = b_{\alpha\beta} = d_{\alpha\beta} = 0$.

Theorem 5.5.1 and Theorem 5.5.2 give the recursive formulas to compute directly the singular point values at the origin of system (5.5.3). By using computer algebra system *Mathematica*, we obtain the terms of the first eight singular point values at infinity of system (5.5.3) as follows:

μ_k	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9
terms	4	30	166	600	1764	4516	10378	21984	$\cdots$

We see from this table that the expressions of the singular point values are very long. We need to simplify them.

When $A_{10} = \lambda$, $A_{01} = B_{01} = B_{10} = 0$, system (5.5.5) can be reduced to

$$\begin{aligned} \frac{dx}{dt} &= -\lambda y - (B_{20} + B_{11} + B_{02})x^2 - 2(A_{20} - A_{02})xy \\ &\quad +(B_{20} - B_{11} + B_{02})y^2 - y(x^2 + y^2), \\ \frac{dy}{dt} &= \lambda x + (A_{20} + A_{11} + A_{02})x^2 - 2(B_{20} - B_{02})xy \\ &\quad -(A_{20} - A_{11} + A_{20})y^2 + x(x^2 + y^2). \end{aligned} \tag{5.5.15}$$

In [Blows etc, 1993], the author discussed the center-focus problem and the bifurcation of limit cycles at origin and infinity of system (5.5.15) where the parameters of (5.5.15) are real. We next assume that the parameters of (5.5.15) are complex.

The associated system of (5.5.15) is given by

$$\begin{aligned} \frac{dz}{dT} &= \lambda z + a_{20}z^2 + a_{11}zw + a_{02}w^2 + z^2 w, \\ \frac{dw}{dT} &= -\lambda w - b_{20}w^2 - b_{11}wz - b_{02}w^2 - w^2 z. \end{aligned} \tag{5.5.16}$$

From Theorem 5.5.1, we have

Lemma 5.5.1. *System* (5.5.16) *has exactly* 14 *elementary generalized rotation invariants which are listed as follows*

$$\begin{aligned}
&\lambda, \quad a_{20}b_{20}, \quad a_{11}b_{11}, \quad a_{02}b_{02}, \quad a_{20}a_{11}, \quad b_{20}b_{11},\\
&a_{20}^3a_{02}, \quad a_{20}^2b_{11}a_{02}, \quad a_{20}b_{11}^2a_{02}, \quad b_{11}^3a_{02},\\
&b_{20}^3b_{02}, \quad b_{20}^2a_{11}b_{02}, \quad b_{20}a_{11}^2b_{02}, \quad a_{11}^3b_{02}.
\end{aligned} \tag{5.5.17}$$

By Theorem 5.5.1 and Theorem 5.5.2, we use computer algebra system-Mathematica to calculate the first 6 singular point values at infinity of system (5.5.16) for which there exist the numbers of terms 2, 14, 64, 180, 416, 846, respectively. Simplifying them, we obtain the following theorem.

Theorem 5.5.4. *The first* 6 *singular point values at infinity of system* (5.5.16) *are given by*

$$\begin{aligned}
&\mu_1 = a_{20}a_{11} - b_{20}b_{11},\\
&\mu_2 \sim \frac{1}{3}(4I_0 - I_2),\\
&\mu_3 \sim \frac{4(2I_1 - I_2)(38\lambda - 7h_{02}) - (2I_2 - I_3)(15h_{20} + 4h_{02} - 16\lambda)}{96},\\
&\mu_4 \sim \frac{1}{1680}(2I_2 - I_3)f_4,\\
&\mu_5 \sim \frac{(2I_2 - I_3)(2085h_{20} - 1258h_{02} + 4776\lambda)}{139917048898943877120000}f_5,\\
&\mu_6 \sim \frac{1}{168872513965820052111360000}h_{02}^2(2I_2 - I_3)f_6,
\end{aligned} \tag{5.5.18}$$

where

$$\begin{aligned}
f_4 =\ & 210h_{20}h_{11} - 541h_{20}h_{02} + 18h_{11}h_{02} + 42h_{02}^2 - 168h_{02}\lambda,\\
f_5 =\ & 25303285752847200 0J_3 + 211453703933250591 9721h_{20}h_{11}\\
&+6921878766377155200h_{20}h_{02} - 5109991674897004938 00h_{11}^2\\
&-3487106223475849744 1h_{11}h_{02} + 1639708991825843200h_{02}^2\\
&+6384378367684800000h_{20}\lambda + 16213285854089676352 4h_{11}\lambda\\
&-4181742245609548800h_{02}\lambda - 9251325634682880000\lambda^2,\\
f_6 =\ & -43354482540693424129161616296852h_{20}h_{11}\\
&+48821955243299261837340424965 76h_{20}h_{02}\\
&+10838886466163652594580429391013h_{11}^2\\
&-12205739657261075689186197675 04h_{11}h_{02}\\
&+1276436235569312563200 00h_{02}^2.
\end{aligned} \tag{5.5.19}$$

In (5.5.18) *and* (5.5.19), μ_1 *and*

$$\begin{aligned} I_0 &= a_{20}^3a_{02} - b_{20}^3b_{02}, \qquad I_1 = a_{20}^2b_{11}a_{02} - b_{20}^2a_{11}b_{02}, \\ I_2 &= a_{20}b_{11}^2a_{02} - b_{20}a_{11}^2b_{02}, \quad I_3 = b_{11}^3a_{02} - a_{11}^3b_{02} \end{aligned} \tag{5.5.20}$$

are anti-symmetric generalized rotation invariants, while λ *and*

$$J_3 = b_{11}^3a_{02} + a_{11}^3b_{02}, \quad h_{20} = a_{20}b_{20}, \quad h_{11} = a_{11}b_{11}, \quad h_{02} = a_{02}b_{02} \tag{5.5.21}$$

are self-symmetric generalized rotation invariants.

Theorem 5.5.5. *For system* (5.5.16), *the first* 6 *singular point values are zero if and only if one of the following six conditions holds:*

$$\begin{aligned} &C_1: I_0 = I_1 = I_2 = I_3 = 0, \quad a_{20}a_{11} - b_{20}b_{11} = 0; \\ &C_2: 2a_{20} - b_{11} = 0, \quad 2b_{20} - a_{11} = 0; \\ &C_3: \lambda = 0, \ b_{11} = -2a_{20}, \ a_{11} = b_{20} = b_{02} = 0, \quad a_{20}a_{02} \neq 0; \\ &C_3^*: \lambda = 0, \ a_{11} = -2b_{20}, \ b_{11} = a_{20} = a_{02} = 0, \quad b_{20}b_{02} \neq 0; \\ &C_4: \lambda = b_{20} = a_{20} = 0, \quad b_{02} = 0, \quad b_{11}a_{02} \neq 0; \\ &C_4^*: \lambda = a_{20} = b_{20} = 0, \quad a_{02} = 0, \quad a_{11}b_{02} \neq 0. \end{aligned} \tag{5.5.22}$$

Proposition 5.5.1. *If one of condition* C_1 *and* C_2 *in Theorem* 5.5.5 *holds, then the infinity of system* (5.5.16) *is an complex center.*

Proof. If C_1 holds, from Lemma 5.5.1, we know that the coefficients of system (5.5.16) satisfy the condition of the extend symmetry principle. If C_2 holds, system (5.5.16) is a *Hamiltonian* system. Thus, this proposition holds. □

Proposition 5.5.2. *If one of condition* C_3 *and* C_3^* *in Theorem* 5.5.5 *is satisfied, then then infinity of system* (5.5.16) *is an complex center.*

Proof. If C_3 holds, system (5.5.16) can be reduced to

$$\frac{dz}{dT} = a_{20}z^2 + a_{02}w^2 + z^2w, \quad \frac{dw}{dT} = 2a_{20}zw - zw^2. \tag{5.5.23}$$

This system has the following integral:

$$\frac{(w - 2a_{20})^3(3z^2 + 2a_{02}w)}{w} = constant. \tag{5.5.24}$$

Thus, infinity of system (5.5.16) is an complex center.

Similarly, when C_3^* holds, the conclusion of Proposition 5.5.2 is true. □

Proposition 5.5.3. *If one of condition* C_4 *and* C_4^* *in Theorem* 5.5.5 *holds, then infinity of system* (5.5.16) *is an complex center.*

Proof. When C_4 holds, system (5.5.16) becomes

$$\frac{dz}{dT} = w(a_{11}z + a_{02}w + z^2), \quad \frac{dw}{dT} = -zw(b_{11} + w). \tag{5.5.25}$$

By means of transformation (5.5.8), system (5.5.25) can be changed into

$$\begin{aligned}\frac{d\xi}{d\mathcal{T}} &= \frac{1}{3}\xi(3 + 2b_{11}\xi^2\eta + a_{11}\xi\eta^2 + a_{02}\eta^3),\\ \frac{d\eta}{d\mathcal{T}} &= -\frac{1}{3}\eta(3 + b_{11}\xi^2\eta + 2a_{11}\xi\eta^2 + 2a_{02}\eta^3).\end{aligned} \tag{5.5.26}$$

By Theorem 1.6.7, system (5.5.26) is linearizable in a neighborhood of the origin. Thus, when C_4 holds, the conclusion of Proposition 5.5.3 holds.

Similarly, when C_4^* is satisfied, the conclusion of Proposition 5.5.3 is true. □

Propositions 5.5.1, 5.5.2, 5.5.3 and Theorem 5.5.5 follow that

Theorem 5.5.6. *Infinity of system* (5.5.16) *is an complex center, if and only if the first* 6 *singular point values are zero, i.e., one of the six conditions in Theorem* 5.5.5 *is satisfied.*

We next discuss the conditions of infinity of system (5.5.16) to be a 6-*order* weak singular point. From Theorem 5.5.4, we have

Theorem 5.5.7. *For system* (5.5.16)*, infinity is a* 6-order *weak singular point if and only if one the following conditions holds:*

$$\begin{aligned}&C_5: \quad \Big\{ a_{11} + 2b_{20} = b_{11} + 2a_{20} = 0,\ \lambda = \frac{1}{960}(99 \mp \sqrt{2761})h_{20},\ h_{02} \neq 0,\\ &h_{11} = \frac{1}{60}(61 \pm \sqrt{2761})h_{02},\ J_3 = -\frac{12638443 \pm 238497\sqrt{2761}}{378000}h_{02}^2;\\ &C_6: \quad a_{02}b_{02} \neq 0,\ \lambda = \frac{1}{4}a_{02}b_{02},\ a_{11} = 0, \quad 63b_{11}^3 + 4a_{02}b_{02}^2 = 0;\\ &C_6^*: \quad a_{02}b_{02} \neq 0,\ \lambda = \frac{1}{4}a_{02}b_{02},\ b_{11} = 0, \quad 63a_{11}^3 + 4b_{02}a_{02}^2 = 0.\end{aligned} \tag{5.5.27}$$

Theorem 5.5.8. *If x, y, t are real variables and all the coefficients of system* (5.5.15) *are real, then it is impossible that infinity of system* (5.5.15) *is a* 6-th *weak focus.*

Proof. From the conditions given in Theorem 5.5.8, we have

$$b_{20} = \bar{a}_{20}, \quad b_{11} = \bar{a}_{11}, \quad b_{02} = \bar{a}_{02}. \tag{5.5.28}$$

By Theorem 5.5.7, we only need to prove that when one of conditions of C_5, C_6 and C_6^* holds, it is impossible that (5.5.28) is satisfied.

In fact, if condition C_5 holds, then

$$\frac{J_3^2 - 4h_{11}^3 h_{02}}{h_{02}^4} = \frac{157420332562049 \pm 2995809288171\sqrt{2761}}{71442000000} > 0. \qquad (5.5.29)$$

On the other hand, from (5.5.20) and (5.5.21), we know that when (5.5.28) is satisfied, I_3 is a pure imaginary. Then, $I_3^2 = J_3^2 - 4h_{11}^3 h_{02} < 0$. It implies that if condition C_5 is satisfied, then (5.5.28) does not hold.

In addition, it is easy to see that when one of condition C_6 and C_6^* holds, we have $a_{11}b_{11} = 0$, but not all a_{11} and b_{11} are zero. Thus, if one of condition C_6 and C_6^* is satisfied, then (5.5.28) does not hold. □

5.6 Bifurcation of Limit Cycles at Infinity

Consider the following perturbed system of (5.5.1) with two small parament ε, δ

$$\frac{dx}{dt} = \sum_{k=0}^{2n+1} X_k(x, y, \varepsilon, \delta), \quad \frac{dy}{dt} = \sum_{k=0}^{2n+1} Y_k(x, y, \varepsilon, \delta), \qquad (5.6.1)$$

where $X_k(x, y, \varepsilon, \delta), Y_k(x, y, \varepsilon, \delta)$ are homogeneous polynomials of degree k in x, y, and the coefficients are power series in ε, δ having nonzero convergent radius. Assume that there is an integer d, such that

$$xY_{2n+1}(x, y.0, 0) - yX_{2n+1}(x, y, 0, 0) \geqslant d(x^2 + y^2)^{n+1} \qquad (5.6.2)$$

and

$$\int_0^{2\pi} \frac{\cos\theta X_{2n+1}(\cos\theta, \sin\theta, 0, 0) + \sin\theta Y_{2n+1}(\cos\theta, \sin\theta, 0, 0)}{\cos\theta Y_{2n+1}(\cos\theta, \sin\theta, 0, 0) - \sin\theta X_{2n+1}(\cos\theta, \sin\theta, 0, 0)} d\theta = 0. \qquad (5.6.3)$$

By means of transformation (5.1.5), system (5.6.1) can be changed into

$$\begin{aligned}\frac{dr}{d\theta} &= -r\frac{\varphi_{2n+2}(\theta, \varepsilon, \delta) + \displaystyle\sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta, \varepsilon, \delta) r^k}{\psi_{2n+2}(\theta, \varepsilon, \delta) + \displaystyle\sum_{k=1}^{2n+1} \psi_{2n+2-k}(\theta, \varepsilon, \delta) r^k} \\ &= \frac{-\varphi_{2n+2}(\theta, \varepsilon, \delta)}{\psi_{2n+2}(\theta, \varepsilon, \delta)} r + o(r),\end{aligned} \qquad (5.6.4)$$

where

$$\begin{aligned}\varphi_k(\theta, \varepsilon, \delta) &= \cos\theta X_{k-1}(\cos\theta, \sin\theta, \varepsilon, \delta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta, \varepsilon, \delta), \\ \psi_k(\theta, \varepsilon, \delta) &= \cos\theta Y_{k-1}(\cos\theta, \sin\theta, \varepsilon, \delta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta, \varepsilon, \delta).\end{aligned} \qquad (5.6.5)$$

For a sufficiently small h, we write the solution of (5.6.4) with the initial condition $r|_{\theta=0} = h$ and the Poincaré succession function respectively as follows:

$$r = \tilde{r}(\theta, h, \varepsilon, \delta) = \sum_{k=1}^{\infty} \nu_k(\theta, \varepsilon, \delta) h^k,$$
$$\Delta(h, \varepsilon, \delta) = \tilde{r}(2\pi, h, \varepsilon, \delta) - h. \tag{5.6.6}$$

Clearly, $\nu_1(\theta)$ can be expressed as

$$\nu_1(\theta, \varepsilon, \delta) = \exp \int_0^{\theta} \frac{-\varphi_{2n+2}(\vartheta, \varepsilon, \delta)\, d\vartheta}{\psi_{2n+2}(\vartheta, \varepsilon, \delta)}. \tag{5.6.7}$$

From (5.6.3) and (5.6.7), we have

$$\nu_1(2\pi, 0, 0) = 1. \tag{5.6.8}$$

Particularly, if $X_{2n+1},\ Y_{2n+1}$ are given by (5.2.1), then $\nu_1(\theta, \varepsilon, \delta) = e^{-\delta\theta}$.

Obviously, equation (5.6.4) is a particular case of equation (4.1.7).

If $\delta = \delta(\varepsilon)$ in the right hand of system (5.6.1) is a power series of ε having nonzero convergent radius, and $\delta(0) = 0$, we can obtain a quasi succession function $L(h, \varepsilon)$ by computing the focal values at infinity. The method mentioned in Chapter 3 can be used to study the bifurcation of limit cycles in a neighborhood of infinity of system (5.6.1). For an example, we discuss a class of real planar cubic system

$$\frac{dx}{dt} = (\delta x - y)(x^2 + y^2) + X_2(x, y),$$
$$\frac{dy}{dt} = (x + \delta y)(x^2 + y^2) + Y_2(x, y), \tag{5.6.9}$$

where $X_2(x, y),\ Y_2(x, y)$ are homogeneous polynomials of degree 2 in x, y. By means of transformation (5.1.5), system (5.6.9) can be changed into

$$\frac{dr}{d\theta} = -r \frac{\delta + [\cos\theta X_2(\cos\theta, \sin\theta) + \sin\theta Y_2(\cos\theta, \sin\theta)] r}{1 + [\cos\theta Y_2(\cos\theta, \sin\theta) - \sin\theta X_2(\cos\theta, \sin\theta)] r}. \tag{5.6.10}$$

It is interesting that under the polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, real planar quadratic system (4.4.1) can reduce to

$$\frac{dr}{d\theta} = r \frac{\delta + [\cos\theta X_2(\cos\theta, \sin\theta) + \sin\theta Y_2(\cos\theta, \sin\theta)] r}{1 + [\cos\theta Y_2(\cos\theta, \sin\theta) - \sin\theta X_2(\cos\theta, \sin\theta)] r}. \tag{5.6.11}$$

The vector fields defined by (5.5.10) and (5.6.11) are opposite oriented. Therefore, we can use known conclusions of the center-focus problem and bifurcations of limit cycles for the quadratic system.

By means of transformation $z = x + iy,\ w = x - iy,\ T = it,\ i = \sqrt{-1}$, system (5.6.9) reduce to

$$\begin{aligned}\frac{dz}{dT} &= (1 - i\delta)z^2w + a_{20}z^2 + a_{11}zw + a_{02}w^2,\\ \frac{dw}{dT} &= -(1 + i\delta)w^2z - b_{20}w^2 - b_{11}wz - b_{02}z^2.\end{aligned} \tag{5.6.12}$$

where the relations between $a_{\alpha\beta}, b_{\alpha\beta}$ and the coefficients of $X_2(x, y), Y_2(x, y)$ are given by (3.4.3) and (3.4.4).

Obviously, system $(5.6.12)_{\delta=0}$ is the case of $\lambda = 0$ of system (5.5.16), namely it is the particular case of the system (5.5.3) with $a_{10} = a_{01} = b_{10} = b_{01}$. From Theorem 5.5.1, we have

Lemma 5.6.1. *When* $\delta = 0$, *system* (5.6.12) *has exactly* 13 *elementary generalized rotation invariants with time exponent* 1 *at infinity, which are listed as follows:*

degree	generalized rotation invariant
-1	$a_{20}b_{20},\ a_{11}b_{11},\ a_{02}b_{02}$ (self-symmetry) $a_{20}a_{11},\ b_{20}b_{11}$
-2	$a_{20}^3a_{02},\ a_{20}^2b_{11}a_{02},\ a_{20}b_{11}^2a_{02},\ b_{11}^3a_{02}$ $b_{20}^3b_{02},\ b_{20}^2a_{11}b_{02},\ b_{20}a_{11}^2b_{02},\ a_{11}^3b_{02}$

Remark 5.6.1. *From Corollary* 2.5.1 *and Lemma* 5.6.1, *system* $(5.6.12)_{\delta=0}$ *and the quadric system* 2.5.4 *have the same elementary generalized rotation invariants. But, the order of the same generalized rotation invariant in the two systems are difference as the time exponents are difference.*

By computing the singular point values at infinity of system (5.6.12) and simplifying them, we have

Theorem 5.6.1. *When* $\delta = 0$, *the first* 4 *singular point values at infinity of system* (5.6.12) *are listed as follows*:

$$\begin{aligned}\mu_1 &= a_{20}a_{11} - b_{20}b_{11},\\ \mu_2 &\sim \frac{1}{3}(4I_0 - I_1),\\ \mu_3 &\sim \frac{1}{48}(2a_{02}b_{02} - 3a_{20}b_{20})(-14I_1 + 5I_2 + I_3),\\ \mu_4 &\sim \frac{1}{1800}a_{02}^2b_{02}^2(-364I_1 + 220I_2 - 19I_3),\end{aligned} \tag{5.6.13}$$

where I_k *are given by* (5.5.20).

From Theorem 5.6.1, we obtain

Theorem 5.6.2. *When $\delta = 0$, the first 4 singular point values at infinity of system* (5.6.12) *are zero if and only if one of the following two conditions holds:*

$$a_{20}a_{11} - b_{20}b_{11} = I_0 = I_1 = I_2 = I_3 = 0, \tag{5.6.14}$$

$$b_{11} = 2a_{20}, \quad a_{11} = 2b_{20}. \tag{5.6.15}$$

From Lemma 5.6.1, we have

Theorem 5.6.3. *When $\delta = 0$, if* (5.6.14) *holds, the coefficients of system* (5.6.12) *give rise to the condition of the extend symmetric principle. While if* (5.6.15) *holds, system* (5.6.12) *is a Hamiltonian system.*

Theorem 5.6.1, 5.6.2 and 5.6.3 follow that

Theorem 5.6.4. *When $\delta = 0$, infinity of system* (5.6.12) *is a complex center if and only if $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, namely, one of the two conditions in Theorem* 5.6.3 *holds.*

We next discuss the bifurcation of limit cycles created from infinity of system (5.6.9). Assume that $a_{\alpha\beta} = A_{\alpha\beta} + iB_{\alpha\beta}$, $b_{\alpha\beta} = A_{\alpha\beta} - iB_{\alpha\beta}$ and

$$\begin{aligned} &\delta = \frac{1}{2}\varepsilon^{10+N}, \\ &A_{20} = -1 - \frac{33}{40}\varepsilon^3, \qquad B_{20} = \frac{-1}{4}\varepsilon^{6+N}, \\ &A_{11} = 2, \qquad B_{11} = 0, \\ &A_{02} = \frac{\sqrt{18150 - 15972\varepsilon - 625\varepsilon^{2N}}}{110}, \quad B_{02} = \frac{-5}{22}\varepsilon^N. \end{aligned} \tag{5.6.16}$$

From Theorem 5.6.1, we have

Lemma 5.6.2. *When* (5.6.16) *holds, for infinity of system* (5.6.9)*, we have*

$$\begin{aligned} &\nu_1(2\pi, \varepsilon, \delta(\varepsilon)) - 1 = -\pi\varepsilon^{10+N} + o(\varepsilon^{10+N}), \\ &\nu_3(2\pi, \varepsilon, \delta(\varepsilon)) \simeq \pi\varepsilon^{6+N}, \quad \nu_5(2\pi, \varepsilon, \delta(\varepsilon)) \simeq -\pi\varepsilon^{3+N}, \\ &\nu_7(2\pi, \varepsilon, \delta(\varepsilon)) \simeq \pi\varepsilon^{1+N}, \quad \nu_9(2\pi, \varepsilon, \delta(\varepsilon)) \simeq -\pi\varepsilon^{N}. \end{aligned} \tag{5.6.17}$$

By Lemma 5.6.2, if (5.6.16) is satisfied, then when $\varepsilon = 0$, if $N = 0$, infinity of system (5.6.9) is a 4-order weak focus. If $N > 0$, infinity of system (5.6.9) is a center, for which the vector field is symmetric with respect to $x-$axis. To obtain the quasi succession function at infinity, we need to prove the following conclusions.

Lemma 5.6.3. *If* (5.6.16) *holds, for any positive integer $k > 4$, the k-th focal value at infinity of system* (5.6.9) *satisfies the following formula*

$$\nu_{2k+1}(2\pi, \varepsilon, \delta(\varepsilon)) = O(\varepsilon^N). \tag{5.6.18}$$

Proof. When (5.6.16) holds, it is easy to prove that

$$\begin{aligned} &a_{20}a_{11} - b_{20}b_{11} = O(\varepsilon^N), \\ &I_0 = O(\varepsilon^N), \quad I_1 = O(\varepsilon^N), \\ &I_2 = O(\varepsilon^N), \quad I_3 = O(\varepsilon^N). \end{aligned} \tag{5.6.19}$$

Thus, Lemma 5.6.1 and the construction theorem of singular point values at infinity (Theorem 5.4.1) lead to the conclusion of this Lemma. □

From Lemma 5.6.2 and 5.6.3, we have

Lemma 5.6.4. *When* (5.6.16) *holds, the quasi succession function at infinity of system* (5.6.9) *is given by*

$$L(h, \varepsilon) = -\pi(\varepsilon^{10} - \varepsilon^6 h^2 + \varepsilon^3 h^4 - \varepsilon h^6 + h^8). \tag{5.6.20}$$

From Theorem 3.3.4 and Lemma 5.6.4, we obtain

Theorem 5.6.5. *If* (5.6.16) *holds, then when* $0 < \varepsilon \ll 1$*, there exist at least* 4 *limit cycles in a sufficiently small neighborhood of infinity of system* (5.6.9)*, which are close to the circles* $(x^2 + y^2)^{-1} = \varepsilon^k$*,* $k = 1,\ 2,\ 3,\ 4$.

5.7 Isochronous Centers at Infinity of a Polynomial Systems

In this section, we extended the definition of the isochronous center to the case of infinity for a class of polynomial systems.

We consider the following real system:

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{(x^2 + y^2)^n} \sum_{k=0}^{2n} X_k(x, y) - y, \\ \frac{dy}{dt} &= \frac{1}{(x^2 + y^2)^n} \sum_{k=0}^{2n} Y_k(x, y) + x, \end{aligned} \tag{5.7.1}$$

where n is a positive integer, and $X_k(x, y), Y_k(x, y)$ are homogeneous polynomials of degree k.

By a time rescaling $t \rightharpoonup (x^2+y^2)^n t$, system (5.7.1) becomes the system $(5.2.2)_{\delta=0}$.

Definition 5.7.1. *For system* (5.7.1)*, infinity is called an isochronous center, if it is a center and the period of all periodic solutions in a neighborhood of infinity is the same constant.*

By using the transformation

$$x = \frac{u}{(u^2 + v^2)^{n+1}}, \quad y = \frac{v}{(u^2 + v^2)^{n+1}}, \tag{5.7.2}$$

system (5.7.1) can be reduced to the system $(5.2.13)_{\delta=0}$. We immediately have the following conclusion.

Theorem 5.7.1. *Infinity of system* (5.7.1) *is a center* (*isochronous center*) *if and only if the origin of system* $(5.2.13)_{\delta=0}$ (*or system* $(5.3.17)_{\delta=0}$) *is a center* (*isochronous center*).

Definition 5.7.2. *Infinity of system* (5.7.1) *is called a complex isochronous center, if the origin of system* $(5.2.13)_{\delta=0}$ (*or system* $(5.3.17)_{\delta=0}$) *is a complex isochronous center.*

Infinity of system $(5.5.1)_{\delta=0}$ *is called a complex quasi isochronous center, if infinity of system* (5.7.1) *is a complex isochronous center.*

By Theorem 5.7.1 and the above knowledge, to determine center conditions and isochronous center conditions at infinity of system (5.7.1), we only need to consider the the computation problem of singular point values and period constants at the origin for system $(5.3.17)_{\delta=0}$.

To explain the mentioned idea, we consider the following real rational system

$$\frac{dx}{dt} = -y + \frac{X_3(x,y)}{(x^2+y^2)^2}, \quad \frac{dy}{dt} = x + \frac{Y_3(x,y)}{(x^2+y^2)^2}, \tag{5.7.3}$$

where $X_3(x,y)$, $Y_3(x,y)$ are homogeneous polynomials of degree 3 in x, y. The associated system of system (5.7.3) has the form

$$\begin{aligned}\frac{dz}{dT} &= z + \frac{1}{z^2w^2}(a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3),\\ \frac{dw}{dT} &= -w - \frac{1}{z^2w^2}(b_{30}w^3 + b_{21}w^2z + b_{12})wz^2 + b_{03}z^3).\end{aligned} \tag{5.7.4}$$

By means of transformations

$$z = \frac{\xi}{\xi^3\eta^3}, \quad w = \frac{\eta}{\xi^3\eta^3}, \tag{5.7.5}$$

system (5.7.4) becomes

$$\begin{aligned}\frac{d\xi}{dT} =&\, \xi + \frac{3}{5}b_{03}\xi^8\eta^3 + \frac{1}{5}(2a_{30}+3b_{12})\xi^7\eta^4 + \frac{1}{5}(2a_{21}+3b_{21})\xi^6\eta^5\\ &+\frac{1}{5}(2a_{12}+3b_{30})\xi^5\eta^6 + \frac{2}{5}a_{03}\xi^4\eta^7,\\ \frac{d\eta}{dT} =&\, -\eta - \frac{3}{5}a_{03}\eta^8\xi^3 - \frac{1}{5}(2b_{30}+3a_{12})\eta^7\xi^4 - \frac{1}{5}(2b_{21}+3a_{21})\eta^6\xi^5\\ &-\frac{1}{5}(2b_{12}+3a_{30})\eta^5\xi^6 - \frac{2}{5}b_{03}\eta^4\xi^7.\end{aligned} \tag{5.7.6}$$

5.7.1 Conditions of Complex Center for System (5.7.6)

First, we discuss the center conditions. By using Theorem 2.3.6 to compute the singular point values at origin of system (5.7.6) and simplify them, we obtain

Theorem 5.7.2. *The first* 35 *singular point values at the origin of system* (5.7.6) *are given by*

$$\begin{aligned}
\mu_5 &= \frac{1}{5}(-a_{21}+b_{21}),\\
\mu_{10} &\sim \frac{1}{5}(a_{12}a_{30}-b_{12}b_{30}),\\
\mu_{15} &\sim \frac{1}{40}(-9a_{03}a_{30}^2-a_{12}^2b_{03}+a_{03}b_{12}^2+9b_{03}b_{30}^2),\\
\mu_{20} &\sim \frac{1}{20}(a_{21}+b_{21})(3a_{03}a_{30}^2-a_{03}a_{30}b_{12}+a_{12}b_{03}b_{30}-3b_{03}b_{30}^2),\\
\mu_{25} &\sim \frac{1}{120}(16a_{30}b_{30}-3a_{03}b_{03})(3a_{03}a_{30}^2-a_{03}a_{30}b_{12}+a_{12}b_{03}b_{30}-3b_{03}b_{30}^2),\\
\mu_{30} &\sim 0,\\
\mu_{35} &\sim -\frac{88}{675}a_{30}^2b_{30}^2(3a_{03}a_{30}^2-a_{03}a_{30}b_{12}+a_{12}b_{03}b_{30}-3b_{03}b_{30}^2).
\end{aligned} \tag{5.7.7}$$

We know from Theorem 5.7.2 that

Theorem 5.7.3. *For system* (5.7.6)*, the first* 35 *singular point values are zero if and only if one of the following four conditions holds:*

$$\begin{aligned}
&C_1: \quad a_{21}=b_{21},\ a_{12}=3b_{30},\ b_{12}=3a_{30},\\
&C_2: \quad \begin{cases} a_{21}=b_{21},\ |3a_{30}-b_{12}|+|3b_{30}-a_{12}|\neq 0,\ a_{12}a_{30}=b_{12}b_{30},\\ a_{30}^2a_{03}=b_{30}^2b_{03},\ a_{30}b_{12}a_{03}=b_{30}a_{12}b_{03},\ b_{12}^2a_{03}=a_{12}^2b_{03},\end{cases}\\
&C_3: \quad a_{21}=b_{21}=a_{30}=b_{12}=a_{03}=0,\ a_{12}=-3b_{30},\\
&C_3^*: \quad a_{21}=b_{21}=b_{30}=a_{12}=b_{03}=0,\ b_{12}=-3a_{30}.
\end{aligned} \tag{5.7.8}$$

We next discuss the conditions that the origin is a complex center.

(1) When condition C_1 holds, there exist a constant s, such that $a_{21}-b_{21}=s$. Thus system (5.7.6) becomes

$$\begin{aligned}
\frac{d\xi}{dT} &= \xi+\frac{3}{5}b_{03}\xi^8\eta^3+\frac{11}{5}a_{30}\xi^7\eta^4+s\xi^6\eta^5+\frac{9}{5}b_{30}\xi^5\eta^6+\frac{2}{5}a_{03}\xi^4\eta^7,\\
\frac{d\eta}{dT} &= -\eta-\frac{3}{5}a_{03}\eta^8\xi^3-\frac{11}{5}b_{30}\eta^7\xi^4-s\eta^6\xi^5-\frac{9}{5}a_{30}\eta^5\xi^6-\frac{2}{5}b_{03}\eta^4\xi^7.
\end{aligned} \tag{5.7.9}$$

System (5.7.9) has an analytic first integral

$$F=\frac{\xi^{15}\eta^{15}}{4+3b_{03}\xi^7\eta^3+12a_{30}\xi^6\eta^4+6s\xi^5\eta^5+12b_{30}\xi^4\eta^6+3a_{03}\xi^3\eta^7}. \tag{5.7.10}$$

Thus, the origin of system (5.7.9) is a complex center.

(2) When condition C_2 holds, we denote that $3a_{30}-b_{12}=\beta,\ 3b_{30}-a_{12}=\alpha$, then there exist complex p,q,s, such that

$$a_{30}=p\beta,\quad b_{30}=p\alpha,\quad a_{03}=q\alpha^2,\quad b_{03}=q\beta^2,\quad a_{21}=b_{21}=s. \tag{5.7.11}$$

Thus, system (5.7.6) becomes

$$\begin{aligned}\frac{d\xi}{dT} =& \xi + \frac{3}{5}q\beta^2\xi^8\eta^3 + \frac{1}{5}(11p-3)\beta\xi^7\eta^4\\ &+s\xi^6\eta^5 + \frac{1}{5}(9p-2)\alpha\xi^5\eta^6 - \frac{2}{5}q\alpha^2\xi^4\eta^7,\\ \frac{d\eta}{dT} =& -\eta - \frac{3}{5}q\alpha^2\eta^8\xi^3 - \frac{1}{5}(11p-3)\alpha\eta^7\xi^4\\ &-s\eta^6\xi^5 - \frac{1}{5}(9p-2)\beta\eta^5\xi^6 - \frac{2}{5}q\beta^2\eta^4\xi^7.\end{aligned} \tag{5.7.12}$$

For system (5.7.12), the conditions of the extended symmetric principle are satisfied. Thus the origin of system (5.7.12) is a complex center.

(3) When one of conditions C_3 and C_3^* holds, we have

Proposition 5.7.1. *If one of conditions C_3 and C_3^* holds, then the origin of system* (5.7.6) *is a complex isochronous center.*

Proof. When conditions C_3 holds, system (5.7.6) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \frac{3}{5}b_{03}\xi^8\eta^3 - \frac{3}{5}b_{30}\xi^5\eta^6,\\ \frac{d\eta}{dT} &= -\eta + \frac{7}{5}b_{30}\eta^7\xi^4 - \frac{2}{5}b_{03}\eta^4\xi^7.\end{aligned} \tag{5.7.13}$$

System (5.7.13) is linearizable by using the transformation

$$u = \xi\frac{(1-3b_{30}\xi^4\eta^6)^{\frac{1}{10}}}{\left(1+\frac{3}{4}b_{03}\xi^7\eta^3\right)^{\frac{1}{5}}}, \quad v = \eta\frac{\left(1+\frac{3}{4}b_{03}\xi^7\eta^3\right)^{\frac{2}{15}}}{(1-3b_{30}\xi^4\eta^6)^{\frac{7}{30}}}. \tag{5.7.14}$$

Thus the origin of system (5.7.14) is a complex isochronous center.

Similarly, if Condition C_3^* holds, the origin of system (5.7.6) is also a complex isochronous center □

To sum up, we have

Theorem 5.7.4. *All singular point values of the origin of* (5.7.6) *are zero if and only if the first* 35 *singular point values of the origin are zero, i.e., one of the four conditions in Theorem* 5.7.3 *holds.*

Theorem 5.7.5. *For system* (5.7.1)*, infinity is a complex center if and only if one of the four conditions in Theorem* 5.7.3 *holds.*

5.7.2 Conditions of Complex Isochronous Center for System (5.7.6)

We next discuss the isochronous center conditions.

(1) When condition C_1 in Theorem 5.7.3 holds, we have

Proposition 5.7.2. *The first* 20 *period constants of the origin of system* (5.7.9) *are given by*

$$\begin{aligned}&\tau_5 = 2s, \quad \tau_{10} \sim -\frac{1}{2}(a_{03}b_{03} + 16a_{30}b_{30}),\\&\tau_{15} \sim 0, \quad \tau_{20} \sim -80a_{30}^2b_{30}^2.\end{aligned} \tag{5.7.15}$$

From Proposition 5.7.2, we have

Proposition 5.7.3. *The first* 20 *period constants of the origin of system* (5.7.9) *are all zeros, if and only if one of the following conditions holds:*

$$\begin{aligned}C_{11}:&\quad s = 0,\ a_{30} = 0,\ a_{03} = 0;\\C_{11}^*:&\quad s = 0,\ b_{30} = 0,\ b_{03} = 0;\\C_{21}:&\quad s = 0,\ a_{30} = 0,\ b_{03} = 0;\\C_{21}^*:&\quad s = 0,\ b_{30} = 0,\ a_{03} = 0.\end{aligned} \tag{5.7.16}$$

Proposition 5.7.4. *If one of condition* C_{11} *and* C_{11}^* *holds, then the origin of* (5.7.9) *is a complex isochronous center.*

Proof. When condition C_{11} holds, system (5.7.9) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \frac{3}{5}b_{03}\xi^8\eta^3 + \frac{9}{5}b_{30}\xi^5\eta^6,\\\frac{d\eta}{dT} &= -\eta - \frac{2}{5}b_{03}\xi^7\eta^4 - \frac{11}{5}b_{30}\xi^4\eta^7.\end{aligned} \tag{5.7.17}$$

System (5.7.17) is linearizable by using the transformation

$$\begin{aligned}u &= \frac{\xi\sqrt{1 + 3b_{30}\xi^4\eta^6}}{(1 + 3b_{30}\xi^4\eta^6 + \frac{3}{4}b_{03}\xi^7\eta^3)^{\frac{1}{5}}},\\v &= \frac{\eta(1 + 3b_{30}\xi^4\eta^6 + \frac{3}{4}b_{03}\xi^7\eta^3)^{\frac{2}{15}}}{\sqrt{1 + 3b_{30}\xi^4\eta^6}}.\end{aligned} \tag{5.7.18}$$

Thus, the origin of (5.7.17) is a complex isochronous center.

If Condition C_{11}^* is satisfied, then by using the same method as the above, we know that the origin of system (5.7.9) is also a complex isochronous center. □

Proposition 5.7.5. *If one of condition* C_{12} *and* C_{12}^* *holds, then the origin of* (5.7.9) *is a complex isochronous center.*

Proof. When condition C_{12} holds, system (5.7.9) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \frac{9}{5}b_{30}\xi^5\eta^6 + \frac{2}{5}a_{03}\xi^4\eta^7,\\ \frac{d\eta}{dT} &= -\eta - \frac{11}{5}b_{30}\xi^4\eta^7 - \frac{3}{5}a_{03}\xi^3\eta^8.\end{aligned} \tag{5.7.19}$$

By using the transformation

$$u = \xi^4\eta^6, \quad v = \xi^3\eta^7, \tag{5.7.20}$$

system (5.7.20) becomes

$$\frac{du}{dT} = -2u(1 + 3b_{30}u + a_{03}v), \quad \frac{dv}{dT} = -v(4 + 10b_{30}u + 3a_{03}v). \tag{5.7.21}$$

The origin of system (5.7.21) is an integer-ratio node, similar to the proof of Theorem 1.6.1, we can prove that the origin of (5.7.19) is a complex isochronous center.

Similarly, when condition C_{12}^* holds, the origin of (5.7.9) is also a complex isochronous center. □

(2) When condition C_2 in Theorem 5.7.3 holds, we have

Proposition 5.7.6. *The first* 25 *period constants of the origin of system* (5.7.12) *are given by*

$$\begin{aligned}&\tau_5 = 2s, \quad \tau_{10} \sim -\frac{1}{2}\alpha\beta(-4p + 16p^2 + \alpha\beta q^2),\\ &\tau_{15} \sim \frac{1}{4}\alpha^2\beta^2 q(6p - 1), \quad \tau_{20} \sim -\frac{1}{48}\alpha^3\beta^3 q^2.\end{aligned} \tag{5.7.22}$$

From Proposition 5.7.6, we have

Proposition 5.7.7. *The first* 20 *period constants of the origin of system* (5.7.12) *are all zeros, if and only if one of the following conditions holds:*

$$\begin{aligned}&C_{21}: \quad s = 0,\ q = 0,\ p = 0;\\ &C_{22}: \quad s = 0,\ q = 0,\ p = \frac{1}{4};\\ &C_{23}: \quad s = 0,\ \alpha = 0;\\ &C_{23}^*: \quad s = 0,\ \beta = 0.\end{aligned} \tag{5.7.23}$$

Proposition 5.7.8. *If condition C_{21} holds, then the origin of* (5.7.12) *is a complex isochronous center.*

Proof. When condition C_{21} holds, system (5.7.12) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi - \frac{3}{5}\beta\xi^7\eta^4 - \frac{2}{5}\alpha\xi^5\eta^6,\\ \frac{d\eta}{dT} &= -\eta + \frac{3}{5}\alpha\eta^7\xi^4 + \frac{2}{5}\beta\eta^5\xi^6.\end{aligned} \tag{5.7.24}$$

System (5.7.16) is linearizable by using the transformation

$$\xi = \frac{\xi(1-\alpha\xi^4\eta^6)^{\frac{1}{5}}}{(1-\beta\xi^6\eta^4)^{\frac{3}{10}}}, \quad \eta = \frac{\eta(1-\beta\xi^6\eta^4)^{\frac{1}{5}}}{(1-\alpha\xi^4\eta^6)^{\frac{3}{10}}}. \tag{5.7.25}$$

Thus, the origin of system (5.7.24) is an complex isochronous center. □

Proposition 5.7.9. *If condition C_{22} holds, then the origin of* (5.7.12) *is a complex isochronous center.*

Proof. When condition C_{22} holds, system (5.7.12) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi - \frac{1}{20}\beta\xi^7\eta^4, + \frac{1}{20}\alpha\xi^5\eta^6\\ \frac{d\eta}{dT} &= -\eta + \frac{1}{20}\alpha\eta^7\xi^4 - \frac{1}{20}\beta\eta^5\xi^6.\end{aligned} \tag{5.7.26}$$

By using the polar coordinates $\xi = \rho e^{i\theta}, \eta = \rho e^{-i\theta}$ and $T = it$, we have $\dfrac{d\theta}{dt} \equiv 1$. System (5.7.26) has an isochronous center at the origin. □

Proposition 5.7.10. *If one of condition C_{23} and C_{23}^* holds, then the origin of* (5.7.12) *is a complex isochronous center.*

Proof. When condition C_{23} holds, system (5.7.12) becomes

$$\begin{aligned}\frac{d\xi}{dT} &= \xi + \frac{3}{5}\beta^2 q\xi^8\eta^3 + \frac{1}{5}\beta(11p-3)\xi^7\eta^4,\\ \frac{d\eta}{dT} &= -\eta - \frac{2}{5}\beta^2 q\xi^7\eta^4 - \frac{1}{5}\beta(9p-2)\xi^6\eta^5.\end{aligned} \tag{5.7.27}$$

Letting

$$u = \xi^7\eta^3, \quad v = \xi^6\eta^4, \tag{5.7.28}$$

system (5.7.27) becomes

$$\frac{du}{dT} = 4u + 3\beta^2 qu^2 + \beta(10p-3)uv, \quad \frac{dv}{dt} = 2v + 2\beta^2 quv + 2\beta(3p-1)v^2. \tag{5.7.29}$$

The origin of system (5.7.29) is an integer-ratio node, similar to the proof of Theorem 1.6.1, we can prove that the origin of (5.7.29) is a complex isochronous center.

Similarly, when condition C_{23}^* holds, the origin of (5.7.12) is also a complex isochronous center. □

(3) Finally, when one of conditions C_3 and C_3^* in Theorem 5.7.3 holds, according to Proposition 5.7.1, the origin of system (5.7.6) is a complex isochronous center.

The problem of a complex isochronous center for the infinity of system (5.7.3) and (5.7.4) are already solved completely in this section.

The problem of a complex quasi isochronous center for the infinity of system

$$\frac{dx}{dt} = -y(x^2+y^2)^2 + X_3(x,y), \quad \frac{dy}{dt} = x(x^2+y^2)^2 + Y_3(x,y) \tag{5.7.30}$$

are also already solved completely.

Bibliographical Notes

The center-focus problem at infinity and the bifurcation of limit cycles created from Γ_∞ are essentially difficult problems. There are only a few results on the study for these problems before 2000 year (see [Rujenok, 1987; Blows etc, 1993; Han M.A., 1993] et al). Recent years, many papers have been published (see [Liu Y.R. etc, 2003b; Chen H.B. etc, 2002; Liu Y.R. etc, 2002a; Chen H.B. etc, 2003; Huang W.T. etc, 2004b; Zhang L. etc, 2006; Zhang Q. etc, 2006a; Liu Y.R. etc, 2006a; Liu Y.R. etc, 2006c; Zhang Q. etc, 2006b; Wang Q. L. etc, 2007; Huang W.T. etc, 2007] et al.)

The materials of this chapter are taken from [Liu Y.R., 2001] and [Liu Y.R. etc, 2004].

Chapter 6

Theory of Center-Focus and Bifurcations of Limit Cycles for a Class of Multiple Singular Points

For a multiple singular point, the criteria of center-focus is very difficult. There is no general theory for center-focus problem. Only on some particular cases, a multiple singular points can be changed to an elementary singular points by means of suitable transformations, we can treat the criteria problem.

In this chapter, we introduce the theory of center-focus for a class of multiple singular points.

6.1 Succession Function and Focal Values for a Class of Multiple Singular Points

Consider the following real planar system:

$$\frac{dx}{dt} = \sum_{k=2n+1}^{\infty} X_k(x,y), \quad \frac{dy}{dt} = \sum_{k=2n+1}^{\infty} Y_k(x,y), \tag{6.1.1}$$

where n is a positive integer, and $X_k(x,y)$, $Y_k(x,y)$ are homogeneous polynomials of degree k in x, y. Suppose that $xY_{2n+1}(x,y) - yX_{2n+1}(x,y)$ is a positive (or negative) definite function. Without loss of generality, we assume that $xY_{2n+1}(x,y) - yX_{2n+1}(x,y)$ is positive definite, then there exists a positive number d, such that

$$xY_{2n+1}(x,y) - yX_{2n+1}(x,y) \geqslant d(x^2+y^2)^{n+1}. \tag{6.1.2}$$

The origin of system (6.1.1) is a multiple singular point. From (6.1.2), we know that the origin of system (6.1.1) has no Frömmer characteristic directions. It implies that the origin is a center or a focus. By the polar coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{6.1.3}$$

system (6.1.1) becomes

$$\begin{aligned} \frac{dr}{dt} &= r^{2n+1}\sum_{k=0}^{\infty}\varphi_{2n+2+k}(\theta)r^k, \\ \frac{d\theta}{dt} &= r^{2n}\sum_{k=0}^{\infty}\psi_{2n+2+k}(\theta)r^k. \end{aligned} \tag{6.1.4}$$

Thus, we have

$$\frac{dr}{d\theta} = r\,\frac{\varphi_{2n+2}(\theta)+\displaystyle\sum_{k=1}^{\infty}\varphi_{2n+2+k}(\theta)r^k}{\psi_{2n+2}(\theta)+\displaystyle\sum_{k=1}^{\infty}\psi_{2n+2+k}(\theta)r^k}, \tag{6.1.5}$$

where $\varphi_k(\theta),\ \psi_k(\theta)$ are given by (2.1.5). Especially,

$$\begin{aligned} \varphi_{2n+2}(\theta) &= \cos\theta X_{2n+1}(\cos\theta,\sin\theta)+\sin\theta Y_{2n+1}(\cos\theta,\sin\theta), \\ \psi_{2n+2}(\theta) &= \cos\theta Y_{2n+1}(\cos\theta,\sin\theta)-\sin\theta X_{2n+1}(\cos\theta,\sin\theta). \end{aligned} \tag{6.1.6}$$

From (6.1.2) and (6.1.6), we obtain

$$\psi_{2n+2}(\theta) \geqslant d > 0. \tag{6.1.7}$$

Since for all k, $\varphi_k(\theta)$, $\psi_k(\theta)$ are homogeneous polynomials of degree k in $\cos\theta$, $\sin\theta$, we see that

$$\varphi_k(\theta+\pi) = (-1)^k\varphi_k(\theta), \quad \psi_k(\theta+\pi) = (-1)^k\psi_k(\theta). \tag{6.1.8}$$

Obviously, we know from (6.1.8) that system (6.1.5) is a particular case of system (2.1.7).

For sufficiently small h, let

$$r = \tilde{r}(\theta,h) = \sum_{k=1}^{\infty}\nu_k(\theta)h^k. \tag{6.1.9}$$

be the solution of system (6.1.5) satisfying the initial condition $r|_{\theta=0} = h$. From (6.1.5) and (6.1.9), we obtain the expression of $\nu_1(\theta)$ as follows:

$$\nu_1(\theta) = \exp\int_0^{\theta}\frac{\varphi_{2n+2}(\omega)\ d\omega}{\psi_{2n+2}(\omega)}. \tag{6.1.10}$$

Corollary 2.1.1 follows that if $\nu_1(2\pi) = 1$, then the first integer k satisfying $k > 1$ and $\nu_k(2\pi) \neq 0$ is an odd number.

Definition 6.1.1. *For any positive integer* k, $\nu_{2k+1}(2\pi)$ *is called* k*-th focal value at the origin of system* (6.1.1).

Definition 6.1.2. *For system* (6.1.1):

(1) *If* $\nu_1(2\pi) \neq 1$, *then the origin is called a rough focus*;

(2) *If* $\nu_1(2\pi) = 1$ *and there exist a positive integer* k, *such that* $\nu_2(2\pi) = \nu_3(2\pi) = \cdots = \nu_{2k-1}(2\pi) = 0$ *and* $\nu_{2k+1}(2\pi) \neq 0$, *then the origin is called a weak focus of order* k;

(3) *If* $\nu_1(2\pi) = 1$ *and for all positive integers* k, $\nu_{2k+1}(2\pi) = 0$, *then the origin is called a center.*

By Corollary 2.1.1 and the geometric properties of Poincaré succession function $\Delta(h) = \tilde{r}(2\pi, h) - h$, we have the following conclusion.

Theorem 6.1.1. *For system* (6.1.1):

(1) *If the origin is a rough focus , then when* $\nu_1(2\pi) < 1$ (> 1), *the origin is stable* (*unstable*);

(2) *If the origin is a weak focus of order* k, *then when* $\nu_{2k+1}(2\pi) < 0$ (> 0), *the origin is stable* (*unstable*);

(3) *If the origin is a center, in a neighborhood of the origin, there exists a family of closed orbits of system* (6.1.1).

6.2 Conversion of the Questions

In this section, we assume that in the right hand of system (6.1.1), the first homogeneous polynomial of degree $2n+1$ is as follows:

$$\begin{aligned} X_{2n+1}(x,y) &= (\delta x - y)(x^2+y^2)^n, \\ Y_{2n+1}(x,y) &= (x+\delta y)(x^2+y^2)^n. \end{aligned} \tag{6.2.1}$$

Hence, system (6.1.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= (\delta x - y)(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} X_k(x,y), \\ \frac{dy}{dt} &= (x+\delta y)(x^2+y^2)^n + \sum_{k=2n+2}^{\infty} Y_k(x,y). \end{aligned} \tag{6.2.2}$$

Now, (6.1.6) reduces to

$$\varphi_{2n+2}(\theta) \equiv \delta, \quad \varphi_{2n+2}(\theta) \equiv 1. \tag{6.2.3}$$

Under the polar coordinates, system (6.2.2) becomes

$$\frac{dr}{d\theta} = r\,\frac{\delta + \sum_{k=1}^{\infty} \varphi_{2n+2+k}(\theta) r^k}{1 + \sum_{k=1}^{\infty} \psi_{2n+2+k}(\theta) r^k}. \tag{6.2.4}$$

It is easy to prove that the following propositions.

Proposition 6.2.1. *For system* (6.2.2), *we have* $\nu_1(\theta) = e^{\delta\theta}$ *and when* $\delta < 0$ (> 0), *the origin is a stable* (*an unstable*) *focus.*

From Lemma 2.1.2, we obtain

Proposition 6.2.2. *For system* (6.2.2), *if* $\delta = 0$, *then all* $\nu_k(\theta)$ *are polynomials in* θ, $\sin\theta$ *and* $\cos\theta$ *and the coefficients of* $\nu_k(\theta)$ *are polynomials in* $A_{\alpha\beta}$, $B_{\alpha\beta}$. *Especially, for all* k, $\nu_k(\pi)$, $\nu_k(2\pi)$ *are polynomials in* $A_{\alpha\beta}$, $B_{\alpha\beta}$.

Similar to Theorem 5.2.2, we have the following theorem.

Theorem 6.2.1. *By the transformation*

$$\begin{aligned} &x = u(u^2+v^2)^{n+1}, \quad y = v(u^2+v^2)^{n+1}, \\ &\frac{dt}{d\tau} = (u^2+v^2)^{-n(2n+3)}, \end{aligned} \tag{6.2.5}$$

system (6.2.2) *can be reduced to the following analytic system:*

$$\begin{aligned} \frac{du}{d\tau} &= \frac{\delta u}{2n+3} - v + \sum_{k=1}^{\infty} P_{2nk+3k+1}, \\ \frac{dv}{d\tau} &= u + \frac{\delta v}{2n+3} + \sum_{k=1}^{\infty} Q_{2nk+3k+1}, \end{aligned} \tag{6.2.6}$$

for which the origin is an elementary focus, where

$$\begin{aligned} P_{2nk+3k+1} = &\left[\left(\frac{1}{2n+3}u^2 + v^2\right) X_{2n+1+k}(u,v)\right. \\ &\left. - \frac{2n+2}{2n+3} uv Y_{2n+1+k}(u,v)\right] (u^2+v^2)^{(k-1)(n+1)}, \\ Q_{2nk+3k+1} = &\left[\left(u^2 + \frac{1}{2n+3}v^2\right) Y_{2n+1+k}(\xi,\eta)\right. \\ &\left. - \frac{2n+2}{2n+3} uv X_{2n+1+k}(u,v)\right] (u^2+v^2)^{(k-1)(n+1)} \end{aligned} \tag{6.2.7}$$

are homogeneous polynomials of degree $2nk+3k+1$ *in* u, v.

This theorem tell us that our problems at a multiple singular point can be changed to the corresponding problems at the elementary singular point $O(0,0)$ of system (6.2.6). Since (6.2.6) is a particular system of (2.1.1), therefore, all conclusions in the theory of center-focus about system (2.1.1) can be used to system (6.2.6).

Notice that as a special system of system (2.1.1), the right hand of system (6.2.6) have the following particular properties:

(1) The subscripts of $P_{2nk+3k+1}$, $Q_{2nk+3k+1}$ form an arithmetic sequence with the common difference $2n+3$ $(k=1,2,\cdots,2n+1)$.

(2) $P_{2nk+3k+1}$ and $Q_{2nk+3k+1}$ have common factor $(u^2+v^2)^{(k-1)(n+1)}$.

(3) System (6.2.6) has a pair of conjugated complex straight line solutions $u \pm iv = 0$.

On the basis of the above properties, we can obtain some new results for the theory of center-focus of the origin of system (6.2.6).

6.3 Formal Series, Integral Factors and Singular Point Values for a Class of Multiple Singular Points

Under the polar coordinates

$$u = \rho\cos\theta, \quad v = \rho\sin\theta, \tag{6.3.1}$$

system (6.2.6) has the form

$$\frac{d\rho}{d\theta} = \frac{\rho}{2n+3} \cdot \frac{\delta + \sum_{k=1}^{\infty} \varphi_{2n+2+k}(\theta)\rho^{k(2n+3)}}{1 + \sum_{k=1}^{\infty} \psi_{2n+2+k}(\theta)\rho^{k(2n+3)}}. \tag{6.3.2}$$

Substituting (6.1.3) and (6.2.5) in (6.3.1), we have

$$r = \rho^{2n+3}. \tag{6.3.3}$$

Of course, (6.3.2) can also be obtained from (6.2.4) by using transformation (6.3.3).

Write the solution of (6.3.2) satisfying the initial condition $\rho|_{\theta=0} = \rho_0$ as $\rho = \tilde{\rho}(\theta,\rho_0)$. The properties of the right hand of (6.2.6) follows that

Proposition 6.3.1. *$\tilde{\rho}(\theta,\rho_0)\rho_0^{-1}$ is a power series in ρ_0^{2n+3}, that is, $\tilde{\rho}(\theta,\rho_0)$ has the form*

$$\tilde{\rho}(\theta,\rho_0) = \sum_{m=1}^{\infty} \sigma_{(m-1)(2n+3)+1}(\theta)\rho_0^{(m-1)(2n+3)+1}. \tag{6.3.4}$$

Proof. Let the solution of (6.2.4) satisfying the initial condition $r|_{\theta=0} = h$ be $r = \tilde{r}(\theta, h) = \sum_{m=1}^{\infty} \nu_m(\theta)h^m$. From (6.3.3), we have

$$\tilde{\rho}^{2n+3}(\theta, \rho_0) = \tilde{r}(\theta, \rho_0^{2n+3}). \tag{6.3.5}$$

Taking $h_0 = \rho_0^{2n+3}$, then from (6.3.5), we obtain

$$\frac{\tilde{\rho}(\theta, \rho_0)}{\rho_0} = \left(\frac{\tilde{r}(\theta, h_0)}{h_0}\right)^{\frac{1}{2n+3}} = \left[\sum_{m=1}^{\infty} \nu_m(\theta)h_0^{m-1}\right]^{\frac{1}{2n+3}}. \tag{6.3.6}$$

Since the right function of (6.3.6) can be expanded as a power series in h_0. Thus, the conclusion of Proposition 6.3.1 holds. □

It is easy to see that

$$\begin{aligned} \nu_1(2\pi) - 1 &= e^{2\pi\delta} - 1 = 2\pi\delta + o(\delta), \\ \sigma_1(2\pi) - 1 &= e^{\frac{2\pi\delta}{2n+3}} - 1 = \frac{2\pi\delta}{2n+3} + o(\delta). \end{aligned} \tag{6.3.7}$$

Theorem 6.3.1. *If $\delta = 0$, then for any positive integer k, we have*

$$\begin{aligned} &\sigma_{2k(2n+3)+1}(2\pi) \sim \frac{1}{2n+3}\nu_{2k+1}(2\pi), \\ &\sigma_{(2k-1)(2n+3)+1}(2\pi) \sim 0, \end{aligned} \tag{6.3.8}$$

where $\nu_{2k+1}(2\pi)$ is the k-th focal value of the origin of system (6.2.2), while $\sigma_{2k(2n+3)+1}(2\pi)$ is the $k(2n+3)$-th focal value of the origin of system (6.2.6). Moreover, when m is not an integer multiple of $2n+3$, we take $\sigma_{m+1}(2\pi) = 0$.

Proof. If $\delta = 0$, from (6.3.7), we have $\nu_1(2\pi) = \sigma_1(2\pi) = 1$. From (6.3.4), we obtain

$$\begin{aligned} &\tilde{\rho}^{2n+3}(2\pi, \rho_0) - \rho_0^{2n+3} \\ &= \sum_{j=0}^{2n+2} \rho_0^{2n+2-j}\tilde{\rho}^j(2\pi, \rho_0)[\tilde{\rho}(2\pi, \rho_0) - \rho_0] \\ &= (2n+3)\rho_0^{2n+2}G(\rho_0)[\tilde{\rho}(2\pi, \rho_0) - \rho_0] \\ &= (2n+3)G(\rho_0)\sum_{m=2}^{\infty} \sigma_{(m-1)(2n+3)+1}(2\pi)\rho_0^{m(2n+3)}, \end{aligned} \tag{6.3.9}$$

where $G(\rho_0)$ is a unit formal power series in ρ_0 (see Definition 1.2.3).

On the other hand, from (6.3.5), we have

$$\begin{aligned} \tilde{\rho}^{2n+3}(2\pi, \rho_0) - \rho_0^{2n+3} &= \tilde{r}(2\pi, \rho_0^{2n+3}) - \rho_0^{2n+3} \\ &= \sum_{m=2}^{\infty} \nu_m(2\pi)\rho_0^{m(2n+3)}. \end{aligned} \tag{6.3.10}$$

(6.3.9) and (6.3.10) imply that

$$\sum_{m=2}^{\infty} \sigma_{(m-1)(2n+3)+1}(2\pi)\rho_0^{m(2n+3)}$$
$$= \frac{1}{(2n+3)G(\rho_0)} \sum_{m=2}^{\infty} \nu_m(2\pi)\rho_0^{m(2n+3)}. \tag{6.3.11}$$

Comparing the coefficients of ρ_0^k on the two sides of (6.3.11), it leads to the conclusion of Theorem 6.3.1. □

By making transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \tag{6.3.12}$$

system (6.2.2) becomes

$$\begin{aligned} \frac{dz}{dT} &= (1 - i\delta)z^{n+1}w^n + \sum_{k=2n+2}^{\infty} Z_k(z, w), \\ \frac{dw}{dT} &= -(1 + i\delta)w^{n+1}z^n - \sum_{k=2n+2}^{\infty} W_k(z, w), \end{aligned} \tag{6.3.13}$$

where

$$\begin{aligned} Z_k(z, w) &= \sum_{\alpha+\beta=k} a_{\alpha\beta}z^\alpha w^\beta = Y_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) - iX_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right), \\ W_k(z, w) &= \sum_{\alpha+\beta=k} b_{\alpha\beta}w^\alpha z^\beta = Y_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right) + iX_k\left(\frac{z+w}{2}, \frac{z-w}{2i}\right). \end{aligned} \tag{6.3.14}$$

We say that (6.2.2) and (6.3.13) are associated.

Let

$$\xi = u + iv, \quad \eta = u - iv, \quad \mathcal{T} = i\tau. \tag{6.3.15}$$

From (6.3.12), (6.3.15) and (6.2.5), we obtain

$$z = \xi(\xi\ \eta)^{n+1}, \quad w = \eta(\xi\ \eta)^{n+1}, \quad \frac{dT}{d\mathcal{T}} = (\xi\ \eta)^{-n(2n+3)}. \tag{6.3.16}$$

By transformation (6.3.16), system (6.3.13) can be changed to

$$\begin{aligned} \frac{d\xi}{d\mathcal{T}} &= \left(1 - \frac{i\delta}{2n+3}\right)\xi + \xi\sum_{k=1}^{\infty} \Phi_{k(2n+3)}(\xi, \eta), \\ \frac{d\eta}{d\mathcal{T}} &= -\left(1 + \frac{i\delta}{2n+3}\right)\eta - \eta\sum_{k=1}^{\infty} \Psi_{k(2n+3)}(\xi, \eta), \end{aligned} \tag{6.3.17}$$

where

$$
\begin{aligned}
\Phi_{k(2n+3)}(\xi,\eta)=&\left[\frac{n+2}{2n+3}\ \eta\ Z_{2n+1+k}(\xi,\eta)\right.\\
&\left.+\frac{n+1}{2n+3}\ \xi\ W_{2n+1+k}(\xi,\eta)\right](\xi\ \eta)^{(k-1)(n+1)},\\
\Psi_{k(2n+3)}(\xi,\eta)=&\left[\frac{n+2}{2n+3}\ \xi\ W_{2n+1+k}(\xi,\eta)\right.\\
&\left.+\frac{n+1}{2n+3}\ \eta\ Z_{2n+1+k}(\xi,\eta)\right](\xi\ \eta)^{(k-1)(n+1)}
\end{aligned}
\tag{6.3.18}
$$

are homogeneous polynomials of degree $k(2n+3)$ in ξ,η.

Clearly, system (6.3.17) can also be obtained from system (6.2.6) by using transformation (6.3.15). Thus, system (6.2.6) and system (6.3.17) are associated.

We now discuss the case of $\delta=0$. When $\delta=0$, system (6.2.2), (6.2.6), (6.3.13) and (6.3.17) can be reduced to the following forms, respectively:

$$
\begin{aligned}
\frac{dx}{dt}&=\ -y(x^2+y^2)^n+\sum_{k=2n+2}^{\infty}X_k(x,y)=X(x,y),\\
\frac{dy}{dt}&=\ x(x^2+y^2)^n+\sum_{k=2n+2}^{\infty}Y_k(x,y)=Y(x,y);
\end{aligned}
\tag{6.3.19}
$$

$$
\begin{aligned}
\frac{du}{d\tau}&=-v+\sum_{k=1}^{\infty}P_{2nk+3k+1}(u,v)=P(u,v),\\
\frac{dv}{d\tau}&=u+\sum_{k=1}^{\infty}Q_{2nk+3k+1}(u,v)=Q(u,v);
\end{aligned}
\tag{6.3.20}
$$

$$
\begin{aligned}
\frac{dz}{dT}&=z^{n+1}w^n+\sum_{k=2n+2}^{\infty}Z_k(z,w)=Z(z,w),\\
\frac{dw}{dT}&=-w^{n+1}z^n-\sum_{k=2n+2}^{\infty}W_k(z,w)=-W(z,w);
\end{aligned}
\tag{6.3.21}
$$

$$
\begin{aligned}
\frac{d\xi}{d\mathcal{T}}&=\xi+\xi\sum_{k=1}^{\infty}\Phi_{k(2n+3)}(\xi,\eta)=\Phi(\xi,\eta),\\
\frac{d\eta}{d\mathcal{T}}&=-\eta-\eta\sum_{k=1}^{\infty}\Psi_{k(2n+3)}(\xi,\eta)=-\Psi(\xi,\eta).
\end{aligned}
\tag{6.3.22}
$$

The right hand of system (6.3.22) have the following particular properties:

(1) The subscripts of $\Phi_{k(2n+3)}$ and $\Psi_{k(2n+3)}$ form a arithmetic sequence with the common difference $2n+3$ $(k=1,2,\cdots)$.

(2) $\Phi_{k(2n+3)}$ and $\Psi_{k(2n+3)}$ have the common factor $(\xi\eta)^{(k-1)(n+1)}$.

(3) System (6.3.22) have a pair of straight line solutions $\xi=0$ and $\eta=0$.

On the basis of the above properties, we have

Theorem 6.3.2. *For system* (6.3.22), *one can derive uniquely and successively the terms of the following formal series*

$$F(\xi,\eta)=(\xi\eta)^{2n+3}\left[1+\sum_{m=1}^{\infty}f_{m(2n+3)}(\xi,\eta)\right], \tag{6.3.23}$$

such that

$$\left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)}=\sum_{m=1}^{\infty}\mu_m(\xi\eta)^{(m+1)(2n+3)}, \tag{6.3.24}$$

where

$$f_{m(2n+3)}(\xi,\eta)=\sum_{\alpha+\beta=m(2n+3)}c_{\alpha\beta}\xi^{\alpha}\eta^{\beta} \tag{6.3.25}$$

are homogenous polynomials of degree $m(2n+3)$ *in* ξ,η, $m=1,2,\cdots$ *and we take*

$$c_{00}=1,\quad c_{k(2n+3),k(2n+3)}=0,\quad k=1,2,\cdots. \tag{6.3.26}$$

Definition 6.3.1. *For any positive integer* m, μ_m *in* (6.3.24) *is called the* m*-th singular point value at the origin of system* (6.3.21).

Theorem 6.3.3. *In* (6.3.25), *for all* α, β, *when* $\alpha\neq\beta$ *and* $\alpha+\beta\geqslant 1$, $c_{\alpha\beta}$ *is determined by the recursive formula*

$$\begin{aligned}c_{\alpha\beta}=&\frac{1}{(2n+3)(\beta-\alpha)}\\&\times\sum_{k,j}\Big\{\big[(n+2)\alpha-(n+1)+(n+2-k)(2n+3)\big]a_{k,j-1}\\&-\big[(n+2)\beta-(n+1)\alpha+(n+2-j)(2n+3)\big]b_{j,k-1}\Big\}\\&\times c_{\alpha-(n+2)k-(n+1)j+(n+1)(2n+3),\beta-(n+2)j-(n+1)k+(n+1)(2n+3)}.\end{aligned} \tag{6.3.27}$$

For any positive integer m, μ_m *is given by the recursive formula*

$$\begin{aligned}\mu_m=&\sum_{k+j=2n+3}^{2m+2n+2}\big[(m+n+2-k)a_{k,j-1}-\ (m+n+2-j)b_{j,k-1}\big]\\&\times c_{(m+n+1)(2n+3)-(n+2)k-(n+1)j,(m+n+1)(2n+3)-(n+2)j-(n+1)k},\end{aligned} \tag{6.3.28}$$

where for all (α,β), *when* $\alpha<0$ *or* $\beta<0$ *take* $a_{\alpha\beta}=b_{\alpha\beta}=c_{\alpha\beta}=0$.

Proof. We see from (6.3.23) that

$$(\xi\eta)^{-(2n+3)} \left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)} = \sum_{m=1}^{\infty}\left(\frac{\partial f_{m(2n+3)}}{\partial \xi}\,\xi - \frac{\partial f_{m(2n+3)}}{\partial \eta}\,\eta\right)$$
$$+\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\left[\xi\frac{\partial\, f_{(m-s)(2n+3)}}{\partial \xi} + (2n+3)f_{(m-s)(2n+3)}\right]\Phi_{s(2n+3)}$$
$$-\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\left[\eta\frac{\partial f_{(m-s)(2n+3)}}{\partial \eta} + (2n+3)f_{(m-s)(2n+3)}\right]\Psi_{s(2n+3)}. \qquad (6.3.29)$$

From (6.3.25) and (6.3.29), we have

$$(\xi\eta)^{-(2n+3)} \left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)} = \sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta\\=m(2n+3)}}(\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}$$
$$+\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{\substack{\alpha+\beta\\=(m-s)(2n+3)}}\big[(\alpha+2n+3)\Phi_{s(2n+3)}$$
$$-(\beta+2n+3)\Psi_{s(2n+3)}\big]\,c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}. \qquad (6.3.30)$$

By using (6.3.18) and (6.3.30), we have

$$(\xi\eta)^{-(2n+3)} \left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)} = \sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta\\=m(2n+3)}}(\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}$$
$$+\frac{1}{2n+3}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{\substack{\alpha+\beta\\=(m-s)(2n+3)}}\big[(n\alpha-n\beta+2\alpha-\beta+2n+3)\eta Z_{2n+1+s}$$
$$-(n\beta-n\alpha+2\beta-\alpha+2n+3)\xi W_{2n+1+s}\big]c_{\alpha\beta}\xi^{\alpha+(s-1)(n+1)}\eta^{\beta+(s-1)(n+1)}. \qquad (6.3.31)$$

We see from (6.3.14) that

$$Z_{2n+1+s}(\xi,\eta) = \sum_{k+j=2n+2+s} a_{k,j-1}\xi^{k}\eta^{j-1},$$
$$W_{2n+1+s}(\xi,\eta) = \sum_{k+j=2n+2+s} b_{j,k-1}\xi^{k-1}\eta^{j}. \qquad (6.3.32)$$

From (6.3.31) and (6.3.32), we obtain

$$
\begin{aligned}
&(\xi\eta)^{-(2n+3)}\left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)} = \sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta\\=m(2n+1)}}(\alpha-\beta)c_{\alpha\beta}\xi^{\alpha}\eta^{\beta}\\
&+\frac{1}{2n+3}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\sum_{\substack{\alpha+\beta\\=(m-s)(2n+3)}}\sum_{\substack{k+j=\\2n+2+s}}\big[(n\alpha-n\beta+2\alpha-\beta+2n+3)a_{k,j-1}\\
&-(n\beta-n\alpha+2\beta-\alpha+2n+3)b_{j,k-1}\big]c_{\alpha\beta}\xi^{\alpha+(s-1)(n+1)+k}\eta^{\beta+(s-1)(n+1)+j}.
\end{aligned}
\tag{6.3.33}
$$

Let

$$
\begin{aligned}
\alpha_1 &= \alpha+(s-1)(n+1)+k,\\
\beta_1 &= \beta+(s-1)(n+1)+j.
\end{aligned}
\tag{6.3.34}
$$

Then if $k+j=2n+2+s$, $\alpha+\beta=(m-s)(2n+3)$, we have

$$
\begin{aligned}
&\alpha_1+\beta_1 = m(2n+3),\\
&\alpha = \alpha_1-(n+2)k-(n+1)j+(n+1)(2n+3),\\
&\beta = \beta_1-(n+2)j-(n+1)k+(n+1)(2n+3)
\end{aligned}
\tag{6.3.35}
$$

and

$$
\begin{aligned}
&\quad n\alpha-n\beta+2\alpha-\beta+2n+3\\
&=(n+2)\alpha_1-(n+1)\beta_1+(n+2-k)(2n+3),\\
&\quad n\beta-n\alpha+2\beta-\alpha+2n+3\\
&=(n+2)\beta_1-(n+1)\alpha_1+(n+2-j)(2n+3).
\end{aligned}
\tag{6.3.36}
$$

Substituting (6.3.34), (6.3.35) and (6.3.36) into (6.3.33) and using α,β instead of α_1,β_1, we obtain

$$
\left.\frac{dF}{d\mathcal{T}}\right|_{(6.3.22)} = (\xi\eta)^{2n+3}\sum_{m=1}^{\infty}\sum_{\substack{\alpha+\beta=\\m(2n+3)}}\left[(\alpha-\beta)c_{\alpha\beta}+H_{\alpha\beta}\right]\xi^{\alpha}\eta^{\beta},
\tag{6.3.37}
$$

where

$$
\begin{aligned}
H_{\alpha\beta}=&\frac{1}{2n+3}\\
&\times\sum_{k,j}\{[(n+2)\alpha-(n+1)\beta+(n+2-k)(2n+3)]a_{k,j-1}\\
&-[(n+2)\beta-(n+1)\alpha+(n+2-j)(2n+3)]b_{j,k-1}]\}\\
&\times c_{\alpha-(n+2)k-(n+1)j+(n+1)(2n+3),\beta-(n+2)j-(n+1)k+(n+1)(2n+3)}.
\end{aligned}
\tag{6.3.38}
$$

Thus, (6.3.24) and (6.3.37) follow the conclusion of this theorem. □

Remark 6.3.1. *We see from* (6.3.25) *that* $\alpha+\beta$ *is an integer multiple of* $2n+3$. *Since for all* (α,β), *when* $\alpha<0$ *or* $\beta<0$, *we have taken* $c_{\alpha\beta}=0$. *Thus, in the right sides of* (6.3.27) *and* (6.3.38), *we have*

$$2n+3 \leqslant k+j \leqslant \frac{\alpha+\beta}{2n+3}+2n+2. \tag{6.3.39}$$

For any positive integer m, we write the m-*th* singular point value at the origin of system (6.3.22) as μ'_m.

Theorem 6.3.4. *For any positive integer* k, *the* $k(2n+3)$-*th singular point value* $\mu'_{k(2n+1)}$ *at the origin of system* (6.3.22) *and the* k-*th singular point value* μ_k *at the origin* (*multiple singular point*) *of system* (6.3.21) *have the following relation:*

$$\mu'_{k(2n+3)} \sim \frac{\mu_k}{2n+3} \tag{6.3.40}$$

and when m *is not an integer multiple of* $2n+3$, *we have* $\mu'_m=0$.

Proof. For function $F(\xi,\eta)$ defined by (6.3.23), we denote that

$$\hat{F}(\xi,\eta)=F^{\frac{1}{2n+3}}(\xi,\eta)=\xi\eta\left[1+\sum_{m=1}^{\infty}f_{m(2n+3)}(\xi,\eta)\right]^{\frac{1}{2n+3}}. \tag{6.3.41}$$

We see from (6.3.24) and (6.3.41) that

$$\left.\frac{d\hat{F}}{d\mathcal{T}}\right|_{(6.3.22)}=\frac{\displaystyle\sum_{m=1}^{\infty}\mu_m(\xi\eta)^{m(2n+3)+1}}{(2n+3)\left[1+\displaystyle\sum_{m=1}^{\infty}f_{m(2n+3)}(\xi,\eta)\right]^{(2n+2)/(2n+3)}}. \tag{6.3.42}$$

Thus, (6.3.42) follows the conclusion of Theorem 6.3.4. □

From Theorem 6.3.1, Theorem 6.3.4 and Theorem 2.3.2, we have

Theorem 6.3.5. *For any positive integer* k, *we have*

$$\begin{aligned}\sigma_{2k(2n+3)+1}(2\pi) &\sim \frac{i\pi}{2n+3}\mu_k,\\ \nu_{2k+1}(2\pi) &\sim i\pi\mu_k,\end{aligned} \tag{6.3.43}$$

where $\sigma_{2k(2n+3)+1}(2\pi)$ *is the* $k(2n+3)$-*th focal value at the origin of system* (6.3.22), *while* $\nu_{2k+1}(2\pi)$ *is the* k-*th focal value at the origin of system* (6.3.19).

Theorem 2.3.5, Theorem (2.3.7) and the properties of the right side functions of system (6.3.22) imply that

Theorem 6.3.6. *For system* (6.3.22), *one can determine successively the terms of the following formal series*

$$M(\xi,\eta) = 1+\sum_{m=1}^{\infty} g_{m(2n+3)}(\xi,\eta), \tag{6.3.44}$$

such that

$$\frac{\partial(M\Phi)}{\partial\xi}-\frac{\partial(M\Psi)}{\partial\eta} = \sum_{m=1}^{\infty}\frac{2mn+3m+1}{2n+3}\lambda_m(\xi\eta)^{m(2n+3)}, \tag{6.3.45}$$

where for any positive integer m,

$$g_{m(2n+3)}(\xi,\eta)=\sum_{\alpha+\beta=m(2n+3)} d_{\alpha\beta}\xi^{\alpha}\eta^{\beta} \tag{6.3.46}$$

are homogenous polynomials of degree $m(2n+3)$ *in* ξ,η *and*

$$\lambda_m \sim (2n+3)\mu'_{m(2n+3)} \sim \mu_m. \tag{6.3.47}$$

Similar to the proof of Theorem 6.3.3, we have

Theorem 6.3.7. *In the right hand of* (6.3.44), *let* $d_{00}=1$ *and* $d_{k(2n+3),k(2n+3)}$ $(k=1,\ 2,\ \cdots)$ *is arbitrarily chosen. Then for all* (α,β), *when* $\alpha\neq\beta$ *and* $\alpha+\beta\geqslant 1$, $d_{\alpha\beta}$ *is given by the recursive formula*

$$\begin{aligned} d_{\alpha\beta}=&\frac{1}{(2n+3)(\beta-\alpha)}\sum_{k,j}\{[(n+2)\alpha-(n+1)\beta+1]a_{k,j-1}\\ &-[(n+2)\beta-(n+1)\alpha+1]b_{j,k-1}]\}\\ &\times d_{\alpha-(n+2)k-(n+1)j+(n+1)(2n+3),\beta-(n+2)j-(n+1)k+(n+1)(2n+3)}. \end{aligned} \tag{6.3.48}$$

For any positive integer m, λ_m *is given by the recursive formula*

$$\begin{aligned} \lambda_m=&\sum_{k+j=2n+3}^{2(m+n+1)}(a_{k,j-1}-b_{j,k-1})\\ &\times d_{(1+m+n)(3+2n)-k(2+n)-j(1+n),\ (1+m+n)(3+2n)-j(2+n)-k(1+n)}, \end{aligned} \tag{6.3.49}$$

where for all (α,β), *when* $\alpha<0$ *or* $\beta<0$, *we take* $a_{\alpha\beta}=b_{\alpha\beta}=d_{\alpha\beta}=0$.

Theorem 6.3.3 and Theorem 6.3.7 give the recursive formulas to compute directly the singular point values at the origin by applying the coefficients of system (6.3.21).

Theorem 6.3.8. *For system* (6.3.21), *one can derive successively the terms of the following formal series*

$$\mathcal{F}(z,w) \ = zw\left[1+\sum_{m=1}^{\infty}\frac{f_{m(2n+3)}(z,w)}{(zw)^{m(n+1)}}\right], \tag{6.3.50}$$

such that

$$\left.\frac{d\mathcal{F}}{dT}\right|_{(6.3.21)} = (zw)^n\sum_{m=1}^{\infty}\mu_m(zw)^{m+1}, \tag{6.3.51}$$

where μ_m *is the m-th singular point value at the origin* (*infinity*) *of system* (6.3.21), $m=1,2,\cdots$.

Proof. The inverse transform of transformation (6.3.16) is

$$\xi = z(zw)^{\frac{-(n+1)}{2n+3}}, \quad \eta = w(zw)^{\frac{-(n+1)}{2n+3}}, \quad \frac{d\mathcal{T}}{dT} = (zw)^n. \tag{6.3.52}$$

From (6.3.23) and (6.3.50), we have

$$\mathcal{F}(z,w) = F\left(z(zw)^{\frac{-(n+1)}{2n+3}}, w(zw)^{\frac{-(n+1)}{2n+3}}\right). \tag{6.3.53}$$

Hence, from (6.3.52), (6.3.53) and (6.3.24) we have the conclusion of the theorem. □

Theorem 6.3.9. *For system* (6.3.21), *one can derive successively the terms of the following formal series*

$$\mathcal{M}(z,w) \ = \ (zw)^{-n-1+\frac{1}{2n+3}}\left[1+\sum_{m=1}^{\infty}\frac{g_{m(2n+3)}(z,w)}{(zw)^{m(n+1)}}\right], \tag{6.3.54}$$

such that

$$\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w} = (zw)^{-1+\frac{1}{2n+3}}\sum_{m=1}^{\infty}\frac{2mn+3m+1}{2n+3}\lambda_m(zw)^m, \tag{6.3.55}$$

where $\lambda_m \sim \mu_m,\ m=1,2,\cdots$.

Proof. First, from (6.3.44), we have

$$\mathcal{M}(z,w) = M\left(z(zw)^{\frac{-(n+1)}{2n+3}}, w(zw)^{\frac{-(n+1)}{2n+3}}\right). \tag{6.3.56}$$

Consider the system

$$\begin{aligned}\frac{dz}{d\mathcal{T}} &= \frac{MZ}{(zw)^n} = \mathcal{Z}(z,w),\\ \frac{dw}{d\mathcal{T}} &= -\frac{MW}{(zw)^n} = -\mathcal{W}(z,w).\end{aligned} \tag{6.3.57}$$

By the transformation

$$\xi = z(zw)^{\frac{-(n+1)}{2n+3}}, \quad \eta = w(zw)^{\frac{-(n+1)}{2n+3}}, \tag{6.3.58}$$

system (6.3.57) becomes

$$\frac{d\xi}{d\mathcal{T}} = M(\xi,\eta)\Phi(\xi,\eta), \quad \frac{d\eta}{d\mathcal{T}} = -M(\xi,\eta)\Psi(\xi,\eta). \tag{6.3.59}$$

The Jacobian determinant of transformation (6.3.58) is given by

$$J = \frac{\partial\xi}{\partial z}\frac{\partial\eta}{\partial w} - \frac{\partial\xi}{\partial w}\frac{\partial\eta}{\partial z} = \frac{1}{2n+3}(zw)^{-1+\frac{1}{2n+3}}. \tag{6.3.60}$$

From (6.3.56), (6.3.57) and (6.3.60), we have

$$\frac{1}{2n+3}\mathcal{M}Z = J\mathcal{Z}, \quad \frac{1}{2n+3}\mathcal{M}W = J\mathcal{W}. \tag{6.3.61}$$

Thus,

$$\frac{1}{2n+3}\left[\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w}\right] = \frac{\partial(J\mathcal{Z})}{\partial z} - \frac{\partial(J\mathcal{W})}{\partial w}. \tag{6.3.62}$$

Using Proposition (1.1.3) to system (6.3.57) and (6.3.59), from (6.3.60), we obtain

$$\begin{aligned}\frac{\partial(J\mathcal{Z})}{\partial z} - \frac{\partial(J\mathcal{W})}{\partial w} &= J\left[\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta}\right] \\ &= \frac{1}{2n+3}(zw)^{-1+\frac{1}{2n+3}}\left[\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta}\right].\end{aligned} \tag{6.3.63}$$

From (6.3.62) and (6.3.63), we get

$$\frac{\partial(\mathcal{M}Z)}{\partial z} - \frac{\partial(\mathcal{M}W)}{\partial w} = (zw)^{-1+\frac{1}{2n+3}}\left[\frac{\partial(M\Phi)}{\partial\xi} - \frac{\partial(M\Psi)}{\partial\eta}\right]. \tag{6.3.64}$$

Thus, (6.3.64), (6.3.45) and (6.3.58) imply the conclusion of Theorem 6.3.9. □

Consider the following formal series

$$H(z,w) = 1 + \sum_{m=1}^{\infty} \frac{h_{m(2n+3)}(z,w)}{(zw)^{m(n+1)}}, \tag{6.3.65}$$

where

$$h_{m(2n+3)}(\xi,\eta) = \sum_{\alpha+\beta=m(2n+3)} e_{\alpha\beta} z^\alpha w^\beta \tag{6.3.66}$$

are homogenous polynomials of degree $m(2n+3)$ in z, w ($m = 1, 2, \cdots$), and $h_0 = e_{00} = 1$.

The following two theorems are given by [Liu Y.R., 2001].

Theorem 6.3.10. *For all $s \neq 0$ and $\gamma \neq 0$, one can derive successively the terms of the following formal series*

$$\tilde{F}(z,w) = (zw)^s H^{\frac{1}{\gamma}}(z,w), \tag{6.3.67}$$

such that

$$\left.\frac{d\tilde{F}}{dT}\right|_{(6.3.21)} = \frac{1}{\gamma}(zw)^{n+s} H^{\frac{1}{\gamma}-1} \sum_{m=1}^{\infty} \lambda'_m (zw)^m \tag{6.3.68}$$

and for any positive integer m, λ'_m is given by

$$\lambda'_m \sim s\gamma\mu_m. \tag{6.3.69}$$

Theorem 6.3.11. *Let s, γ be two constants. If for any positive integer m, $\gamma(s+n+1+m) \neq 0$, then one can derive successively the terms of the following formal series*

$$\tilde{M}(z,w) = (zw)^s H^{\frac{1}{\gamma}}(z,w), \tag{6.3.70}$$

such that

$$\frac{\partial(\tilde{M}Z)}{\partial z} - \frac{\partial(\tilde{M}W)}{\partial w} = \frac{1}{\gamma}(zw)^{n+s} H^{\frac{1}{\gamma}-1} \sum_{m=1}^{\infty} \lambda''_m (zw)^m \tag{6.3.71}$$

and for any positive integer m, λ''_m is given by

$$\lambda''_m \sim \gamma(s+n+1+m)\mu_m. \tag{6.3.72}$$

Similar to Theorem 6.3.3, we can prove the following theorem.

Theorem 6.3.12. *For the formal series $\tilde{F}$ given by Theorem 6.3.10, $e_{k(2n+3),k(2n+3)}$ can be arbitrarily chosen, $k = 1, 2, \cdots$. When $\alpha \neq \beta$ and $\alpha + \beta \geqslant 1$, $e_{\alpha\beta}$ is given by the following recursive formula*

$$\begin{aligned} e_{\alpha\beta} = & \frac{1}{(2n+3)(\beta-\alpha)} \\ & \times \sum_{k,j} \Big\{ \big[(n+2)\alpha - (n+1)\beta + (n+1-k+\gamma s)(2n+3)\big] a_{k,j-1} \\ & - \big[(n+2)\beta - (n+1)\alpha + (n+1-j+\gamma s)(2n+3)\big] b_{j,k-1} \Big\} \\ & \times e_{\alpha-(n+2)k-(n+1)j+(n+1)(2n+3),\beta-(n+2)j-(n+1)k+(n+1)(2n+3)} \end{aligned} \tag{6.3.73}$$

and for any positive integer m, λ'_m is given by the recursive formula

$$\begin{aligned} \lambda'_m = & \sum_{k+j=2n+3}^{2m+2n+2} \big[(m+n+1-k+\gamma s) a_{k,j-1} \\ & -(m+n+1-j+\gamma s) b_{j,k-1}\big] \\ & \times e_{(m+n+1)(2n+3)-(n+2)k-(n+1)j,(m+n+1)(2n+3)-(n+2)j-(n+1)k}, \end{aligned} \tag{6.3.74}$$

where for all (α, β), *when* $\alpha < 0$ *or* $\beta < 0$ *we take* $a_{\alpha\beta} = b_{\alpha\beta} = e_{\alpha\beta} = 0$.

Theorem 6.3.13. *For the formal series* $\tilde{M}$ *in Theorem* 6.3.11, $e_{k(2n+3),k(2n+3)}$ *can be arbitrarily chosen*, $k = 1, 2, \cdots$. *When* $\alpha \neq \beta$ *and* $\alpha + \beta \geqslant 1$, $e_{\alpha\beta}$ *is given by the recursive formula*

$$\begin{aligned} e_{\alpha\beta} = & \frac{1}{(2n+3)(\beta-\alpha)} \sum_{k,j} \Big\{ \big[(n+2)\alpha - (n+1)\beta \\ & +(n+1-k+\gamma k+\gamma s)(2n+3)\big] a_{k,j-1} - \big[(n+2)\beta \\ & -(n+1)\alpha + (n+1-j+\gamma j+\gamma s)(2n+3)\big] b_{j,k-1} \Big\} \\ & \times e_{\alpha-(n+2)k-(n+1)j+(n+1)(2n+3),\beta-(n+2)j-(n+1)k+(n+1)(2n+3)} \end{aligned} \tag{6.3.75}$$

and for any positive integer m, λ''_m *is given by the recursive formula*

$$\begin{aligned} \lambda''_m = & \sum_{k+j=2n+3}^{2m+2n+2} \big[(m+n+1-k+\gamma k+\gamma s) a_{k,j-1} \\ & -(m+n+1-j+\gamma j+\gamma s) b_{j,k-1}\big] \\ & \times e_{(m+n+1)(2n+3)-(n+2)k-(n+1)j,(m+n+1)(2n+3)-(n+2)j-(n+1)k}, \end{aligned} \tag{6.3.76}$$

where for all (α, β), *when* $\alpha < 0$ *or* $\beta < 0$, *we take* $a_{\alpha\beta} = b_{\alpha\beta} = e_{\alpha\beta} = 0$.

From Theorem 6.3.10 and Theorem 6.3.11, we have

Theorem 6.3.14. *The origin of system* (6.3.21) *is a center if and only if there exists a first integral* $\tilde{F}(z, w)$ *with the form* (6.3.67).

Theorem 6.3.15. *The origin of system* (6.3.21) *is a center if and only if there exists a integral factor* $\tilde{M}(z, w)$ *with the form* (6.3.70).

6.4 The Algebraic Structure of Singular Point Values of a Class of Multiple Singular Points

In Section 5.4, we defined the generalized rotation and similar transformation having time exponent n and discussed its generalized rotation invariant. By transformation (5.4.1), system (6.3.21) becomes

$$\begin{aligned} \frac{d\hat{z}}{d\hat{T}} &= (\hat{z})^{n+1}(\hat{w})^n + \sum_{\alpha+\beta=2n+2}^{\infty} \hat{a}_{\alpha\beta}(\hat{z})^{\alpha}(\hat{w})^{\beta}, \\ \frac{d\hat{w}}{d\hat{T}} &= -(\hat{w})^{n+1}(\hat{z})^n - \sum_{\alpha+\beta=2n+2}^{\infty} \hat{b}_{\alpha\beta}(\hat{w})^{\alpha}(\hat{z})^{\beta}, \end{aligned} \tag{6.4.1}$$

where $\hat{z}, \hat{w}, \hat{T}$ are new variables, ρ, ϕ are complex constants, $\rho \neq 0$ and for all (α, β)

$$\hat{a}_{\alpha\beta} = a_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{i(\alpha-\beta-1)\phi},$$
$$\hat{b}_{\alpha\beta} = b_{\alpha\beta}\rho^{\alpha+\beta-2n-1}e^{-i(\alpha-\beta-1)\phi}. \tag{6.4.2}$$

Lemma 6.4.1. *For any positive integer m, the m-th singular point values μ_m at the origin of system* (6.3.21) *is m-order generalized rotation invariants of transformation* (5.4.1) *with time exponent n, namely,*

$$\hat{\mu}_m = \rho^{2m}\mu_m. \tag{6.4.3}$$

Proof. For function $\mathcal{F}(z, w)$ given by Theorem 6.3.8, let $\hat{\mathcal{F}} = \rho^{-2}\mathcal{F}(\rho e^{i\phi}\hat{z}, \rho e^{-i\phi}\hat{w})$. Then (5.4.1), (6.3.50) and (6.3.51) follow that

$$\hat{\mathcal{F}} = \hat{z}\hat{w}\left[1 + \sum_{m=1}^{\infty} \frac{\rho^m f_{m(2n+3)}(\hat{z}e^{i\phi}, \hat{w}e^{-i\phi})}{(\hat{z}\hat{w})^{m(n+1)}}\right] \tag{6.4.4}$$

and

$$\left.\frac{d\hat{\mathcal{F}}}{d\hat{T}}\right|_{(6.4.1)} = (\hat{z}\hat{w})^n \sum_{m=1}^{\infty} \rho^{2m}\mu_m(\hat{z}\hat{w})^{m+1}. \tag{6.4.5}$$

(6.4.5) leads to (6.4.3). Hence, Lemma 6.4.1 holds. □

Lemma 6.4.2. *For any positive integer m, the m-th singular point values μ_m at the origin of system* (6.3.21) *is self-antisymmetry, i.e.,*

$$\hat{\mu}_m^* = -\mu_m. \tag{6.4.6}$$

Proof. By the following transformation of antisymmetry:

$$z = w^*, \quad w = z^*, \quad T = -T, \tag{6.4.7}$$

system (6.3.21) can be transformed into the following complex system:

$$\frac{dz^*}{dT^*} = (z^*)^{n+1}(w^*)^n + \sum_{k=0}^{2n} W_k(w^*, z^*) = W(w^*, z^*),$$
$$\frac{dw^*}{dT^*} = -(w^*)^{n+1}(z^*)^n - \sum_{k=0}^{2n} Z_k(w, z) = -Z(w^*, z^*). \tag{6.4.8}$$

For function $\mathcal{F}(z, w)$ given by Theorem 6.3.8, letting $\mathcal{F}^* = \mathcal{F}(w^*, z^*)$, then from (6.4.7), (6.3.50) and (6.3.51) we have

$$\mathcal{F}^* = \frac{1}{z^*w^*}\left[1 + \sum_{m=1}^{\infty} \frac{f_{m(2n+3)}(w^*, z^*)}{(z^*w^*)^{m(2n+1)}}\right] \tag{6.4.9}$$

and

$$\left.\frac{d\mathcal{F}^*}{dT^*}\right|_{(6.4.1)} = (z^*w^*)^n \sum_{m=1}^{\infty} \frac{(-\mu_m^*)}{(z^*w^*)^{m+1}}. \tag{6.4.10}$$

It follows (6.4.6). So that, the conclusion of this lemma holds. □

From Lemma 6.4.1 and Lemma 6.4.2, we obtain

Theorem 6.4.1 (The construction theorem of singular point values of the Multiple Singular Point). *For system* (6.3.21)*, the m-th singular point value μ_m at the origin can be represented as a linear combination of m-order monomial generalized invariants and their antisymmetry forms, i.e.,*

$$\mu_m = \sum_{j=1}^{N} \gamma_{kj}(g_{kj} - g_{kj}^*), \quad k = 1,\ 2,\ \cdots, \tag{6.4.11}$$

where N is a positive integer and γ_{kj} is a rational number, g_{kj} and g_{kj}^* are m-order monomial generalized rotation invariants with time exponent n.

From Theorem 6.4.1 we have

Theorem 6.4.2 (The extended symmetric principle of the multiple singular point). *For system* (6.3.21)*, if all elementary generalized rotation invariants g satisfy symmetric condition $g = g^*$, then all singular point values at the origin are zero.*

6.5 Bifurcation of Limit Cycles From a Class of Multiple Singular Points

Consider the following perturbed system of (6.1.1) depending on two small parameters ε, δ as follows:

$$\begin{aligned} \frac{dx}{dt} &= \sum_{k=2n+1}^{\infty} X_k(x,y,\varepsilon,\delta) = X(x,y,\varepsilon,\delta),\\ \frac{dy}{dt} &= \sum_{k=2n+1}^{\infty} Y_k(x,y,\varepsilon,\delta) = Y(x,y,\varepsilon,\delta), \end{aligned} \tag{6.5.1}$$

where $X(x,y,\varepsilon,\delta), Y(x,y,\varepsilon,\delta)$ are power series in $x, y, \varepsilon, \delta$ having non-zero convergent radius and real coefficients. We assume that there is an integer d, such that

$$xY_{2n+1}(x,y.0,0) - yX_{2n+1}(x,y,0,0) \geqslant d(x^2+y^2)^{n+1} \tag{6.5.2}$$

and

$$\int_0^{2\pi} \frac{\cos\theta X_{2n+1}(\cos\theta,\sin\theta,0,0) + \sin\theta Y_{2n+1}(\cos\theta,\sin\theta,0,0)}{\cos\theta Y_{2n+1}(\cos\theta,\sin\theta,0,0) - \sin\theta X_{2n+1}(\cos\theta,\sin\theta,0,0)}\, d\theta = 0. \tag{6.5.3}$$

Under the polar coordinate (6.1.3), system (6.5.1) takes the form

$$\begin{aligned}\frac{dr}{d\theta} &= r\frac{\varphi_{2n+2}(\theta,\varepsilon,\delta)+\sum\limits_{k=1}^{\infty}\varphi_{2n+2+k}(\theta,\varepsilon,\delta)r^k}{\psi_{2n+2}(\theta,\varepsilon,\delta)+\sum\limits_{k=1}^{\infty}\psi_{2n+2+k}(\theta,\varepsilon,\delta)r^k}\\ &= \frac{\varphi_{2n+2}(\theta,\varepsilon,\delta)}{\psi_{2n+2}(\theta,\varepsilon,\delta)}r+o(r),\end{aligned} \tag{6.5.4}$$

where $\varphi_k(\theta,\varepsilon,\delta),\ \psi_k(\theta,\varepsilon,\delta)$ are given by (5.6.6).

For sufficiently small h, let the solution of equation (6.5.4) satisfying the initial condition $r|_{\theta=0}=h$ and the Poincaré succession function be

$$\begin{aligned} r &= \tilde{r}(\theta,h,\varepsilon,\delta)=\sum_{k=1}^{\infty}\nu_k(\theta,\varepsilon,\delta)h^k,\\ \Delta(h,\varepsilon,\delta) &= \tilde{r}(2\pi,h,\varepsilon,\delta)-h,\end{aligned} \tag{6.5.5}$$

where

$$\nu_1(\theta,\varepsilon,\delta)=\exp\int_0^{\theta}\frac{\varphi_{2n+2}(\vartheta,\varepsilon,\delta)\ d\vartheta}{\psi_{2n+2}(\vartheta,\varepsilon,\delta)}. \tag{6.5.6}$$

From (6.5.3) and (6.5.6) we have

$$\nu_1(2\pi,0,0)=1. \tag{6.5.7}$$

Specially, if $X_{2n+1},\ Y_{2n+1}$ are given by (6.2.1), then $\nu_1(\theta,\varepsilon,\delta)=e^{\delta\theta}$.

Obviously, equation (6.5.4) is the special case of equation (3.1.7).

If $\delta=\delta(\varepsilon)$ given by (6.5.1) is the power series in ε having non-zero convergent radius and real coefficients, and $\delta(0)=0$, then by computing the focal value at the origin of system (6.5.1) we can obtain a quasi succession function $L(h,\varepsilon)$, and use the method mentioned in Chapter 3 to study bifurcation of limit cycles in a neighborhood of the origin of system (6.5.1).

In next section, we consider a quartic system as an example.

6.6 Bifurcation of Limit Cycles Created from a Multiple Singular Point for a Class of Quartic System

As an application of the method described above, we now study the following real planar quartic system:

$$\begin{aligned}\frac{dx}{dt} &= (\delta x-y)(x^2+y^2)+X_4(x,y),\\ \frac{dy}{dt} &= (x+\delta y)(x^2+y^2)+Y_4(x,y),\end{aligned} \tag{6.6.1}$$

where

$$\begin{aligned}X_4(x,y)=&-(q_1+q_2-p_1q_2)x^4-2(p_1-p_2+2p_1p_2)x^3y-6p_1q_2x^2y^2\\&+2(-p_1+p_2+2p_1p_2)xy^3-(-q_1-q_2-p_1q_2)y^4,\\Y_4(x,y)=&(1+p_1)(1+p_2)x^4-2(q_1-q_2-2p_1q_2)x^3y-2(-1+3p_1p_2)x^2y^2\\&-2(q_1-q_2+2p_1q_2)xy^3+(-1+p_1)(-1+p_2)y^4.\end{aligned}\tag{6.6.2}$$

Making the transformation $z=x+iy, w=x-iy, T=it$, system $(6.6.1)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT}&=z^2w+p_1(p_2-iq_2)z^4+(p_1+iq_1)z^3w+w^2z^2+(p_2+iq_2)w^3z,\\\frac{dw}{dT}&=-w^2z-p_1(p_2+iq_2)w^4-(p_1-iq_1)w^3z-w^2z^2-(p_2-iq_2)wz^3.\end{aligned}\tag{6.6.3}$$

Letting

$$z=\xi^3\eta^2,\quad w=\eta^3\xi^2,\quad d\mathcal{T}=(\xi\eta)^5dT,\tag{6.6.4}$$

system (6.6.3) is transformed into the following polynomial system of degree 6:

$$\begin{aligned}\frac{d\xi}{d\mathcal{T}}=&\xi+\frac{1}{5}(2+3p_1)(p_2-iq_2)\xi^5\eta+\frac{1}{5}(2+3p_1+3iq_1)\xi^4\eta^2\\&+\frac{1}{5}(3+2p_1-2iq_1)\xi^3\eta^3+\frac{1}{5}(3+2p_1)(p_2+iq_2)\xi^2\eta^4,\\\frac{d\eta}{d\mathcal{T}}=&-\eta-\frac{1}{5}(2+3p_1)(p_2+iq_2)\eta^5\xi-\frac{1}{5}(2+3p_1-3iq_1)\eta^4\xi^2\\&-\frac{1}{5}(3+2p_1+2iq_1)\eta^3\xi^3-\frac{1}{5}(3+2p_1)(p_2-iq_2)\eta^2\xi^4.\end{aligned}\tag{6.6.5}$$

for which the origin is an elementary singular point.

From Theorem 6.3.1 and Theorem 2.3.1, for any positive integer k, the k-th focal value $\nu_{2k+1}(2\pi)$ at the origin of system $(6.6.1)_{\delta=0}$ and the $5k$-th singular point value μ'_{5k} at the origin of system (6.6.5) have the following relation

$$\nu_{2k+1}(2\pi)\sim 5i\pi\mu'_{5k}.\tag{6.6.6}$$

By calculating singular point values at the origin of system (6.6.5) and from (6.6.6), we have

Theorem 6.6.1. *The first* 4 *focal value at the origin of system* $(6.6.1)_{\delta=0}$ *are as follows:*

$$\begin{aligned}\nu_3(2\pi)&=2\pi q_1,\\\nu_5(2\pi)&\sim\frac{2}{3}\pi(p_1-1)(p_1+1)(p_1+2)q_2,\\\nu_7(2\pi)&\sim\frac{1}{4}\pi(p_1-1)(p_1+1)(p_2^2+q_2^2-5)q_2,\\\nu_9(2\pi)&\sim\frac{35}{4}\pi(p_1-1)(p_1+1)q_2.\end{aligned}\tag{6.6.7}$$

It follows the following result.

Theorem 6.6.2. *The first four focal values at the origin of system* $(6.6.1)_{\delta=0}$ *are zero if and only if one of the following three conditions holds*

$$\begin{aligned} &C_1: \quad q_1 = 0, \quad q_2 = 0, \\ &C_2: \quad q_1 = 0, \quad p_1 = -1, \\ &C_3: \quad q_1 = 0, \quad p_1 = 1. \end{aligned} \tag{6.6.8}$$

Thus, we have

Theorem 6.6.3. *If condition* C_1 *holds, the vector fields defined by system* $(6.6.1)_{\delta=0}$ *is symmetry with respect to x-axis.*

If condition C_2 *holds, then system* $(6.6.1)_{\delta=0}$ *has a first integral* $f_1^{-3}f_2^2$ *in a neighborhood of the origin, where*

$$\begin{aligned} &f_1 = x^2 + y^2, \\ &f_2 = 3(x^2+y^2) - 2(p_2+3)x^3 - 6q_2x^2y + 6(p_2-1)xy^2 + 2q_2y^3. \end{aligned} \tag{6.6.9}$$

If condition C_3 *holds, then the right hand of system* $(6.6.1)_{\delta=0}$ *has a common factor*

$$x^2 + y^2 + 2(p_2+1)x^3 + 6q_2x^2y - 2(3p_2-1)xy^2 - 2q_2y^3 \tag{6.6.10}$$

and there exists a first integral f_1 *in a neighborhood of the origin.*

Form Theorem 6.6.2 and 6.6.3, we obtain

Theorem 6.6.4. *For system* $(6.6.1)_{\delta=0}$*, the origin is a center if and only if the first four focal values of the origin are zero, i.e., one of three conditions in Theorem* 6.6.2 *holds.*

We now construct an example, such that 4 limit cycles can be created from a $4th$ weak focus of system (6.6.1). If the coefficients of system (6.6.1) satisfy

$$\begin{aligned} &\delta = 7560\varepsilon^8, \quad q_1 = -\frac{21525}{2}\varepsilon^6, \quad q_2 = 1, \\ &p_1 = -2 + \frac{28665}{8}\varepsilon^4, \quad p_2 = 2 - \frac{525}{2}\varepsilon^2, \end{aligned} \tag{6.6.11}$$

then by Theorem 6.6.1 we have

$$\begin{aligned} &\nu_1(2\pi) - 1 = 15120\pi\varepsilon^8 + o(\varepsilon^8), \\ &\nu_3(2\pi) = -21525\pi\varepsilon^6 + o(\varepsilon^6), \quad \nu_5(2\pi) = \frac{28665}{4}\pi\varepsilon^4 + o(\varepsilon^4), \\ &\nu_7(2\pi) = -\frac{1575}{2}\pi\varepsilon^2 + o(\varepsilon^2), \quad \nu_9(2\pi) = \frac{105}{4}\pi + o(1). \end{aligned} \tag{6.6.12}$$

Therefore, the quasi succession function of (6.6.1) is given by

$$\begin{aligned}L(h,\varepsilon)&=\frac{105}{4}\pi(h^8-30h^6\varepsilon^2+273h^4\varepsilon^4-820h^2\varepsilon^6+576\varepsilon^8)\\&=\frac{105}{4}\pi(h^2-16\varepsilon^2)(h^2-9\varepsilon^2)(h^2-4\varepsilon^2)(h^2-\varepsilon^2).\end{aligned}\tag{6.6.13}$$

Thus, (6.6.13) and Theorem 3.3.3 imply that

Theorem 6.6.5. *If the coefficients of system* (6.6.1) *are given by* (6.6.11)*, then when* $\varepsilon=0$*, the origin is a* 4*-th weak focus, when* $0<|\varepsilon|\ll 1$*, there exist* 4 *limit cycles in a sufficient small neighborhood of the origin, which are close to the circles* $x^2+y^2=k^2\varepsilon^2$*,* $k=1,2,3,4$*.*

6.7 Quasi Isochronous Center of Multiple Singular Point for a Class of Analytic System

Making the transformation $dt'=(x^2+y^2)dt$ and $dT'=(zw)^n dT$, system $(6.2.2)_{\delta=0}$ and $(6.3.13)_{\delta=0}$ can be respectively become

$$\begin{aligned}\frac{dx}{dt'}&=-y+\frac{1}{(x^2+y^2)^n}\sum_{k=2n+2}^{\infty}X_k(x,y),\\\frac{dy}{dt'}&=x+\frac{1}{(x^2+y^2)^n}\sum_{k=2n+2}^{\infty}Y_k(x,y)\end{aligned}\tag{6.7.1}$$

and

$$\begin{aligned}\frac{dz}{dT}&=z+\frac{1}{(zw)^n}\sum_{k=2n+2}^{\infty}Z_k(z,w),\\\frac{dw}{dT}&=-w-\frac{1}{(zw)^n}\sum_{k=2n+2}^{\infty}W_k(z,w).\end{aligned}\tag{6.7.2}$$

Definition 6.7.1. (1) *We say that the origin of* (6.7.1) (*or* (6.7.2)) *is a complex isochronous center, if the origin of* $(6.2.6)_{\delta=0}$ (*or* $(6.3.17)_{\delta=0}$) *is a complex isochronous center.*

(2) *We say that origin of system* $(6.2.2)_{\delta=0}$ (*or* $(6.3.13)_{\delta=0}$) *is a complex quasi isochronous center, if the origin of* (6.7.1) (*or* (6.7.2)) *is a complex isochronous center.*

Clearly, the functions of the right hands of system (6.7.1) and system (6.7.2) are non-analytic at the origin. However, it is possible that the origin of these system are a complex isochronous center. We study the following system

$$\frac{dx}{dt'}=-y+\frac{X_4(x,y)}{x^2+y^2},\quad \frac{dy}{dt'}=x+\frac{Y_4(x,y)}{x^2+y^2},\tag{6.7.3}$$

where X_4, Y_4 are given by (6.6.2).

We see from Theorem 6.6.4 that the origin of system $(6.6.1)_{\delta=0}$ is a center if and only one of three conditions in Theorem 6.6.2 holds.

Proposition 6.7.1. *If condition C_1 in Theorem 6.6.2 hods, then the origin of system (6.7.1) is an isochronous centers if and only if $p_1 = -1$.*

Proof. When condition C_1 of Theorem *6.6.2* hods, then for the origin of system $(6.6.5)_{\delta=0}$, we have

$$\begin{aligned} &\tau_5 = -\frac{2}{3}(1+p_1)(3+2p_2^2+p_1p_2^2),\\ &\tau_{10} \sim \frac{1}{6}(1+p_1)(81+3p_1-40p_2-3p_1^2-12p_1p_2+4p_1^2p_2+9p_2^2),\\ &\tau_{15} \sim -\frac{1}{226800}(1+p_1)\\ &\times(99679615+53139921p_1-9621062p_1^2-4057890p_1^3+326167p_1^4+49953p_1^5). \end{aligned} \tag{6.7.4}$$

We see from (6.7.4) that if $\tau_5 = \tau_{10} = \tau_{15} = 0$, then $p_1 = -1$. In addition, if $p_1 = -1$ and condition C_1 in Theorem 6.6.2 hods, then for system $(6.6.5)_{\delta=0}$, under the polar coordinate $\xi = \rho e^{i\theta}$, $\eta = \rho e^{-i\theta}$, $T = it$, we have $\dfrac{d\theta}{dt} \equiv 1$. It implies the conclusions of this proposition. □

Proposition 6.7.2. *If condition C_2 in Theorem 6.6.2 hods, then the origin of system (6.7.1) is an isochronous centers.*

Proof. When condition C_2 in Theorem 6.6.2 hods, then for the origin of system $(6.6.5)_{\delta=0}$, under the polar coordinate $\xi = \rho e^{i\theta}$, $\eta = \rho e^{-i\theta}$, $T = it$, we have $\dfrac{d\theta}{dt} \equiv 1$. It implies the conclusions of this proposition. □

Proposition 6.7.3. *If condition C_3 in Theorem 6.6.2 hods, then the origin of system (6.7.1) can not be an isochronous centers.*

Proof. When condition C_3 in Theorem 6.6.2 hods, then for the origin of system $(6.6.5)_{\delta=0}$, we have

$$\tau_5 = -4(1+p_2^2+q_2^2), \quad \tau_{10} \sim -8(2p_2-3), \quad \tau_{15} \sim 180. \tag{6.7.5}$$

It implies the conclusions of this proposition.

To sum up, we have

Theorem 6.7.1. *The origin of system $(6.6.1)_{\delta=0}$ is a quasi isochronous center(i.e., the origin of system (6.7.1) is an isochronous center), if and only if $q_1 = 0$ and $p_1 = -1$.*

□

Bibliographical Notes

The materials of this chapter are taken from [Liu Y.R., 2001] and [Liu Y.R. etc, 2004]. There exist very few papers to concern with the problems of center-focus and bifurcation of limit cycles for this kind of multiple singular points.

Chapter 7

On Quasi Analytic Systems

For a nonanalytic real planar dynamical system, there is a few papers to concern with the study of the center problem and bifurcations of limit cycles. In this chapter, we investigate a class of quasi-analytic systems.

7.1 Preliminary

We consider the following class of systems:

$$\begin{aligned}\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} X_k(x,y),\\ \frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} Y_k(x,y),\end{aligned} \tag{7.1.1}$$

where for any positive integer k,

$$X_k(x,y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^{\alpha}y^{\beta}, \quad Y_k(x,y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^{\alpha}y^{\beta} \tag{7.1.2}$$

are homogeneous polynomials of degree k of x and y, λ is a real constant and $\lambda \neq 0$.

Clearly, when $\lambda = 1$, system (7.1.1) becomes

$$\begin{aligned}\frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{\infty} X_k(x,y) = X(x,y),\\ \frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{\infty} Y_k(x,y) = Y(x,y).\end{aligned} \tag{7.1.3}$$

We assume that $X(x,y)$ and $Y(x,y)$ are power series of x and y with non-zero convergent radius.

Generally, for $\lambda \neq (2s+1)$ where s is an positive integer, the functions of the right hand of system (7.1.1) are non-analytic. We say that system (7.1.1) is a quasi-analytic system corresponding to system (7.1.3).

For $\lambda > 0$ (or < 0), the linear terms of (7.1.1) are lowest (or highest) order terms in the right hand of (7.1.1). Hence, when $\lambda > 0$, the origin of (7.1.1) is a center or a focus. When $\lambda < 0$, (7.1.1) has no real singular point in the equator of Poincarè compactification. The point at infinity is a center or a focus. Therefore, it is necessary to determine whether the origin (or the infinity) is a center (or a weak focus) or not for all $\lambda \neq 0$.

Making the transformation

$$x = r^{\frac{1}{\lambda}} \cos\theta, \quad y = r^{\frac{1}{\lambda}} \sin\theta, \tag{7.1.4}$$

system (7.1.1) becomes

$$\begin{aligned} \frac{dr}{dt} &= \lambda r \left[\delta + \sum_{k=1}^{\infty} \varphi_{k+2}(\theta) r^k \right], \\ \frac{d\theta}{dt} &= 1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) r^k, \end{aligned} \tag{7.1.5}$$

where

$$\begin{aligned} \varphi_k(\theta) &= \cos\theta X_{k-1}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta), \\ \psi_k(\theta) &= \cos\theta Y_{k-1}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta). \end{aligned} \tag{7.1.6}$$

From (7.1.5), we have

$$\frac{dr}{d\theta} = \lambda r \frac{\delta + \sum_{k=1}^{\infty} \varphi_{k+2}(\theta) r^k}{1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) r^k}. \tag{7.1.7}$$

Obviously, equation (2.1.6), i.e., the polar coordinate form of (7.1.3), differs from (7.1.7) only in a constant factor λ.

Suppose that a solution of (7.1.3) satisfying the initial condition $r|_{\theta=0} = r_0$ has the form

$$r = \tilde{r}(\theta, r_0, \delta) = \sum_{k=1}^{\infty} \nu_k(\theta, \delta) r_0^k, \tag{7.1.8}$$

where

$$\nu_1(\theta, \delta) = e^{\delta\lambda\theta}, \quad \nu_k(0, \delta) = 0, \quad k = 2,\ 3,\ \cdots. \tag{7.1.9}$$

We know from the theory of Chapter 2 that if $\delta = 0$, then, for the first non-zero $\nu_k(2\pi, 0)$, we have $k = 2s + 1$, where s is a positive integer.

Definition 7.1.1. *For system* (7.1.1), *suppose that* $\lambda > 0$ (*or* $\lambda < 0$).

(1) *If* $\delta \neq 0$, *the origin* (*or the infinity*) *is called a rough focus.*

(2) *If* $\delta = 0$ *and there exists a positive integer* k, *such that* $\nu_2(2\pi, 0) = \nu_3(2\pi, 0) = \cdots = \nu_{2k-1}(2\pi, 0) = 0$, *and* $\nu_{2k+1}(2\pi, 0) \neq 0$, *then the origin* (*or the infinity*) *is called a* $k-$*order weak focus. The* $\nu_{2k+1}(2\pi, 0)$ *is called a* $k-$*order focal value.*

(3) *If* $\delta = 0$ *and for all* k, *we have* $\nu_{2k+1}(2\pi, 0) = 0$, *then the origin* (*or the infinity*) *is called a center.*

It is easy to see the following conclusions hold.

Theorem 7.1.1. *In the case of* $\lambda > 0$, *for system* (7.1.1):

(1) *If the origin is a rough focus, then when* $\delta < 0$ (> 0), *it is stable* (*unstable*);

(2) *If the origin is a* k*-order weak focus, then when* $\nu_{2k+1}(2\pi, 0) < 0$ (> 0), *the origin is stable* (*unstable*);

(3) *If the origin is a center, then there exists a family of closed orbits enclosing the origin.*

Theorem 7.1.2. *In the case of* $\lambda < 0$, *for system* (7.1.1):

(1) *If the infinity is a rough focus, then, when* $\delta > 0$ (< 0), *it is stable* (*unstable*);

(2) *If the infinity is a* k*-order weak focus, then when* $\nu_{2k+1}(2\pi, 0) < 0$ (> 0), *the infinity is stable* (*unstable*);

(3) *If the infinity is a center, then there exists a family of closed orbits which lies in an inner neighborhood of the equator in Poincarè compactification.*

Remark 7.1.1. *For* $\lambda < 0$, *if the infinity in Poincarè compactification of system* (7.1.1) *is a stable* (*unstable*) *focus, then the equator* Γ_∞ *in Poincaré compactification of system* (7.1.1) *is a internal stable* (*unstable*) *limit cycle. If the infinity of system* (7.1.1) *is a center, then, there exists a family of closed orbits which lies in an inner neighborhood of the equator* Γ_∞.

Definition 7.1.2. (1) *For* $\lambda > 0$, *if the origin of* (7.1.1) *is a center and the period of any closed orbit enclosing the center is* 2π, *then, the origin is called an isochronous center of* (7.1.1).

(2) *For* $\lambda < 0$, *if the infinity of* (7.1.1) *is a center and the period of any closed orbit in an inner neighborhood of the equator is* 2π, *then, the infinity is called an isochronous center of* (7.1.1).

Clearly, for $\lambda > 0$ (< 0), when the origin (the infinity) of (7.1.1) is an center, the origin (the infinity) of (7.1.1) is an isochronous center if and only if $\mathcal{T}(r_0) \equiv 2\pi$, where

$$\mathcal{T}(r_0) = \int_0^{2\pi} \frac{d\theta}{1 + \sum\limits_{k=1}^{\infty} \psi_{k+2}(\theta)\tilde{r}^k(\theta, r_0, 0)}. \tag{7.1.10}$$

It is difficult to study directly the problems of center-focus, isochronous center and bifurcations of limit cycles by using (7.1.7). In next section, we shall use a necessary transformation, such that system (7.1.1) becomes an equivalent analytic system.

7.2 Reduction of the Problems

Let

$$\xi = x(x^2 + y^2)^{\frac{\lambda-3}{6}}, \quad \eta = y(x^2 + y^2)^{\frac{\lambda-3}{6}}. \tag{7.2.1}$$

Taking $x = r^{\frac{1}{\lambda}} \cos\theta, \quad y = r^{\frac{1}{\lambda}} \sin\theta$, we have

$$\xi = r^{\frac{1}{3}} \cos\theta, \quad \eta = r^{\frac{1}{3}} \sin\theta. \tag{7.2.2}$$

Thus, for $\lambda > 0$ (< 0), (7.2.1) makes the origin (the infinity) become the origin in $(\xi, \eta)-$plane and (7.1.1) reduce to

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{\delta\lambda}{3}\xi - \eta + \frac{1}{3}\sum_{k=1}^{\infty}(\xi^2 + \eta^2)^{k-1} P_{k+3}(\xi, \eta),\\ \frac{d\eta}{dt} &= \xi + \frac{\delta\lambda}{3}\eta + \frac{1}{3}\sum_{k=1}^{\infty}(\xi^2 + \eta^2)^{k-1} Q_{k+3}(\xi, \eta),\end{aligned} \tag{7.2.3}$$

where for any positive integer k,

$$\begin{aligned}P_{k+3}(\xi, \eta) &= (\lambda\xi^2 + 3\eta^2)X_{k+1}(\xi, \eta) + (\lambda - 3)\xi\eta Y_{k+1}(\xi, \eta),\\ Q_{k+3}(\xi, \eta) &= (\lambda\eta^2 + 3\xi^2)Y_{k+1}(\xi, \eta) + (\lambda - 3)\xi\eta X_{k+1}(\xi, \eta)\end{aligned} \tag{7.2.4}$$

are homogeneous polynomials of degree $k + 3$ of x and y. Thus, the study on the problems of center-focus and bifurcations of limit cycles for (7.1.1) is transformed to the discussion for (7.2.3).

Notice that the functions on the right hand of (7.2.3) are a special class of (7.1.3). Therefore, it makes sense to do new study for this system.

Remark 7.2.1. *The functions on the right hand of system* (7.2.3) *have the following properties*:

(1) *There exist two complex straight line solutions* $\xi + i\eta = 0$ *and* $\xi - i\eta = 0$ *of* (7.2.3).

(2) *Expanding the two functions of the right hand of* (7.2.3) *as two power series of* ξ *and* η, *then, every monomial has the degree* $3k - 2, \quad k = 1,\ 2,\ \cdots$.

(3) *As a exponent parameter* λ *of* (7.1.1), *it becomes a coefficient parameter of* (7.2.3). *Therefore,* λ *appears in all formulas of the focal values and period constants. This is different from an analytic system.*

By using the polar coordinate transformation

$$\xi = \rho\cos\theta, \quad \eta = \rho\sin\theta, \tag{7.2.5}$$

system (7.2.3) becomes

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{\lambda}{3}\rho\left[\delta + \sum_{k=1}^{\infty}\varphi_{k+2}(\theta)\rho^{3k}\right],\\ \frac{d\theta}{dt} &= 1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta)\rho^{3k}.\end{aligned} \tag{7.2.6}$$

From (7.2.6), we have

$$\frac{d\rho}{d\theta} = \frac{\lambda}{3}\rho \cdot \frac{\delta + \sum_{k=1}^{\infty}\varphi_{k+2}(\theta)\rho^{3k}}{1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta)\rho^{3k}}. \tag{7.2.7}$$

Obviously, (7.2.7) can become the from (7.1.7) by using the transformation $r = \rho^3$.

For each small ρ_0, writing the solution of (7.2.7) satisfying the initial value $\rho|_{\theta=0} = \rho_0$ as $\rho = \tilde{\rho}(\theta, \rho_0, \delta)$, by using (7.2.3), we have the following lemma.

Lemma 7.2.1. *The solution* $\rho = \tilde{\rho}(\theta, \rho_0, \delta)$ *of* (7.2.7) *has the form*

$$\tilde{\rho}(\theta, \rho_0, \delta) = \rho_0 \sum_{k=0}^{\infty}\sigma_{3k+1}(\theta, \delta)\rho_0^{3k}, \tag{7.2.8}$$

where

$$\sigma_1(\theta, \delta) = e^{\frac{\delta\lambda\theta}{3}}, \quad \sigma_{3k+1}(0, \delta) = 0, \quad k = 1,\ 2,\ \cdots. \tag{7.2.9}$$

To find the relationship between the k-order focal value $\nu_{2k+1}(2\pi, 0)$ of (7.1.1) and the $3k$-order focal value $\sigma_{6k+1}(2\pi, 0)$ of (7.2.3), we need the following results.

Theorem 7.2.1. $\sigma_{6k+1}(2\pi, 0)$ *and* $\nu_{2k+1}(2\pi, 0)$ *are algebraic equivalent, i.e.,*

$$\{\nu_{2k+1}(2\pi, 0)\} \sim \{3\sigma_{6k+1}(2\pi, 0)\}. \tag{7.2.10}$$

Proof. Equation (7.2.7) can be become to the from (7.1.5), by using the transformation $r = \rho^3$. So that,

$$\tilde{r}(\theta, \rho_0^3, 0) \equiv \tilde{\rho}^3(\theta, \rho_0, 0). \tag{7.2.11}$$

From (7.2.11), when $r_0 = \rho_0^3$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} \nu_{k+1}(2\pi, 0) r_0^k &= \frac{\tilde{r}(2\pi, r_0, 0)}{r_0} - 1 \\ &= \frac{\tilde{\rho}^3(2\pi, \rho_0, 0)}{\rho_0^3} - 1 = \left(1 + \sum_{k=1}^{\infty} \sigma_{3k+1}(2\pi, 0) r_0^k\right)^3 - 1 \\ &= 3 \sum_{k=1}^{\infty} \sigma_{3k+1}(2\pi, 0) r_0^k G(r_0), \end{aligned} \tag{7.2.12}$$

where

$$G(r_0) = 1 + \left(\sum_{k=1}^{\infty} \sigma_{3k+1}(2\pi, 0) r_0^k\right) + \frac{1}{3}\left(\sum_{k=1}^{\infty} \sigma_{3k+1}(2\pi, 0) r_0^k\right)^2 \tag{7.2.13}$$

is a unit formal power series of r_0. Thus, (7.2.12) and Theorem 2.2.1 give rise to the conclusion of this theorem. □

7.3 Focal Values, Periodic Constants and First Integrals of (7.2.3)

By using the transformation

$$z = \xi + i\eta, \quad w = \xi - i\eta, \quad T = it, \quad i = \sqrt{-1}, \tag{7.3.1}$$

system $(7.2.3)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT} &= \ \ z + \frac{z}{6} \sum_{k=1}^{\infty} (zw)^{k-1} \Phi_{k+2}(z, w) = \mathcal{Z}(z, w), \\ \frac{dw}{dT} &= -w - \frac{w}{6} \sum_{k=1}^{\infty} (zw)^{k-1} \Psi_{k+2}(z, w) = -\mathcal{W}(z, w), \end{aligned} \tag{7.3.2}$$

where

$$\begin{aligned} \Phi_k(z, w) &= (\lambda + 3) w Z_{k-1}(z, w) - (\lambda - 3) z W_{k-1}(z, w), \\ \Psi_k(z, w) &= (\lambda + 3) z W_{k-1}(z, w) - (\lambda - 3) w Z_{k-1}(z, w), \\ Z_k(z, w) &= Y_k\left(\frac{z + w}{2}, \frac{z - w}{2i}\right) - iX_k\left(\frac{z + w}{2}, \frac{z - w}{2i}\right), \\ W_k(z, w) &= Y_k\left(\frac{z + w}{2}, \frac{z - w}{2i}\right) + iX_k\left(\frac{z + w}{2}, \frac{z - w}{2i}\right). \end{aligned} \tag{7.3.3}$$

We next denote that

$$Z_k(z, w) = \sum_{\alpha+\beta=k} a_{\alpha\beta} z^\alpha w^\beta, \quad W_k(z, w) = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^\alpha z^\beta. \tag{7.3.4}$$

Let μ_k be the k-th singular point value of the origin of system (7.3.2), τ_k be the k-th period constant, $k = 1, 2, \cdots$. By the properties of system (7.3.2), we have

Lemma 7.3.1. *For system* (7.3.2)*, we have*

$$\{\mu_{3k-1}\} = \{0\}, \quad \{\mu_{3k-2}\} = \{0\}, \quad \{\tau_{3k-1}\} = \{0\}, \quad \{\tau_{3k-2}\} = \{0\}. \tag{7.3.5}$$

From Theorem 7.2.2 and Theorem 2.3.1, we obtain

Theorem 7.3.1.

$$\{\nu_{2k+1}(2\pi, 0)\} \ \sim \ \{3\sigma_{6k+1}(2\pi, 0)\} \ \sim \ \{3i\pi\mu_{3k}\}. \tag{7.3.6}$$

Denote that

$$\mathcal{T}(r_0) = \pi\left(2 - \sum_{k=1}^{\infty} \mathcal{T}_k r_0^k\right), \tag{7.3.7}$$

where $\mathcal{T}(r_0)$ is given by (7.1.10). Thus, (7.1.10), (7.2.11) and (7.3.7) follow that

$$\mathcal{T}(\rho_0^3) = \pi\left(2 - \sum_{k=1}^{\infty} \mathcal{T}_k \rho_0^{3k}\right) = \int_0^{2\pi} \frac{d\theta}{1 + \sum\limits_{k=1}^{\infty} \psi_{k+2}(\theta)\tilde{\rho}^{3k}(\theta, \rho_0, 0)}. \tag{7.3.8}$$

From (7.3.8) and Theorem 4.1.2, we have

Theorem 7.3.2.

$$\{\mathcal{T}_{2k-1}\} \ \sim \ \{0\}, \quad \{\mathcal{T}_{2k}\} \ \sim \ \{\tau_{3k}\}. \tag{7.3.9}$$

Definition 7.3.1. *For* $\lambda > 0$ (< 0)*,* $\mathcal{T}_{2k}$ *is called k-th period constant of the origin* (*the infinity*) *of system* $(7.1.1)_{\delta=0}$.

From Theorem 7.3.2, we know that

Theorem 7.3.3. *For* $\lambda > 0$ (< 0)*, the origin* (*the infinity*) *of system* $(7.1.1)_{\delta=0}$ *is an isochronous center, if and only if the origin of system* $(7.2.3)_{\delta=0}$ *is an isochronous center.*

Theorem 2.3.5 implies that

Theorem 7.3.4. *For system* (7.3.2)*, one can derive successively the terms of the following formal series*

$$M(z, w) = \sum_{k=0}^{\infty} M_{3k}(z, w), \tag{7.3.10}$$

where

$$M_{3k}(z,w) = \sum_{\alpha+\beta=3k} c_{\alpha\beta} z^\alpha w^\beta \tag{7.3.11}$$

is a $3k$-degree homogeneous polynomial in z, w and

$$c_{00} = 1, \quad c_{3k,3k} = 0, \quad k = 1,\ 2,\ \cdots, \tag{7.3.12}$$

such that

$$\frac{\partial(M\mathcal{Z})}{\partial z} - \frac{\partial(M\mathcal{W})}{\partial w} = \sum_{k=1}^{\infty}(3k+1)\mu'_{3k}(zw)^{3k}. \tag{7.3.13}$$

In addition,

$$\{\mu'_{3k}\} \ \sim \ \{\mu_{3k}\}. \tag{7.3.14}$$

Similar to Theorem 2.3.6, we have

Theorem 7.3.5. *When $\alpha \neq \beta$, $c_{\alpha\beta}$ in (7.3.11) are given successively by*

$$\begin{aligned} c_{\alpha\beta} = \frac{1}{6(\beta-\alpha)} \sum_{k+j=3}^{\frac{1}{3}(\alpha+\beta+6)} \Big\{ & [\ 2\lambda + (\lambda+3)\alpha + (\lambda-3)\beta\ ]\ a_{k,j-1} \\ & -[\ 2\lambda + (\lambda-3)\alpha + (\lambda+3)\beta\ ]\ b_{j,k-1}\Big\}\ c_{\alpha-2k-j+3,\ \beta-2j-k+3} \end{aligned} \tag{7.3.15}$$

and for any positive integer m, μ'_{3m} in (7.3.13) are given successively by

$$\mu'_{3m} = \frac{\lambda}{3} \sum_{k+j=3}^{2m+2} (a_{k,j-1} - b_{j,k-1}) c_{3m-2k-j+3,\ 3m-2j-k+3}, \tag{7.3.16}$$

where for $k < 0$ or $j < 0$, we take $c_{kj} = a_{kj} = b_{kj} = 0$.

From Theorem 4.2.1 and Theorem 4.2.3, we obtain

Theorem 7.3.6. *Suppose that the origin of system (7.3.2) is a complex center, then one can derive successively the terms of the following formal series*

$$f(z,w) = z\sum_{k=0}^{\infty} \sum_{\alpha+\beta=3k} c_{\alpha\beta} z^\alpha w^\beta, \tag{7.3.17}$$

where $c_{00} = 1$, $c_{3k,3k} = 0$, $k = 1,\ 2,\ \cdots$, such that

$$\frac{df}{dT} = f + \frac{z}{2}\sum_{m=1}^{\infty} \tau'_{3m}(zw)^{3m}. \tag{7.3.18}$$

In addition, $\tau'_3 = \tau_3$, and for all positive integer m, when $\tau'_3 = \tau'_6 = \cdots = \tau'_{3(m-1)} = 0$, we have $\tau'_{3m} = \tau_{3m}$.

Theorem 7.3.7. *In Theorem* 7.3.6, *for all pairs* $(\alpha,\ \beta)$ *with* $\alpha \neq \beta$, $c_{\alpha\beta}$ *is given successively by*

$$c_{\alpha\beta} = \frac{1}{6(\beta-\alpha)} \sum_{k+j=3}^{\frac{1}{3}(\alpha+\beta+6)} \Big[\lambda(\alpha+\beta-3k-3j+7)(a_{k,j-1}-b_{j,k-1}) \\ +3(\alpha-\beta-k+j+1)(a_{k,j-1}+b_{j,k-1})\Big]\, c_{\alpha-2k-j+3,\ \beta-2j-k+3}, \quad (7.3.19)$$

and for all positive integer m, τ'_{3m} *is given successively by*

$$\tau'_{3m} = \frac{1}{3} \sum_{k+j=3}^{2m+2} [\ \lambda(6m-3k-3j+7)(a_{k,j-1}-b_{j,k-1}) \\ +3(1-k+j)(a_{k,j-1}+b_{j,k-1})]\ c_{3m-2k-j+3,\ 3m-2j-k+3}, \quad (7.3.20)$$

in which for $k < 0$ *or* $j < 0$, *we take* $c_{kj} = a_{kj} = b_{kj} = 0$.

Clearly, Theorem 7.3.5 and Theorem 7.3.7 give two recursive formulas to determine the focal values and period constants of $(7.2.3)_{\delta=0}$. It can be realized easily by computer program.

From the peculiarity of system (7.3.2), the origin of system (7.3.2) is a complex center, if and only if there exists a first integral $F(z,w)$ of the form

$$F(z,w) = zw \sum_{k=0}^{\infty} F_{3k}(z,w), \quad (7.3.21)$$

where $F_0(z,w) \equiv 1$, $F_{3k}(z,w)$ is a homogeneous polynomial of degree $3k$ in z and w, $k = 0,\ 1,\ \cdots$. In addition, $\sum_{k=0}^{\infty} F_{3k}(z,w)$ are analytic in a neighborhood of the origin.

Let $\mathcal{F}(z,w) = F^s(z,w)$, we have

Theorem 7.3.8. *The origin of system* (7.3.2) *is a complex center, if and only if for any non-zero constant* s, *there exists a first integral* $\mathcal{F}(z,w)$ *of the form*

$$\mathcal{F}(z,w) = (zw)^s \sum_{k=0}^{\infty} \mathcal{F}_{3k}(z,w), \quad (7.3.22)$$

where $\mathcal{F}_0(z,w) \equiv 1$, $\mathcal{F}_{3k}(z,w)$ *is a homogeneous polynomial of degree* $3k$ *in* z *and* w, $k = 0,\ 1,\ \cdots$. *In addition,* $\sum_{k=0}^{\infty} \mathcal{F}_{3k}(z,w)$ *are analytic in a neighborhood of the origin.*

Theorem 7.3.8 implies the following result.

Theorem 7.3.9. *For $\lambda > 0$ (< 0), the origin (the infinity) of system* $(7.1.1)_{\delta=0}$ *is a center, if and only if there exists a first integral in a neighborhood of the the origin (the infinity) having the form*

$$\mathcal{H}(x,y) = (x^2+y^2)^{\frac{\lambda s}{3}} \sum_{k=0}^{\infty} (x^2+y^2)^{\frac{(\lambda-3)k}{2}} \mathcal{H}_{3k}(x,y), \tag{7.3.23}$$

where s is a non-zero constant, $\mathcal{H}_{3k}(x,y) = \mathcal{F}_{3k}(x+iy, x-iy)$ is a homogeneous polynomial of degree $3k$ in x and y, $k = 0, 1, \cdots,$ $\mathcal{H}_0(x,y) \equiv 1$. In addition, $\sum\limits_{k=0}^{\infty} \mathcal{H}_{3k}(x,y)$ are analytic in a neighborhood of the origin.

7.4 Singular Point Values and Bifurcations of Limit Cycles of Quasi-Quadratic Systems

In this section, we consider quasi-quadratic systems

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y + (x^2+y^2)^{\frac{\lambda-1}{2}} X_2(x,y), \\ \frac{dy}{dt} &= x + \delta y + (x^2+y^2)^{\frac{\lambda-1}{2}} Y_2(x,y), \end{aligned} \tag{7.4.1}$$

where

$$\begin{aligned} X_2(x,y) &= A_{20}x^2 + A_{11}xy + A_{02}y^2, \\ Y_2(x,y) &= B_{20}x^2 + B_{11}xy + B_{02}y^2. \end{aligned} \tag{7.4.2}$$

By using (7.2.1), (7.4.1) becomes

$$\begin{aligned} \frac{d\xi}{dt} &= \frac{\delta\lambda}{3}\xi - \eta + \frac{1}{3}[(\lambda\xi^2 + 3\eta^2)X_2(\xi,\eta) + (\lambda-3)\xi\eta Y_2(\xi,\eta)], \\ \frac{d\xi}{dt} &= \xi + \frac{\delta\lambda}{3}\eta + \frac{1}{3}[(\lambda\eta^2 + 3\xi^2)Y_2(\xi,\eta) + (\lambda-3)\xi\eta X_2(\xi,\eta)]. \end{aligned} \tag{7.4.3}$$

Transformation (7.3.1) makes (7.4.3) become

$$\begin{aligned} \frac{dz}{dT} &= \left(1 - \frac{i\delta\lambda}{3}\right) z + \frac{z}{6}[(\lambda+3)wZ_2(z,w) - (\lambda-3)zW_2(z,w)], \\ \frac{dw}{dT} &= -\left(1 + \frac{i\delta\lambda}{3}\right) w - \frac{w}{6}[(\lambda+3)zW_2(z,w) - (\lambda-3)wZ_2(z,w)], \end{aligned} \tag{7.4.4}$$

where

$$\begin{aligned} Z_2(z,w) &= a_{20}z^2 + a_{11}zw + a_{02}w^2, \\ W_2(z,w) &= b_{20}w^2 + b_{11}zw + b_{02}z^2. \end{aligned} \tag{7.4.5}$$

The coefficients of (7.4.4) and (7.4.1) have the following relations

$$
\begin{aligned}
a_{20} &= \frac{1}{4}[(B_{20} - A_{11} - B_{02}) - i(A_{20} + B_{11} - A_{02})],\\
b_{20} &= \frac{1}{4}[(B_{20} - A_{11} - B_{02}) + i(A_{20} + B_{11} - A_{02})],\\
a_{11} &= \frac{1}{2}[(B_{20} + B_{02}) - i(A_{20} + A_{02})],\\
b_{11} &= \frac{1}{2}[(B_{20} + B_{02}) + i(A_{20} + A_{02})],\\
a_{02} &= \frac{1}{4}[(B_{20} + A_{11} - B_{02}) - i(A_{20} - B_{11} - A_{02})],\\
b_{02} &= \frac{1}{4}[(B_{20} + A_{11} - B_{02}) + i(A_{20} - B_{11} - A_{02})].
\end{aligned}
\tag{7.4.6}
$$

The recursive formulas to compute the singular point values of origin of system $(7.4.4)_{\delta=0}$ have been given by [Liu Y.R., 2002], in which we know that

Theorem 7.4.1. *The first 18 singular point values at the origin of system* $(7.4.4)_{\delta=0}$ *are given by*

$$
\begin{aligned}
\mu_3 &= \frac{\lambda(b_{20}b_{11} - a_{20}a_{11})}{3},\\
\mu_6 &= \frac{-\lambda\Delta_2}{36},\\
\mu_9 &= \lambda\frac{\left[60H + (\lambda^2 - 10\lambda - 51)a_{02}b_{02})\right]\Delta_1 - (\lambda-1)(\lambda+3)a_{02}b_{02}\Delta_3}{1152},\\
\mu_{12} &= \frac{-\lambda(\lambda-1)(\lambda+9)g_1(\lambda)a_{02}^2b_{02}^2\Delta_1}{2073600},\\
\mu_{15} &= \frac{\lambda(\lambda-1)(5\lambda^2+6\lambda+81)\left[24216192000J_0 + g_2(\lambda)a_{02}^2b_{02}^2\right]a_{02}b_{02}\Delta_1}{608662978560000},\\
\mu_{18} &= \frac{-\lambda(\lambda-1)(\lambda+9)g_3(\lambda)a_{02}^4b_{02}^4\Delta_1}{172345713609765019484160000},
\end{aligned}
\tag{7.4.7}
$$

where

$$
\begin{aligned}
g_1(\lambda) &= 91\lambda^2 - 326\lambda - 3057,\\
g_2(\lambda) &= 3719329667\lambda^2 - 4236625226\lambda - 85125314061,\\
g_3(\lambda) &= 114059407179219568146253\lambda + 499405812207464098577649,
\end{aligned}
$$

$$
\begin{aligned}
H &= (a_{20}+b_{11})(b_{20}+a_{11}),\\
J_0 &= (a_{20}+b_{11})^3 a_{02} + (b_{20}+a_{11})^3 b_{02},\\
\Delta_1 &= [(\lambda-1)a_{20}-(\lambda+1)b_{11}]^2(2a_{20}-b_{11})a_{02}\\
&\quad -[(\lambda-1)b_{20}-(\lambda+1)a_{11}]^2(2b_{20}-a_{11})b_{02},\\
\Delta_2 &= [(\lambda-1)a_{20}-(\lambda+1)b_{11}][(\lambda-3)a_{20}-(\lambda+3)b_{11}](2a_{20}-b_{11})a_{02}\\
&\quad -[(\lambda-1)b_{20}-(\lambda+1)a_{11}][(\lambda-3)b_{20}-(\lambda+3)a_{11}](2b_{20}-a_{11})b_{02},\\
\Delta_3 &= [(\lambda-3)a_{20}-(\lambda+3)b_{11}]^2(2a_{20}-b_{11})a_{02}\\
&\quad -[(\lambda-3)b_{20}-(\lambda+3)a_{11}]^2(2b_{20}-a_{11})b_{02}. \qquad (7.4.8)
\end{aligned}
$$

Theorem 7.4.1 follows that

Theorem 7.4.2. *The origin of system* $(7.4.4)_{\delta=0}$ *is a* 18*-th weak singular point if and only if the following conditions hold*:

$$
\begin{aligned}
&(\lambda-3)a_{20}-(\lambda+3)b_{11} = (\lambda-3)b_{20}-(\lambda+3)a_{11} = 0,\\
&H = \frac{-(\lambda^2-10\lambda-51)}{60}a_{02}b_{02},\\
&550918368000J_0 + (206742143969\lambda + 905896803117)a_{02}^2b_{02}^2 = 0,\\
&g_1(\lambda) = 0, \quad \Delta_1 a_{02}b_{02} \neq 0. \qquad (7.4.9)
\end{aligned}
$$

Proposition 7.4.1. *If system* $(7.4.3)_{\delta=0}$ *is a real autonomous differential system, then it is impossible that the origin is the* 18*-th weak focus point of the real system* $(7.4.3)_{\delta=0}$.

Proof. Let system $(7.4.3)_{\delta=0}$ be a real autonomous differential system. We have from (7.4.6) that

$$
b_{20} = \bar{a}_{20}, \quad b_{11} = \bar{a}_{11}, \quad b_{02} = \bar{a}_{02}. \qquad (7.4.10)
$$

Therefore, (7.4.10) follows that

$$
a_{02}b_{02} > 0, \quad H > 0, \quad I_0^2 \leqslant 0, \qquad (7.4.11)
$$

where

$$
I_0 = (a_{20}+b_{11})^3 a_{02} - (b_{20}+a_{11})^3 b_{02}. \qquad (7.4.12)
$$

Equation $g_1(\lambda) = 0$ has two roots

$$
\lambda_1 = \frac{163-\sqrt{76189}}{91}, \quad \lambda_2 = \frac{163+\sqrt{76189}}{91}. \qquad (7.4.13)
$$

We have from (7.4.9) that

$$
\begin{aligned}
H|_{\lambda=\lambda_1} &= \frac{-11738}{29917+146\sqrt{76189}}\, a_{02}b_{02} < 0,\\
H|_{\lambda=\lambda_2} &= \frac{2(29917+146\sqrt{76189}}{124215}\, a_{02}b_{02} > 0. \qquad (7.4.14)
\end{aligned}
$$

From (7.4.11) and (7.4.14), we have $\lambda \neq \lambda_1$. When $\lambda = \lambda_2$, we have

$$\begin{aligned} I_0^2 &\equiv J_0^2 - 4H^3 a_{02}b_{02} \\ &= \frac{870488298132255556324979\,3\lambda_2 + 3735555106552110674683836\,9}{690487634654507289600000\,0} a_{02}^4 b_{02}^4 > 0. \end{aligned} \tag{7.4.15}$$

This result is in contradiction with (7.4.11). Hence, it gives rise to the conclusion of this proposition. □

From the Theorem 7.4.1, it is easily proved that

Theorem 7.4.3. *The origin of system* $(7.4.4)_{\delta=0}$ *is a* 15*-th weak singular point if and only if the following conditions holds*:

$$\begin{aligned} &(\lambda-3)a_{20}-(\lambda+3)b_{11}=(\lambda-3)b_{20}-(\lambda+3)a_{11}=0, \\ &\lambda=\lambda_2, \qquad H=\frac{2(29917+146\sqrt{76189})}{124215}a_{02}b_{02}, \\ &(24216192000J_0+g_2(\lambda)a_{02}^2b_{02}^2)a_{02}b_{02}\Delta_1 \neq 0. \end{aligned} \tag{7.4.16}$$

The conditions of Theorem 7.4.3 can be realized in real domain. The following conclusion are given by [Liu Y.R., 2002]

Theorem 7.4.4. *For system* (7.4.3)*, if*

$$\begin{aligned} &a_{20}=b_{20}=\lambda+3, \quad a_{11}=b_{11}=\lambda-3, \quad \lambda=\frac{163+2\sqrt{76189}}{91}, \\ &a_{02}=\frac{3057\sqrt{10}\ i}{\sqrt{107059+1162\sqrt{76189}}}, \quad b_{02}=\frac{-3057\sqrt{10}\ i}{\sqrt{107059+1162\sqrt{76189}}}, \end{aligned} \tag{7.4.17}$$

then, the origin of system $(7.4.3)_{\delta=0}$ *is a stable* 15*-th weak focus point. By small perturbation, there exist* 5 *limit cycles in a sufficiently small neighborhood of the origin.*

7.5 Integrability of Quasi-Quadratic Systems

We know from Theorem 7.4.1 that

Theorem 7.5.1. *For system* $(7.4.4)_{\delta=0}$*, the first* 18 *singular point values are zero if and only if one of the following* 10 *conditions are satisfied*:

$$C_1: \quad 2a_{20}-b_{11}=2b_{20}-a_{11}=0;$$

$$C_2: \quad \begin{cases} a_{20}a_{11}=b_{20}b_{11}, \quad |2a_{20}-b_{11}|+|2b_{20}-a_{11}|\neq 0, \\ a_{20}^3a_{02}=b_{20}^3b_{02}, \quad a_{20}^2b_{11}a_{02}=b_{20}^2a_{11}b_{02}, \\ a_{20}b_{11}^2a_{02}=b_{20}a_{11}^2b_{02}, \quad b_{11}^3a_{02}=a_{11}^3b_{02}; \end{cases}$$

$$
\begin{aligned}
&C_3: \quad \lambda = 1, \quad a_{11} = b_{11} = 0;\\
&C_4: \quad \lambda = 1, \quad a_{20} + 2b_{11} = b_{20} + 2a_{11} = 0, \quad a_{11}b_{11} = a_{02}b_{02};\\
&C_5: \quad \lambda = 3, \quad a_{11} = b_{11} = a_{20} = a_{02} = 0, \quad b_{20}b_{02} \neq 0;\\
&C_5^*: \quad \lambda = 3, \quad b_{11} = a_{11} = b_{20} = b_{02} = 0, \quad a_{20}a_{02} \neq 0;\\
&C_6: \quad \begin{cases} (\lambda-3)a_{20} - (\lambda+3)b_{11} = 0, \\ b_{20} = a_{11} = b_{02} = 0, \quad (\lambda-3)a_{02} \neq 0; \end{cases}\\
&C_6^*: \quad \begin{cases} (\lambda-3)b_{20} - (\lambda+3)a_{11} = 0, \\ a_{20} = b_{11} = a_{02} = 0, \quad (\lambda-3)b_{02} \neq 0; \end{cases}\\
&C_7: \quad \begin{cases} (\lambda-1)a_{20} - (\lambda+1)b_{11} = (\lambda-1)b_{20} - (\lambda+1)a_{11} = 0, \\ b_{02} = 0, \quad (\lambda+3)(\lambda-1)a_{02} \neq 0; \end{cases}\\
&C_7^*: \quad \begin{cases} (\lambda-1)a_{20} - (\lambda+1)b_{11} = (\lambda-1)b_{20} - (\lambda+1)a_{11} = 0, \\ a_{02} = 0, \quad (\lambda+3)(\lambda-1)b_{02} \neq 0. \end{cases}
\end{aligned} \tag{7.5.1}
$$

Proposition 7.5.1. *When C_1 holds, there exists the integral factor of system $(7.4.4)_{\delta=0}$ as follows*:

$$
M_1 = (zw)^{-1}[6 + (3-\lambda)(b_{02}z^3 + 3a_{20}z^2w + 3b_{20}w^2z + a_{02}w^3)]^{-1}. \tag{7.5.2}
$$

Proposition 7.5.2. *When C_2 holds, the coefficients of the right hand satisfy the conditions of the extended symmetric principle.*

Proposition 7.5.3. *When C_3 holds, there exists the integral factor of system $(7.4.4)_{\delta=0}$ as follows*:

$$
\begin{aligned}
M_2 = z^2w^2[&1 + 2zw(a_{20}z + b_{20}w) + (a_{20}^2 + b_{20}b_{02})z^4w^2\\
&+3(a_{20}b_{20} - a_{02}b_{02})z^3w^3 + (b_{20}^2 + a_{20}a_{02})w^4z^2\\
&+(a_{20}b_{20} - a_{02}b_{02})(b_{02}z^3 + a_{20}z^2w + b_{20}w^2z + a_{02}w^3)z^3w^3]^{-1}.
\end{aligned} \tag{7.5.3}
$$

Proposition 7.5.4. *When C_4 holds, system there exists the integral factor of system $(7.4.4)_{\delta=0}$ as follows*:

$$
M_3 = \frac{z^2w^2}{(1-2b_{11}z^2w-2a_{11}w^2z-a_{11}b_{02}z^4w^2+2a_{11}b_{11}z^3w^3-b_{11}a_{02}w^4z^2)^{\frac{5}{2}}}. \tag{7.5.4}
$$

Proposition 7.5.5. *When one of C_5 and C_5^* holds, there exists the integral factor of system $(7.4.4)_{\delta=0}$ as follows*:

$$
M_4 = \frac{1}{z^3w^3}\exp\frac{-2(b_{02}z^3 + a_{02}w^3)}{3}. \tag{7.5.5}
$$

Proposition 7.5.6. *When one of C_6 and C_6^* holds, there exists the integral factor of system $(7.4.4)_{\delta=0}$ as follows*:

$$
M_5 = (zw)^{-3}\left(\frac{6}{3-\lambda} + b_{02}z^3 + a_{02}w^3\right)^{\frac{\lambda+9}{3\lambda-9}}. \tag{7.5.6}
$$

Proposition 7.5.7. *If one of* C_7 *and* C_7^* *holds, then system* $(7.4.4)_{\delta=0}$ *is linearizable in a neighborhood of the origin.*

Proof. When C_7 holds, system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT} &= z + \frac{2}{3}(a_{20}+b_{11})z^3w + \frac{1}{3}(b_{20}+a_{11})z^2w^2 + \frac{\lambda+3}{6}a_{02}zw^3,\\ \frac{dw}{dT} &= -w + \frac{\lambda-3}{6}a_{02}w^4 - \frac{2}{3}(b_{20}+a_{11})w^3z - \frac{1}{3}(a_{20}+b_{11})w^2z^2.\end{aligned} \quad (7.5.7)$$

We see from Theorem 1.6.7 that system (7.5.7) is linearizable in a neighborhood of the origin. Similarly, we can prove that if C_7^* is satisfied, then system $(7.4.4)_{\delta=0}$ is also linearizable in a neighborhood of the origin. □

Theorem 1.8.28, Theorem 7.5.1 and Proposition 7.5.1 ∼ Proposition 7.5.7 imply that

Theorem 7.5.2. *The origin of system* $(7.4.4)_{\delta=0}$ *is a complex center, if and only if the first* 18 *singular point values are all zero, i.e., one of the* 10 *conditions in Theorem* 7.5.1 *is satisfied.*

7.6 Isochronous Center of Quasi-Quadratic Systems

The first 18 singular point values of the origin of system $(7.4.4)_{\delta=0}$ are all zero if and only if one of the 10 conditions of Theorem 7.5.1 is satisfied. In this section, we shall solve completely the problem of complex isochronous centers of the origin for system $(7.4.4)_{\delta=0}$.

7.6.1 The Problem of Complex Isochronous Centers Under the Condition of C_1

When condition C_1 holds, system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT} &= z - \frac{\lambda-3}{6}b_{02}z^4 - \frac{\lambda-9}{6}a_{20}z^3w + \frac{\lambda+9}{6}b_{20}z^2w^2 + \frac{\lambda+3}{6}a_{02}w^3z,\\ \frac{dw}{dT} &= -w + \frac{\lambda-3}{6}a_{02}w^4 + \frac{\lambda-9}{6}b_{20}w^3z - \frac{\lambda+9}{6}a_{20}w^2z^2 - \frac{\lambda+3}{6}b_{02}wz^3.\end{aligned} \quad (7.6.1)$$

Using Theorem 7.3.5, we have

Lemma 7.6.1. *The first* 15 *period constants of the origin of system* (7.6.1) *are given by*

$$\begin{aligned}\tau_3 &= \frac{-1}{3}(a_{02}b_{02} + 9a_{20}2b_{20})(\lambda+3),\\ \tau_6 &\sim \frac{-1}{2}(a_{20}^3a_{02} + b_{20}^3b_{02} - 9a_{20}^2b_{20}^2)(\lambda+1)(\lambda+3)(\lambda+9),\end{aligned}$$

$$\begin{aligned}
\tau_9 &\sim \frac{3}{32}a_{20}b_{20}(a_{20}^3a_{02}+b_{20}^3b_{02})(\lambda+3)(\lambda+6)(\lambda+15)(2\lambda+3)(5\lambda+3),\\
\tau_{12} &\sim \frac{-21}{160}a_{20}^4b_{20}^4\lambda(\lambda+1)(\lambda+3)(\lambda+9)(359\lambda^2+2010\lambda+3231),\\
\tau_{15} &\sim \frac{-6435}{4}a_{20}^5b_{20}^5(\lambda+3)(820\lambda+819).
\end{aligned} \tag{7.6.2}$$

This lemma follows that

Lemma 7.6.2. *The first* 15 *periodic constants of system* (7.6.1) *are all zero if and only if one of the following* 7 *conditions is satisfied*:

$$\begin{aligned}
&C_{11}: \quad \lambda=-3;\\
&C_{12}: \quad a_{20}=b_{02}=0;\\
&C_{12}^*: \quad b_{20}=a_{02}=0;\\
&C_{13}: \quad a_{20}=a_{02}=0, \quad \lambda=-1;\\
&C_{13}^*: \quad b_{20}=b_{02}=0, \quad \lambda=-1;\\
&C_{14}: \quad a_{20}=a_{02}=0, \quad \lambda=-9;\\
&C_{14}^*: \quad b_{20}=b_{02}=0, \quad \lambda=-9.
\end{aligned} \tag{7.6.3}$$

Proposition 7.6.1. *if* C_{11} *holds, then the origin of system* (7.6.1) *is an isochronous center.*

Proof. When C_{11} holds, system (7.6.1) becomes

$$\begin{aligned}
\frac{dz}{dT} &= z(1+b_{02}z^3+2a_{20}z^2w+b_{20}zw^2),\\
\frac{dw}{dT} &= -w(1+a_{02}w^3+2b_{20}w^2z+a_{20}wz^2).
\end{aligned} \tag{7.6.4}$$

It has the first integral

$$\frac{z^3w^3}{1+b_{02}z^3+3a_{20}z^2w+3b_{20}w^2z+a_{02}w^3}=c. \tag{7.6.5}$$

Let $z=re^{i\theta}$, $w=re^{-i\theta}$, $T=it$. (7.6.5) becomes

$$r^3=cg(\theta)+\sqrt{c+c^2g^2(\theta)}, \tag{7.6.6}$$

where

$$g(\theta)=\frac{1}{2}(b_{02}e^{3i\theta}+3a_{20}e^{i\theta}+3b_{20}e^{-i\theta}+a_{02}e^{-3i\theta}). \tag{7.6.7}$$

Thus, system (7.6.4) follows that

$$\frac{dt}{d\theta}=\frac{1}{1+g(\theta)r^3}=1-\frac{cg(\theta)}{\sqrt{c+c^2g^2(\theta)}}. \tag{7.6.8}$$

Hence, we obtain $\displaystyle\int_0^{2\pi}\frac{dt}{d\theta}\,d\theta\equiv 2\pi$. □

Proposition 7.6.2. *If one of* C_{12} *and* C_{12}^* *holds, then the origin of system* (7.6.1) *is a complex isochronous center.*

Proof. When C_{12} holds, system (7.6.1) becomes

$$\frac{dz}{dT} = z + \frac{\lambda+9}{6}b_{20}z^2w^2 + \frac{\lambda+3}{6}a_{02}zw^3,$$
$$\frac{dw}{dT} = -w + \frac{\lambda-3}{6}a_{02}w^4 + \frac{\lambda-9}{6}b_{20}w^3z. \tag{7.6.9}$$

According to Theorem 1.6.8, system (7.6.9) is linearizable in a neighborhood of the origin. So that, the origin of (7.6.9) is a complex isochronous center. Similarly, if C_{12}^* holds, the same conclusion as the above is true. □

Proposition 7.6.3. *If one of* C_{13} *and* C_{13}^* *holds, then the origin of system* (7.6.1) *is a complex isochronous center.*

Proof. When C_{13} holds, system (7.6.1) can be reduced to

$$\frac{dz}{dT} = z + \frac{2}{3}b_{02}z^4 + \frac{4}{3}b_{20}z^2w^2,$$
$$\frac{dw}{dT} = -w - \frac{5}{3}b_{20}w^3z - \frac{1}{3}b_{02}wz^3. \tag{7.6.10}$$

By the transformation

$$\xi = z\left(1+2b_{20}w^2z\right)\left(1+\frac{2}{3}b_{02}z^3+2b_{20}zw^2\right)^{\frac{-1}{3}},$$
$$\eta = w\left(1+2b_{20}zw^2\right)^{-1}\left(1+\frac{2}{3}b_{02}z^3+2b_{20}w^2z\right)^{\frac{1}{6}}, \tag{7.6.11}$$

system (7.6.10) can be linearized. So that, the origin of (7.6.10) is a complex isochronous center. The same conclusion is true for the condition C_{13}^*. □

Proposition 7.6.4. *If one of* C_{14} *and* C_{14}^* *holds, then the origin of system* (7.6.1) *is a complex isochronous center.*

Proof. When C_{14} holds, system (7.6.1) becomes

$$\frac{dz}{dT} = z(1+2b_{02}z^3), \quad \frac{dw}{dT} = -w(1+3b_{20}w^2z - b_{02}z^3). \tag{7.6.12}$$

By the transformation

$$\xi = z(1+2b_{02}z^3)^{\frac{-1}{3}},$$
$$\eta = w(1+2b_{02}z^3)^{\frac{1}{3}}(1+6b_{20}w^2z+2b_{02}z^3)^{\frac{-1}{2}}, \tag{7.6.13}$$

system (7.6.12) can be linearized. So that, the origin of system (7.6.12) is a complex isochronous center. The same conclusion is true for the condition C_{14}^*. □

To sum up, Proposition 7.6.1 $\sim$ Proposition 7.6.4 follow that

Theorem 7.6.1. *The origin of system* (7.6.1) *is a complex isochronous center if and only if the first* 15 *period constants are all zero, i.e., one of the seven conditions in Lemma* 7.6.2 *is satisfied.*

7.6.2 The Problem of Complex Isochronous Centers Under the Condition of C_2

Write that $2a_{20}-b_{11}=B,\ 2b_{20}-a_{11}=A$. When C_2 is satisfied, there exist constants p and q, such that

$$a_{20}=p\beta,\quad b_{20}=p\alpha,\quad a_{02}=q\alpha^3,\quad b_{02}=q\beta^3. \tag{7.6.14}$$

In this case, system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT}=&z-\frac{1}{6}r-3)q\beta^3(z^4-\frac{1}{6}(3-9p-r+pr)\beta z^3w\\&+\frac{1}{6}(-3+9p-r+pr)\alpha z^2w^2+\frac{1}{6}(r+3)q\alpha^3zw^3,\\\frac{dw}{dT}=&-w+\frac{1}{6}(r-3)q\alpha^3w^4+\frac{1}{6}(3-9p-r+pr)\alpha w^3z\\&-\frac{1}{6}(-3+9p-r+pr)\beta w^2z^2-\frac{1}{6}(3+r)q\beta^3wz^3.\end{aligned} \tag{7.6.15}$$

By calculating period constants, we have

Lemma 7.6.3. *The first* 12 *period constants of the origin of system* (7.6.15) *are all zero if and only if one of the following* 8 *conditions hold*:

$$\begin{aligned}&C_{21}: \quad p=\frac{1}{3},\quad q=0,\quad \alpha\beta\neq 0;\\&C_{22}: \quad q=0,\quad -1+3p-\lambda+p\lambda=0;\\&C_{23}: \quad p=\frac{1}{2},\quad q=\frac{-1}{2\alpha\beta},\ \lambda=3,\quad \alpha\beta\neq 0;\\&C_{24}: \quad p=\frac{3}{10},\quad q=\frac{-3}{10\alpha\beta},\ \lambda=-1,\quad \alpha\beta\neq 0;\\&C_{25}: \quad p=\frac{5}{12},\quad q=\frac{-1}{4\alpha\beta},\ \lambda=1,\quad \alpha\beta\neq 0;\\&C_{26}: \quad p=\frac{7}{20},\quad q=\frac{3}{20\alpha\beta},\ \lambda=1,\quad \alpha\beta\neq 0;\\&C_{27}: \quad \beta=0;\\&C_{27}^*: \quad \alpha=0.\end{aligned} \tag{7.6.16}$$

Proposition 7.6.5. *If* C_{21} *holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. When C_{21} holds, system (7.6.15) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{1}{9}z(9+\lambda\beta z^2w-\lambda\alpha zw^2),\\ \frac{dw}{dT} &= -\frac{1}{9}w(9+\lambda\alpha w^2z-\lambda\beta wz^2).\end{aligned} \tag{7.6.17}$$

Let $z = re^{i\theta}$, $w = re^{-i\theta}$, $T = it$. We have $\frac{d\theta}{dt} \equiv 1$. Thus, this proposition is true. □

Proposition 7.6.6. *If C_{22} holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. When C_{22} holds, system (7.6.15) becomes

$$\begin{aligned}\frac{dz}{dT} &= z+\frac{1}{3}(3p-1)(2\beta z+\alpha w)z^2w,\\ \frac{dw}{dT} &= -w-\frac{1}{3}(3p-1)(2\alpha w+\beta z)w^2z.\end{aligned} \tag{7.6.18}$$

Let

$$\xi = zf_1^{\frac{-2}{3}}f_2^{\frac{1}{3}}, \quad \eta = wf_1^{\frac{1}{3}}f_2^{\frac{-2}{3}}, \tag{7.6.19}$$

where

$$f_1 = 1+(3p-1)\beta z^2w, \quad f_2 = 1+(3p-1)\alpha zw^2. \tag{7.6.20}$$

By using (7.6.19), system (7.6.18) can be linearized. It follows the conclusion of this proposition. □

Proposition 7.6.7. *If C_{23} holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. When C_{23} holds, system (7.6.15) becomes

$$\frac{dz}{dT} = z+\frac{\beta z^3w}{2}-\frac{\alpha^2zw^3}{2\beta}, \quad \frac{dw}{dT} = -w-\frac{\alpha w^3z}{2}+\frac{\beta^2wz^3}{2\alpha}. \tag{7.6.21}$$

System (7.6.21) has the following function of the time-angle difference:

$$G_1 = \frac{(\alpha w-\beta z)^3i}{12\alpha\beta}. \tag{7.6.22}$$

By Theorem 4.4.4, the conclusion of this proposition holds. □

Proposition 7.6.8. *If C_{24} holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. Suppose that the condition C_{24} is satisfied. Then system (7.6.15) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{z}{30\alpha\beta}(30\alpha\beta - 6\beta^3 z^3 - 5\alpha\beta^2 z^2 w + 2\beta\alpha^2 z w^2 - 3\alpha^3 w^3),\\ \frac{dw}{dT} &= \frac{-w}{30\alpha\beta}(30\alpha\beta - 6\alpha^3 w^3 - 5\beta\alpha^2 w^2 z + 2\alpha\beta^2 w z^2 - 3\beta^3 z^3).\end{aligned} \tag{7.6.23}$$

System (7.6.23) has the following function of the time-angle difference:

$$G_2 = \frac{i}{4}\log\frac{\beta(5\alpha - \beta^2 z^3 - \alpha\beta z^2 w)}{\alpha(5\beta - \alpha^2 w^3 - \alpha\beta w^2 z)}. \tag{7.6.24}$$

Thus, Theorem 4.4.4 implies the conclusion of this proposition. □

Proposition 7.6.9. *If* C_{25} *holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. Suppose that the condition C_{25} is satisfied. Then, (7.6.15) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{z}{36\alpha\beta}(36\alpha\beta - 3\beta^3 z^3 + 8\alpha\beta^2 z^2 w + \beta\alpha^2 z w^2 - 6\alpha^3 w^3),\\ \frac{dw}{dT} &= -\frac{w}{36\alpha\beta}(36\alpha\beta - 3\alpha^3 w^3 + 8\beta\alpha^2 w^2 z + \alpha\beta^2 w z^2 - 6\beta^3 z^3).\end{aligned} \tag{7.6.25}$$

There exists a transformation

$$\xi = z f_3^{\frac{-1}{3}} f_4^{\frac{2}{3}} f_5^{\frac{-1}{6}}, \quad \eta = w f_3^{\frac{2}{3}} f_4^{\frac{-1}{3}} f_5^{\frac{-1}{6}}, \tag{7.6.26}$$

where

$$\begin{aligned}f_3 &= 1 - \frac{\beta^2 z^3}{12\alpha} + \frac{\beta z^2 w}{6} - \frac{\alpha z w^2}{12},\\ f_4 &= 1 - \frac{\alpha^2 w^3}{12\beta} + \frac{\alpha w^2 z}{6} - \frac{\beta z^2}{12},\\ f_5 &= 1 + \frac{2\beta z^2 w}{3} + \frac{2\alpha z w^2}{3},\end{aligned} \tag{7.6.27}$$

such that (7.6.25) reduces to a linear system. This gives the conclusion of the Proposition 7.6.9. □

Proposition 7.6.10. *If* C_{26} *holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. Suppose that the condition C_{26} is satisfied. Then, (7.6.15) becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{z}{60\alpha\beta}(60\alpha\beta + 3\beta^3 z^3 + 8\alpha\beta^2 z^2 w - 5\beta\alpha^2 z w^2 + 6\alpha^3 w^3),\\ \frac{dw}{dT} &= -\frac{w}{60\alpha\beta}(60\alpha\beta + 3\alpha^3 w^3 + 8\beta\alpha^2 w^2 z - 5\alpha\beta^2 w z^2 + 6\beta^3 z^3).\end{aligned} \tag{7.6.28}$$

There exists a transformation

$$\xi = z f_6^{\frac{-1}{3}} f_7^{\frac{2}{3}} f_8^{\frac{-2}{3}}, \quad \eta = w f_6^{\frac{2}{3}} f_7^{\frac{-1}{3}} f_8^{\frac{-2}{3}}, \tag{7.6.29}$$

where

$$\begin{aligned} f_6 &= 1 + \frac{\beta^2 z^3}{20\alpha} + \frac{\beta z^2 w}{10} + \frac{\alpha z w^2}{20}, \\ f_7 &= 1 + \frac{\alpha^2 w^3}{20\beta} + \frac{\alpha w^2 z}{10} + \frac{\beta w z^2}{20}, \\ f_8 &= 1 + \frac{\beta z^2 w}{5} + \frac{\alpha z w^2}{5}, \end{aligned} \tag{7.6.30}$$

such that (7.6.9) reduces to a linear system. This gives the conclusion of the Proposition 7.6.10. □

Proposition 7.6.11. *If one of C_{27} and C_{27}^* holds, then the origin of system* (7.6.15) *is a complex isochronous center.*

Proof. Suppose that the condition C_{27} is satisfied. Then, system (7.6.15) becomes

$$\begin{aligned} \frac{dz}{dT} &= z + \frac{1}{6}(-3 + 9p - \lambda + p\lambda)\alpha z^2 w^2 + \frac{1}{6}(3 + \lambda) q \alpha^3 z w^3, \\ \frac{dw}{dT} &= -w + \frac{1}{6}(3 - 9p - \lambda + p\lambda)\alpha z w^3 + \frac{1}{6}(-3 + \lambda) q \alpha^3 w^4. \end{aligned} \tag{7.6.31}$$

According to Theorem 1.6.8, system (7.6.31) is linearizable in a neighborhood of the origin. So that, the origin of (7.6.31) is a complex isochronous center. Similarly, we can prove that the origin of system (7.6.15) is also a complex isochronous center, if C_{27}^* holds. □

Thus, Proposition 7.6.5 $\sim$ Proposition 7.6.11 imply that

Theorem 7.6.2. *The origin is a complex isochronous center of system* (7.6.15) *if and only if the first* 12 *period constants of the origin are all zero, i.e., one of the* 8 *conditions holds in Lemma* 7.6.3.

7.6.3 The Problem of Complex Isochronous Centers Under the Other Conditions

If C_3 holds, we can figure out that the third period constant of the origin of system $(7.4.4)_{\delta=0}$ is $\tau_3 = \dfrac{-4}{3} a_{02} b_{02}$. Thus, we have

Lemma 7.6.4. *If condition C_3 holds and $\tau_3 = 0$, then one of the following conditions holds:*

$$\begin{aligned} C_{31}: &\quad \lambda = 1, \quad a_{11} = b_{11} = b_{02} = 0; \\ C_{31}^*: &\quad \lambda = 1, \quad a_{11} = b_{11} = a_{02} = 0. \end{aligned} \tag{7.6.32}$$

Theorem 7.6.3. *If one of C_{31} and C_{31}^* holds, then the origin of system* $(7.4.4)_{\delta=0}$ *is a complex isochronous center.*

Proof. If C_{31} holds, then system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{z}{3}(3 + 2a_{20}z^2w + b_{20}zw^2 + 2a_{02}w^3),\\ \frac{dw}{dT} &= \frac{-w}{3}(3 + a_{02}w^3 + 2b_{20}w^2z + a_{20}wz^2).\end{aligned} \tag{7.6.33}$$

According to Theorem 1.6.7, system (7.6.33) is linearizable in a neighborhood of the origin. So that, the origin of (7.6.33) is a complex isochronous center. Similarly, we can prove that the origin of system (7.6.1) is also a complex isochronous center, if C_{31}^* holds. □

If C_4 holds, we can figure out that the third period constant of the origin of system $(7.4.4)_{\delta=0}$ is $\tau_3 = \frac{2}{3}a_{02}b_{02}$. Thus, we have

Lemma 7.6.5. *If condition C_4 holds and $\tau_3 = 0$, then one of the following conditions holds:*

$$\begin{aligned}C_{41}:&\quad a_{20} = b_{11} = b_{02} = 0,\ b_{20} + 2a_{11} = 0,\ \lambda = 1;\\ C_{41}^*:&\quad b_{20} = a_{11} = a_{02} = 0,\ a_{20} + 2b_{11} = 0,\ \lambda = 1;\\ C_{42}:&\quad b_{20} = a_{11} = b_{02} = 0,\ a_{20} + 2b_{11} = 0,\ \lambda = 1;\\ C_{42}^*:&\quad a_{20} = b_{11} = a_{02} = 0,\ b_{20} + 2a_{11} = 0,\ \lambda = 1.\end{aligned} \tag{7.6.34}$$

Proposition 7.6.12. *If one of C_{41} and C_{41}^* holds, then the origin of system* $(7.4.4)_{\delta=0}$ *is a complex isochronous center.*

Proof. If C_{41} holds, then system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT} &= \frac{z}{3}(3 + 2a_{02}w^3),\\ \frac{dw}{dT} &= \frac{-w}{3}(3 + a_{02}w^3 - 3a_{11}zw^2).\end{aligned} \tag{7.6.35}$$

According to Theorem 1.6.8, system (7.6.35) is linearizable in a neighborhood of the origin. So that, the origin of (7.6.35) is a complex isochronous center. Similarly, we can prove that the origin of system (7.6.1) is also a complex isochronous center, if C_{41}^* holds. □

Proposition 7.6.13. *If one of C_{42} and C_{42}^* holds, then the origin of system* $(7.4.4)_{\delta=0}$ *is a complex isochronous center.*

Proof. If C_{42} holds, then system $(7.4.4)_{\delta=0}$ becomes

$$\frac{dz}{dT} = z - b_{11}z^3w + \frac{2}{3}a_{02}zw^3, \quad \frac{dw}{dT} = w + \frac{1}{3}a_{02}w^4. \tag{7.6.36}$$

There exists a transformation

$$\xi = xf_9^{\frac{-1}{2}} f_{10}^{\frac{2}{3}}, \quad \eta = wf_{10}^{\frac{-1}{3}}, \tag{7.6.37}$$

where

$$f_9 = 1 - 2b_{11}z^2w - a_{02}b_{11}z^2w^4, \quad f_{10} = 1 + \frac{1}{3}a_{02}w^3, \tag{7.6.38}$$

such that (7.6.36) reduce to a linear system. Hence, the origin of system (7.6.36) is a complex isochronous center. Similarly, we can prove that the origin of system $(7.4.4)_{\delta=0}$ is also a complex isochronous center if C_{42}^* holds. □

From Lemma 7.6.5, Proposition 7.6.12 and Proposition 7.6.13, we have

Theorem 7.6.4. *Suppose that the condition C_4 is satisfied. Then, the origin of system $(7.4.4)_{\delta=0}$ is a complex isochronous center, if and only if one of four conditions in Lemma* 7.6.5 *holds.*

Theorem 7.6.5. *If one of C_5 and C_5^* holds, then the origin of system* $(7.4.4)_{\delta=0}$ *is a complex isochronous center.*

Proof. Suppose that the condition C_5 is satisfied. Then, $(7.4.4)_{\delta=0}$ becomes

$$\frac{dz}{dT} = z, \quad \frac{dw}{dT} = -w(1 + b_{02}z^3 + b_{20}zw^2). \tag{7.6.39}$$

There exists a transformation

$$\xi = z, \quad v = wf_{11}^{\frac{-1}{2}}, \tag{7.6.40}$$

where

$$f_{11} = (1 + 2b_{20}zw^2)e^{\frac{-2}{3}b_{02}z^3} + 4b_{20}b_{02}z^2w^2 \int_0^z ze^{\frac{-2}{3}b_{02}z^3} dz, \tag{7.6.41}$$

such that (7.6.39) reduce to a linear system. Hence, the origin of system (7.6.39) is a complex isochronous center. Similarly, we can prove that the origin of system $(7.4.4)_{\delta=0}$ is also a complex isochronous center if C_5^* holds. □

Theorem 7.6.6. *If one of C_6 and C_6^* holds, then the origin of system* $(7.4.4)_{\delta=0}$ *is a complex isochronous center.*

Proof. Suppose that the condition C_6 is satisfied. Then, system $(7.4.4)_{\delta=0}$ becomes

$$\begin{aligned}\frac{dz}{dT}&=z+\frac{2\lambda}{\lambda-3}b_{11}z^3w+\frac{\lambda+3}{6}a_{02}zw^3,\\ \frac{dw}{dT}&=-w+\frac{\lambda-3}{6}a_{02}w^4.\end{aligned}\tag{7.6.42}$$

There exists a transformation

$$\xi=zf_{12}^{\frac{-1}{3}}f_{13}^{\frac{-1}{2}},\quad \eta=wf_{12}^{\frac{-1}{3}},\tag{7.6.43}$$

where

$$\begin{aligned}&f_{12}=6-(\lambda-3)a_{02}w^3,\\ f_{13}=\left(f_{12}+\frac{24\lambda}{\lambda-3}b_{11}z^2w\right)&f_{12}^{\frac{\lambda+9}{3\lambda-9}}+\frac{24\lambda(\lambda+9)}{\lambda-3}a_{02}b_{11}z^2w^2\int_0^w wf_{12}^{\frac{18-2\lambda}{3\lambda-9}}dw,\end{aligned}\tag{7.6.44}$$

such that (7.6.42) reduce to a linear system. So that, the origin of system (7.6.42) is a complex isochronous center. Similarly, we can prove that the origin of system $(7.4.4)_{\delta=0}$ is also a complex isochronous center if C_6^* holds. □

Finally, when one of C_7 and C_7^* holds, Proposition 7.5.7 gives directly rise to the conclusion that the origin is a complex isochronous center.

From tha above discission, we know that the isochronous center problem of the quasi-quadratic systems have been solved completely.

7.7 Singular Point Values and Center Conditions for a Class of Quasi-Cubic Systems

Consider the following quasi-cubic systems

$$\begin{aligned}\frac{dz}{dT}&=(1-i\delta)z+(zw)^{\lambda-1}Z_3(z,w),\\ \frac{dw}{dT}&=-(1+i\delta)w-(zw)^{\lambda-1}W_3(z,w),\end{aligned}\tag{7.7.1}$$

where

$$\begin{aligned}Z_3(z,w)&=a_{30}z^3+a_{21}z^2w+a_{12}zw^2+a_{03}w^3,\\ W_3(z,w)&=b_{30}w^3+b_{21}w^2z+b_{12}wz^2+b_{03}z^3.\end{aligned}\tag{7.7.2}$$

By the transformation $z=x+iy,\ w=x-y,\ T=it$, system (7.7.1) becomes

$$\begin{aligned}\frac{dx}{dt}&=-y+\delta x+(x^2+y^2)^{\lambda-1}X_3(x,y),\\ \frac{dy}{dt}&=x+\delta y+(x^2+y^2)^{\lambda-1}Y_3(x,y),\end{aligned}\tag{7.7.3}$$

where $X_3(x,y), Y_3(x,y)$ are homogeneous polynomials of degree 3 of x and y. By using the transformation (7.2.1), system (7.7.3) becomes

$$\begin{aligned} \frac{d\xi}{dt} &= \frac{\delta\lambda}{3}\xi - \eta + \frac{1}{3}(\xi^2+\eta^2)P_5(\xi,\eta),\\ \frac{d\xi}{dt} &= \xi + \frac{\delta\lambda}{3}\eta + \frac{1}{3}(\xi^2+\eta^2)Q_5(\xi,\eta), \end{aligned} \tag{7.7.4}$$

where

$$\begin{aligned} P_5(\xi,\eta) &= (\lambda\xi^2+3\eta^2)X_3(\xi,\eta) + (\lambda-3)\xi\eta Y_3(\xi,\eta),\\ Q_5(\xi,\eta) &= (\lambda\eta^2+3\xi^2)Y_3(\xi,\eta) + (\lambda-3)\xi\eta X_3(\xi,\eta). \end{aligned} \tag{7.7.5}$$

The associated system of system (7.7.4) is given by

$$\begin{aligned} \frac{dz}{dT} &= \left(1 - \frac{i\delta\lambda}{3}\right)z \;+ \frac{1}{6}z^2 w\Phi_5(z,w),\\ \frac{dw}{dT} &= -\left(1 + \frac{i\delta\lambda}{3}\right)w - \frac{1}{6}w^2 z\Psi_5(z,w), \end{aligned} \tag{7.7.6}$$

where

$$\begin{aligned} \Phi_5(z,w) &= (\lambda+3)wZ_3(z,w) - (\lambda-3)zW_3(z,w),\\ \Psi_5(z,w) &= (\lambda+3)zW_3(z,w) - (\lambda-3)wZ_3(z,w). \end{aligned} \tag{7.7.7}$$

Theorem 7.7.1. *The first* 27 *singular point values at the origin of system* $(7.7.6)_{\delta=0}$ *are given by*

$$\begin{aligned} \mu_3 &= \frac{1}{3}\lambda(a_{21}-b_{21}),\\ \mu_6 &\sim \frac{1}{3}\lambda(b_{30}b_{12-}a_{30}a_{12}),\\ \mu_9 &\sim \frac{1}{24}\lambda(2\Delta_1 - \lambda\Delta_2),\\ \mu_{12} &\sim \frac{1}{24}\lambda(a_{21}+b_{21})\Delta_1,\\ \mu_{15} &\sim \frac{1}{288}\lambda[8(a_{30}+b_{12})(b_{30}+a_{12}) + (\lambda^2-\lambda-8)a_{03}b_{03}]\Delta_1,\\ \mu_{18} &\sim 0,\\ \mu_{21} &\sim \frac{1}{46080}\lambda^2(\lambda-1)(3\lambda-8)(5\lambda+13)a_{03}^2b_{03}^2\Delta_1,\\ \mu_{24} &\sim \frac{-7}{13824}\lambda^2(\lambda-1)a_{03}b_{03}J_0\Delta_1,\\ \mu_{27} &\sim \frac{121}{13824000}\lambda^2(\lambda-1)(3\lambda-8)a_{03}^3b_{03}^3\Delta_1, \end{aligned} \tag{7.7.8}$$

where

$$\begin{aligned}\Delta_1 &= (3a_{30}-b_{12})(a_{30}+b_{12})a_{03}-(3b_{30}-a_{12})(b_{30}+a_{12})b_{03},\\ \Delta_2 &= (a_{30}-b_{12})(3a_{30}-b_{12})a_{03}-(3b_{30}-a_{12})(b_{30}-a_{12})b_{03},\\ J_0 &= (a_{30}+b_{12})^2a_{03}+(b_{30}+a_{12})^2b_{03}.\end{aligned} \tag{7.7.9}$$

Theorem 7.7.2. *The first* 27 *singular point values of system* $(7.7.6)_{\delta=0}$ *are zero, if and only if one of the following* 6 *conditions is satisfied*:

$$\begin{aligned}
&C_1: \quad a_{21}-b_{21}=0, \quad 3a_{30}-b_{12}=3b_{30}-a_{12}=0;\\
&C_2: \quad \begin{cases} a_{21}=b_{21}, \quad a_{30}a_{12}=b_{30}b_{12}, \quad a_{30}^2a_{03}=b_{30}^2b_{03},\\ a_{30}b_{12}a_{03}=b_{30}a_{12}b_{03}, \qquad b_{12}^2a_{03}=a_{12}^2b_{03}; \end{cases}\\
&C_3: \quad \begin{cases} \lambda=1,\ a_{21}=b_{21}=0,\ a_{03}b_{03}=4a_{12}b_{12},\\ a_{30}+3b_{12}=b_{30}+3a_{12}=0; \end{cases}\\
&C_4: \quad \begin{cases} \lambda=\dfrac{8}{3},\ a_{21}=b_{21}=0,\ a_{30}-7b_{12}=b_{30}-7a_{12}=0,\\ b_{12}^2a_{03}+a_{12}^2b_{03}=0,\ a_{03}b_{03}=144a_{12}b_{12},\ a_{03}b_{03}\neq 0; \end{cases}\\
&C_5: \quad \begin{cases} a_{21}=b_{21}=b_{30}=a_{12}=b_{03}=0,\\ (\lambda-2)a_{30}-(\lambda+2)b_{12}=0; \end{cases}\\
&C_5^*: \quad \begin{cases} a_{21}=b_{21}=a_{30}=b_{12}=a_{03}=0,\\ (\lambda-2)b_{30}-(\lambda+2)a_{12}=0. \end{cases}
\end{aligned} \tag{7.7.10}$$

Proposition 7.7.1. *If* C_1 *holds, then system* $(7.7.6)_{\delta=0}$ *has the integral factor*

$$M_1=(zw)^{-1}g_1^{-1}, \tag{7.7.11}$$

where

$$g_1=1-\frac{\lambda-2}{4}zw\big[b_{03}z^4+4a_{30}z^3w+(a_{21}+b_{21})z^2w^2+4b_{30}w^3z+a_{30}w^4\big]. \tag{7.7.12}$$

Proposition 7.7.2. *If* C_2 *holds, then the coefficients of the right hand of system* $(7.7.6)_{\delta=0}$ *satisfy the conditions of the extended symmetric principle.*

Proposition 7.7.3. *If* C_3 *holds, then* $(7.7.1)_{\delta=0}$ *is an integrable cubic system.*

Proposition 7.7.4. *If* C_4 *holds, then system* $(7.7.6)_{\delta=0}$ *has the integral factor*

$$M_2=(zw)^{-3}g_2^{\frac{5}{6}}, \tag{7.7.13}$$

where

$$\begin{aligned}g_2 = {}&1-\frac{1}{3}zw(b_{03}z^4+24b_{12}z^3w+24a_{12}zw^3+a_{03}w^4)\\ &-\frac{1}{108}z^2w^2(-3b_{03}^2z^8-240b_{03}b_{12}z^7w-8640b_{12}^2z^6w^2\\ &+1296a_{12}b_{03}z^5w^3+130a_{03}b_{03}z^4w^4+1296a_{03}b_{12}z^3w^5\\ &-8640a_{12}^2z^2w^6-240a_{03}a_{12}zw^7-3a_{03}^2w^8).\end{aligned} \tag{7.7.14}$$

Proposition 7.7.5. *If one of C_5 and C_5^* holds, then system* $(7.7.6)_{\delta=0}$ *has the integral factor*

$$M_3 = (zw)^{\frac{-11}{2}} g_3, \tag{7.7.15}$$

where

$$g_3 = \begin{cases} \left[1 - \dfrac{1}{4}(\lambda - 2)(b_{03}z^5 w + a_{03}w^5 z)\right]^{\frac{\lambda+4}{2(\lambda-2)}}, & if\ \lambda \neq 2, \\ \exp\left[\dfrac{-3}{4}(b_{03}z^5 w + a_{03}w^5 z)\right], & if\ \lambda = 2. \end{cases} \tag{7.7.16}$$

Theorem 1.8.26, Theorem 7.7.2 and Proposition 7.7.1 ∼ Proposition 7.7.5 follow that

Theorem 7.7.3. *The origin of system* $(7.7.6)_{\delta=0}$ *is a complex center, if and only if the first* 27 *singular point values are all zero, i.e., one of the* 6 *conditions in Theorem* 7.7.2 *is satisfied.*

Bibliographical Notes

The singular point values and center conditions of system (7.7.1) and (7.7.3) were discussed in [Xiao P., 2005] and [Llibre etc, 2009a]. The materials of this chapter are taken by [Xiao P., 2005; Liu Y.R., 2002; Liu Y.R. etc, 2008a].

Chapter 8

Local and Non-Local Bifurcations of Perturbed Z_q-Equivariant Hamiltonian Vector Fields

In order to obtain more limit cycles and various configuration patterns of their relative dispositions, we indicated in [Li Jibin etc, 1987-1992] that an efficient method is to perturb the symmetric Hamiltonian systems having maximal number of centers, i.e., to study the weakened Hilbert's 16th problem for the symmetric planar polynomial Hamiltonian systems, since bifurcation and symmetry are closely connected and symmetric systems play pivotal roles as a bifurcation point in all planar Hamiltonian system class. To investigate perturbed Hamiltonian systems, we should first know the global behavior of unperturbed polynomial systems, namely, determine the global property for the families of real planar algebraic curves defined by the Hamiltonian functions. Then by using proper perturbation techniques, we shall obtain the global information of bifurcations for the perturbed non-integrable systems. In this sense, we say that our study method is to consider integrally two parts of Hilbert's 16th problem.

8.1 Z_q-Equivariant Planar Vector Fields and an Example

Let G be a compact Lie group of transformations acting on R^n. A mapping $\Phi : R^n \to R^n$ is called G-equivariant if for all $g \in G$ and $\boldsymbol{x} \in R^n$, $\Phi(g\boldsymbol{x}) = g\Phi(\boldsymbol{x})$. A function $H : R^n \to R$, is called G-invariant function if for all $\boldsymbol{x} \in R^n$, $H(g\boldsymbol{x}) = H(\boldsymbol{x})$. If Φ is a G-equivariant mapping, the vector field $d\boldsymbol{x}/dt = \Phi(\boldsymbol{x})$ is called a G-equivariant vector field.

Consider the (x, y) real planar polynomial system of the degree n

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{E_n}$$

where $X(x, y), Y(x, y)$ are real polynomials of x and y with the degree n. Let q be a positive integer. A group Z_q is called a cyclic group if it is generated by a real planar counterclockwise rotation through $\frac{2\pi}{q}$ about the origin. Making the transformation

$$z = x + iy, \quad w = x - iy, \quad T = it$$

the system (E_n) becomes its associated system

$$\frac{dz}{dT} = Z(z, w), \quad \frac{dw}{dT} = -W(z, w), \tag{E_n^*}$$

where

$$Z(z, w) = Y(x, y) - iX(x, y) = \sum_{\alpha+\beta=0}^{n} a_{\alpha\beta} z^\alpha w^\beta,$$
$$W(z, w) = Y(x, y) + iX(x, y) = \sum_{\alpha+\beta=0}^{n} b_{\alpha\beta} w^\alpha z^\beta, \tag{8.1.1}$$

and $\forall(\alpha, \beta)$, we have $b_{\alpha\beta} = \bar{a}_{\alpha\beta}$.

[Li Jibin and Zhao Xiaohua, 1989] proved the following result.

Theorem 8.1.1. *A vector field defined by* (E_n^*) *is* Z_q*-equivariant, if and only if the functions* $Z(z, w)$ *and* $W(z, w)$ *have the following form*:

$$Z(z, w) = \sum_{j=1} g_j(zw) w^{jq-1} + \sum_{j=0} h_j(zw) z^{jq+1},$$
$$W(z, w) = \sum_{j=1} \bar{g}_j(zw) z^{jq-1} + \sum_{j=0} \bar{h}_j(zw) w^{jq+1}, \tag{8.1.2}$$

where $g_j(\zeta)$ *and* $h_j(\zeta)$ *are polynomials with complex coefficients in* ζ*. In addition,* (E_n^*) *is a Hamiltonian system having* Z_q*-equivariance, if and only if* (8.1.2) *holds and*

$$\frac{\partial Z}{\partial z} - \frac{\partial W}{\partial w} \equiv 0. \tag{8.1.3}$$

Theorem 8.1.2. *A* Z_q*-invariant function* $I(z, w)$ *has the following form*:

$$I(z, w) = \sum_{j=1} g_j(zw) w^{jq} + \sum_{j=0} h_j(zw) z^{jq}. \tag{8.1.4}$$

Corollary 8.1.1. (1) *For the planar polynomial systems of degree 5, all non-trivial* Z_q*-equivariant vector fields have the following forms*:

(a) $q = 6, \quad Z(z, w) = (a_{10} + a_{21}zw + a_{32}z^2w^2)z + a_{05}w^5;$

(b) $q = 5, \quad Z(z, w) = (a_{10} + a_{21}zw + a_{32}z^2w^2)z + a_{04}w^4;$

(c) $q = 4, \quad Z(z, w) = (a_{10} + a_{21}zw + a_{32}z^2w^2)z + (a_{03} + a_{14}zw)w^3 + a_{05}z^5;$

(d) $q = 3$,

$$Z(z,w) = (a_{10} + a_{21}zw + a_{32}z^2w^2)z + (a_{02} + a_{13}zw)w^2 + a_{40}z^4 + a_{05}w^5;$$

(e) $q = 2$,

$$Z(z,w) = (a_{10} + a_{21}zw + a_{32}z^2w^2)z + (a_{01} + a_{12}zw + a_{23}z^2w^2)w + (a_{30} + a_{41}zw)z^3 + (a_{03} + a_{14}zw)w^3 + a_{50}z^5 + a_{05}w^5,$$

where $a_{\alpha\beta}$ are complex. The above $Z(z,w)$ define Z_q-equivariant Hamiltonian vector fields if and only if $a_{10} - b_{10} = a_{21} - b_{21} = a_{32} - b_{32} = 0$ and

$$for\quad q = 4,\quad a_{14} = 5b_{05},$$
$$for\quad q = 3,\quad a_{13} = 4b_{40};$$
$$for\quad q = 2,\quad a_{12} = 3b_{30},\ a_{23} = 2b_{41},\ a_{14} = 5b_{50}.$$

(2) *For the planar polynomial systems of degree $m-1(m \geqslant 7)$, when $q = m, m-1, m-2, m-3$, Z_q-equivariant Hamiltonian vector fields defined by* (8.1.1) *have the following forms*:

$$\text{(a)}q = m,\quad Z(z,w) = z\sum_{\beta=0}^{[\frac{m}{2}]-1} a_{\beta+1,\beta}(zw)^\beta + a_{0,m-1}w^{m-1};$$

$$\text{(b)}q = m-1,\quad Z(z,w) = z\sum_{\beta=0}^{[\frac{m}{2}]-1} a_{\beta+1,\beta}(zw)^\beta + a_{0,m-2}w^{m-2};$$

$$\text{(c)}q = m-2,\quad Z(z,w) = z\sum_{\beta=0}^{[\frac{m}{2}]-1} a_{\beta+1,\beta}(zw)^\beta + a_{0,m-1}z^{m-1} + \big[a_{0,m-3} - (m-1)b_{1,m-2}zw\big]w^{m-3};$$

$$\text{(d)}q = m-3,\quad Z(z,w) = z\sum_{\beta=0}^{[\frac{m}{2}]-1} a_{\beta+1,\beta}(zw)^\beta + a_{0,m-2}z^{m-2} + \big[a_{0,m-4} - (m-2)b_{1,m-3}zw\big]w^{m-4},$$

where $a_{\beta+1,\beta}$ are all real.

Corollary 8.1.2. *System (E_n^*) is Z_q-equivariant if and only if: $\forall(\alpha,\beta)$, when $(\alpha-\beta-1)/q$ is not a integer, we have $a_{\alpha\beta} = b_{\alpha\beta} = 0$.*

Obviously, system (E_n^*) define Hamiltonian vector fields if and only if $\forall(\alpha,\beta)$, we have $(\alpha+1)a_{\alpha+1,\beta} = (\beta+1)b_{\beta+1,\alpha}$.

A group D_q is called dihedral group of order $2q$ which is characterized as the symmetry group of the regular q-gon. It is generated by two elements, the (plane) rotation by an angle $2\pi/q$ and a reflection in R^2.

Theorem 8.1.3. *A vector field defined by (E_n^*) is D_q-eqiuivariant, if and only if all coefficients of the functions $g_j(zw)$ and $h_j(zw)$ in* (8.1.2) *are real numbers.*

The orbits of Hamiltonian polynomial systems given by Corollary 8.1.1 define different families of sextic ($m = 6$) algebraic curves having Z_q-equivariance. One of the main questions in real algebraic geometry is to describe what schemes of the mutual arrangement (schemes or configurations) of ovals can be realized by curves of given degree. By using some Z_q-equivariant Hamiltonian systems, we can realize a lot of configurations of ovals for planar algebraic curves of degree m.

As an example, we consider a Z_2-Equivariant Planar Vector Fields

$$\begin{aligned}\frac{dx}{dt} &= -y(-1 + ay^2 - ay^4 + y^6),\\ \frac{dy}{dt} &= x(-1 + bx^2 - bx^4 + x^6),\end{aligned} \tag{8.1.5}$$

where $b \geqslant a > 3$. Equation (8.1.5) has the Hamiltonian function as follows:

$$H(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{1}{4}(bx^4 + ay^4) + \frac{1}{6}(bx^6 + ay^6) - \frac{1}{8}(x^8 + y^8). \tag{8.1.6}$$

Clearly, the Hamiltonian $H(x, y)$ is negative definite at infinity.

Denote that

$$\begin{aligned}x_1 &= \frac{1}{2}\sqrt{2(b-1) - 2\sqrt{(b-1)^2 - 4}}, \quad & x_3 &= \frac{1}{2}\sqrt{2(b-1) + 2\sqrt{(b-1)^2 - 4}},\\ y_1 &= \frac{1}{2}\sqrt{2(a-1) - 2\sqrt{(a-1)^2 - 4}}, \quad & y_3 &= \frac{1}{2}\sqrt{2(a-1) + 2\sqrt{(a-1)^2 - 4}},\end{aligned}$$

It is easy to see that system (8.1.5) has 25 centers at $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, \pm 1)$, $(x_1, \pm y_1)$, $(x_1, \pm y_3)$, $(x_3, \pm y_1)$, $(x_3, \pm y_3)$ and their Z_2-equivariant symmetric points, 24 saddle points at $(x_1, 0)$, $(x_3, 0)$, $(0, y_1)$, $(0, y_3)$, $(x_1, \pm 1)$, $(x_3, \pm 1)$, $(1, \pm y_1)$, $(1, \pm y_3)$ and their Z_2-equivariant symmetric points. we denote that

$$\begin{aligned}&h_0^c = H(0, 0), && h_1^c = H(1, 0), && h_2^c = H(0, 1), && h_3^c = H(1, 1),\\ &h_4^c = H(x_1, y_1), && h_5^c = H(x_3, y_3), && h_6^c = H(x_3, y_1), && h_7^c = H(x_1, y_3),\\ &h_1^s = H(x_1, 0), && h_2^s = H(0, y_1), && h_3^s = H(1, y_1), && h_4^s = H(x_1, 1),\\ &h_5^s = H(x_2, 1), && h_6^s = H(x_3, 0), && h_7^s = H(1, y_3), && h_8^s = H(0, y_3),\end{aligned}$$

where

$$H(0, 0) = 0, \quad H(1, 0) = \frac{3}{8} - \frac{1}{12}b, \quad H(0, 1) = \frac{3}{8} - \frac{1}{12}a, \quad H(1, 1) = h_1^c + h_2^c.$$

Obviously, for $b > a > 3$, $h_1^s < h_2^s$, $h_3^s < h_4^s$, $h_8^s < h_5^s$, $h_7^s < h_6^s$. To compare $h_i^s (i = 1 \sim 8)$ in the first quadrant of the (a, b) parametric plane, we have the following four curves :

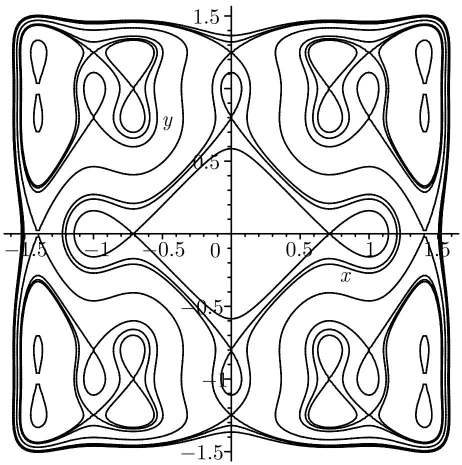
(C_1) $b = a$;

(C_2) $b = 4.5$, in which $h_2^s = h_3^s$, $h_7^s = h_8^s$;

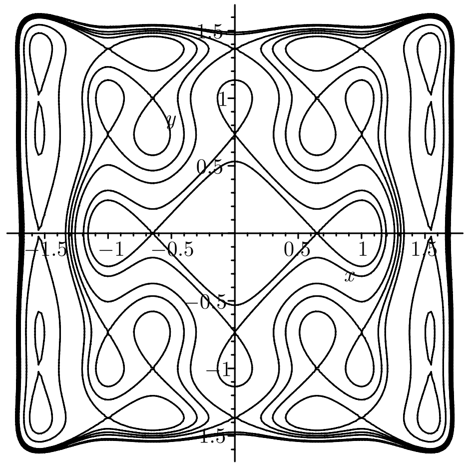
$$(C_3)\ b = \frac{1}{4}\left[18 + 2(a-3)(a^2-2a-3)^{\frac{3}{2}}\right], \quad \text{in which } h_2^s = h_7^s;$$

$$(C_4)\ b = \frac{1}{4}\left[18 - 2(a-3)(a^2-2a-3)^{\frac{3}{2}}\right], \quad \text{in which } \ h_3^s = h_8^s.$$

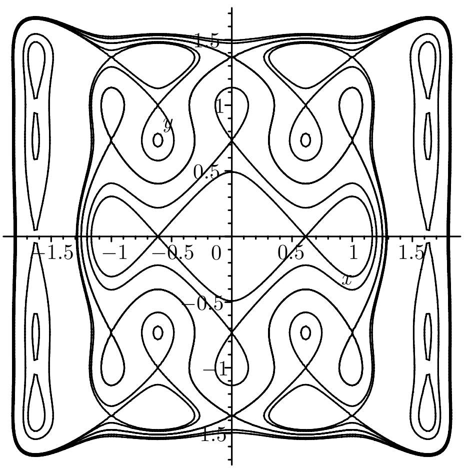
There are other bifurcation curves by comparing some h_i^s. These curves partition the $\frac{\pi}{4}$-angle region of the first quadrant in (a,b)-parametric plane into different regions. For example, we give 9 different phase portraits of (8.1.5), which are shown in figures (1)~(9) of Fig.8.1.1, when h_i $(i = 1 \sim 8)$ have the following orders:



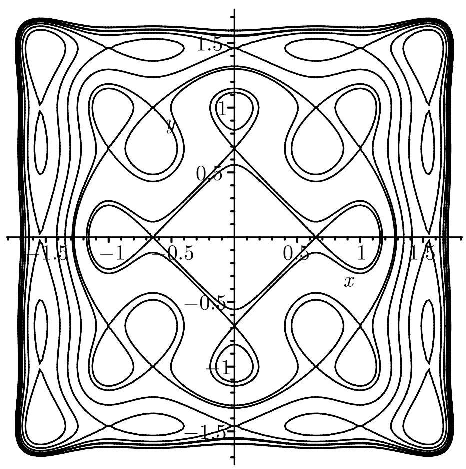
(1)



(2)



(3) $(a,b) \in (C_4)$



(4)

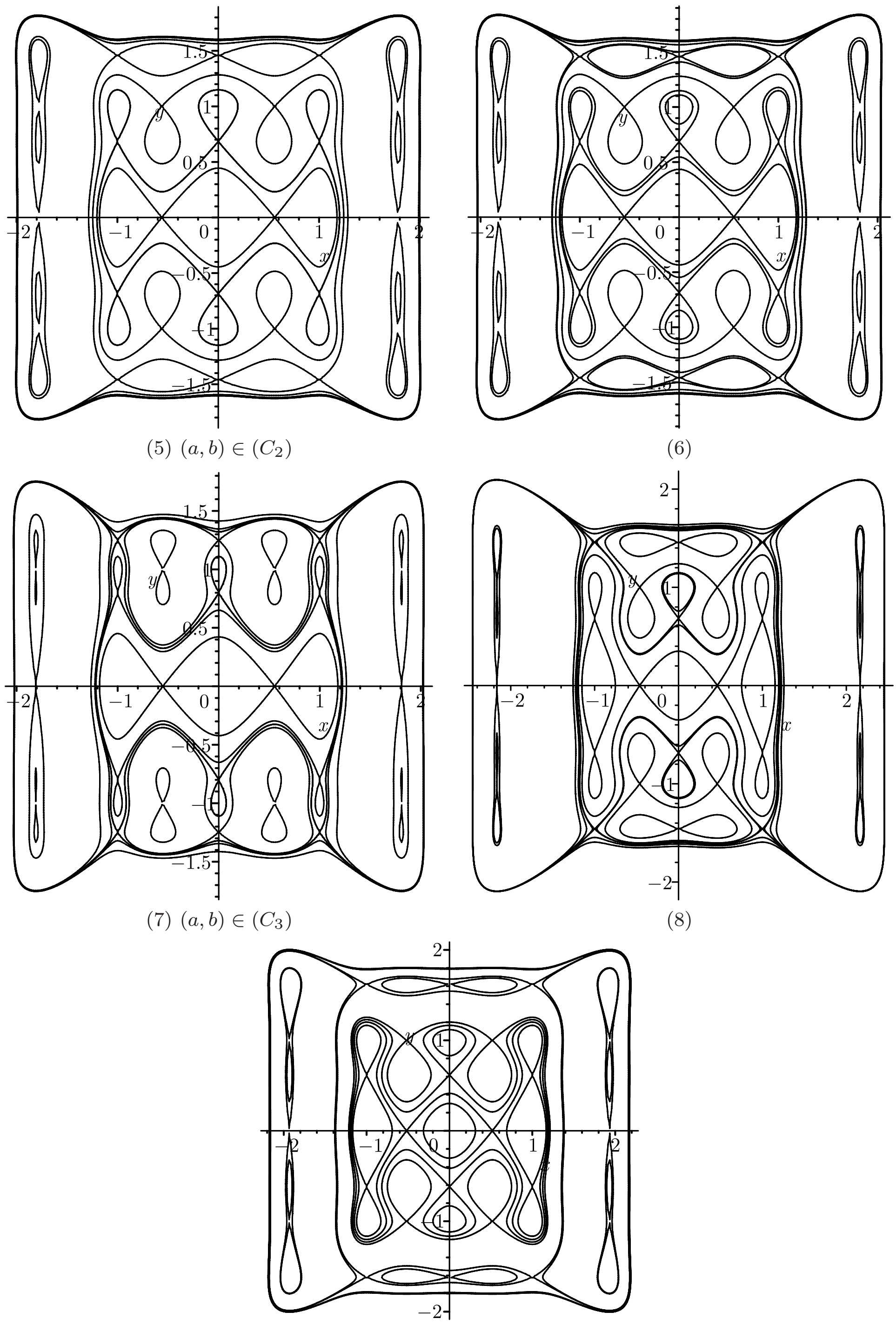


Fig.8.1.1 Some phase portraits of system (8.1.5)

(1) $-\infty < h_1^s < h_2^s < h_8^s < h_6^s < h_3^s < h_7^s < h_4^s < h_5^s$;

(2) $-\infty < h_1^s < h_2^s < h_8^s < h_3^s < h_4^s < h_7^s < h_6^s < h_5^s$;

(3) $(a,b) \in (C_4),\ -\infty < h_1^s < h_2^s < h_8^s = h_3^s < h_4^s < h_7^s < h_6^s < h_5^s$;

(4) $-\infty < h_1^s < h_2^s < h_3^s < h_4^s < h_8^s < h_7^s < h_6^s < h_5^s$;

(5) $(a,b) \in (C_2), -\infty < h_1^s < h_2^s = h_3^s < h_4^s < h_8^s = h_7^s < h_6^s < h_5^s$;

(6) $-\infty < h_1^s < h_3^s < h_2^s < h_4^s < h_7^s < h_8^s < h_6^s < h_5^s$;

(7) $(a,b) \in (C_3), -\infty < h_3^s < h_1^s < h_7^s = h_2^s < h_4^s < h_8^s < h_6^s < h_5^s$;

(8) $-\infty < h_3^s < h_1^s = h_7^s < h_2^s < h_4^s = h_8^s < h_6^s < h_5^s$;

(9) $-\infty < h_3^s < h_1^s < h_2^s < h_4^s < h_7^s = h_8^s < h_6^s < h_5^s$.

As h is varied, the level curves $H(x,y) = h$ of the Hamiltonian defined by (8.1.6) give rise to different eighth algebraic curves in the affine real plane.

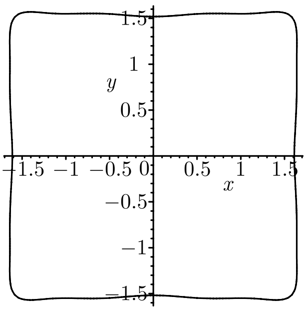
For an example, we consider figure (1) of Fig.8.1.1, i.e., $G = (a,b) = (3.2, 3.48)$. We have

$$\xi_1 = 0.7118903630,\ \xi_2 = 1.404710686, \eta_1 = 0.8010882788,\ \eta_2 = 1.248301875,$$

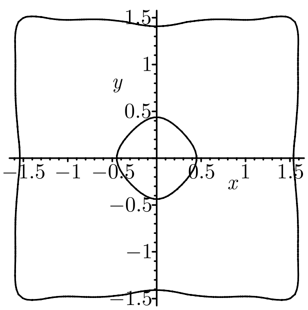
$$\begin{array}{lll} h_1^c = 0.085, & h_2^c = 0.1083333333, & h_3^c = 0.1933333333, \\ h_4^c = 0.2083547972, & h_5^c = 0.2778383751, & h_6^c = 0.2714227698, \\ h_7^c = 0.2147704034, & h_1^s = 0.09719593344, & h_2^s = 0.1111588638, \\ h_3^s = 0.1961588638, & h_4^s = 0.2055292667, & h_5^s = 0.2685972390, \\ h_6^s = 0.160263906, & h_7^s = 0.2025744701, & h_8^s = 0.1175744701, \end{array}$$

$$\begin{aligned} -\infty < 0 < h_1^c < h_1^s < h_2^c < h_2^s < h_8^s < h_6^s < h_3^c < h_3^s \\ < h_7^s < h_4^s < h_4^c < h_7^c < h_5^s < h_6^c < h_5^c. \end{aligned}$$

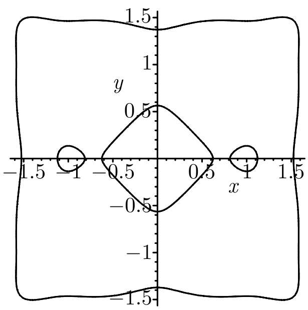
As h increases from $-\infty$ to h_5^c, the schemes of ovals of the eighth algebraic curves will be varied. This change process is shown in figures (1)∼(24) of Fig.8.1.2. Similarly, we can discuss other phase portraits in Fig.8.1.1



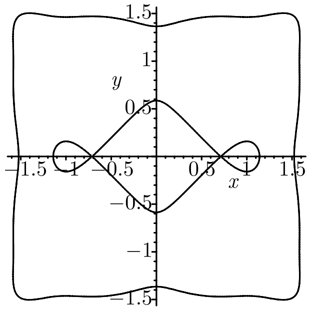
(1) $-\infty < h < 0$



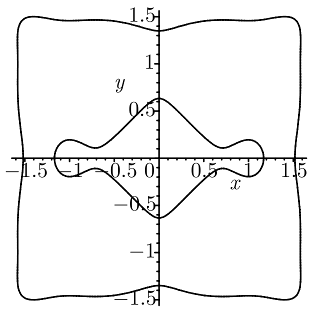
(2) $0 < h < h_1^c$



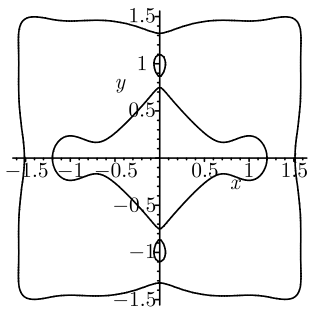
(3) $h_1^c < h < h_1^s$



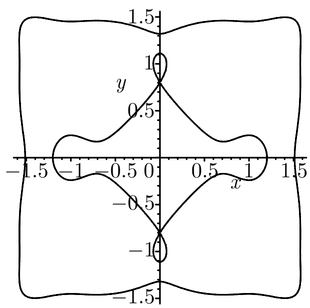
(4)$h = h_1^s$



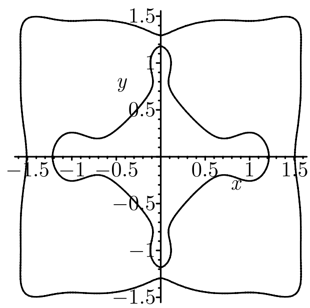
(5) $h_1^s < h < h_2^c$



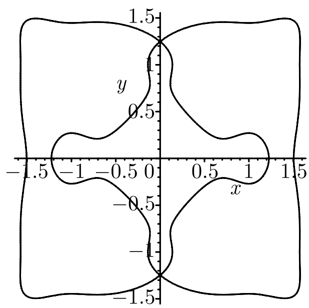
(6) $h_2^c < h < h_2^s$



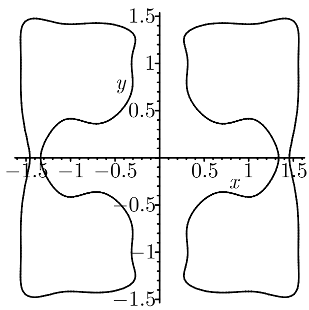
(7) $h = h_2^s$



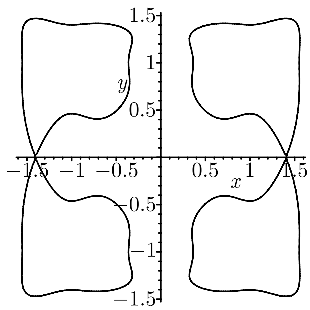
(8) $h_2^s < h < h_8^s$



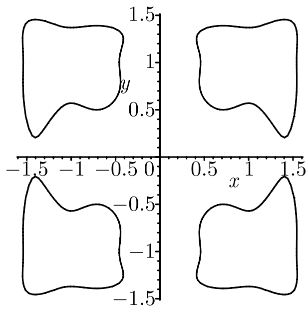
(9) $h = h_8^s$



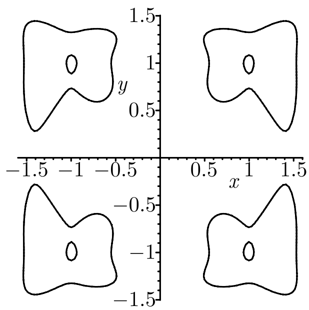
(10) $h_8^s < h < h_6^s$



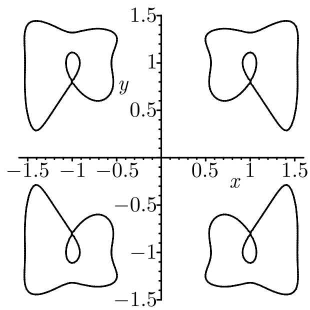
(11) $h = h_6^s$



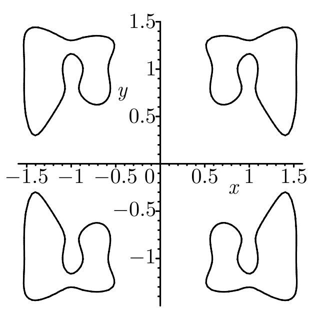
(12) $h_6^s < h < h_3^c$



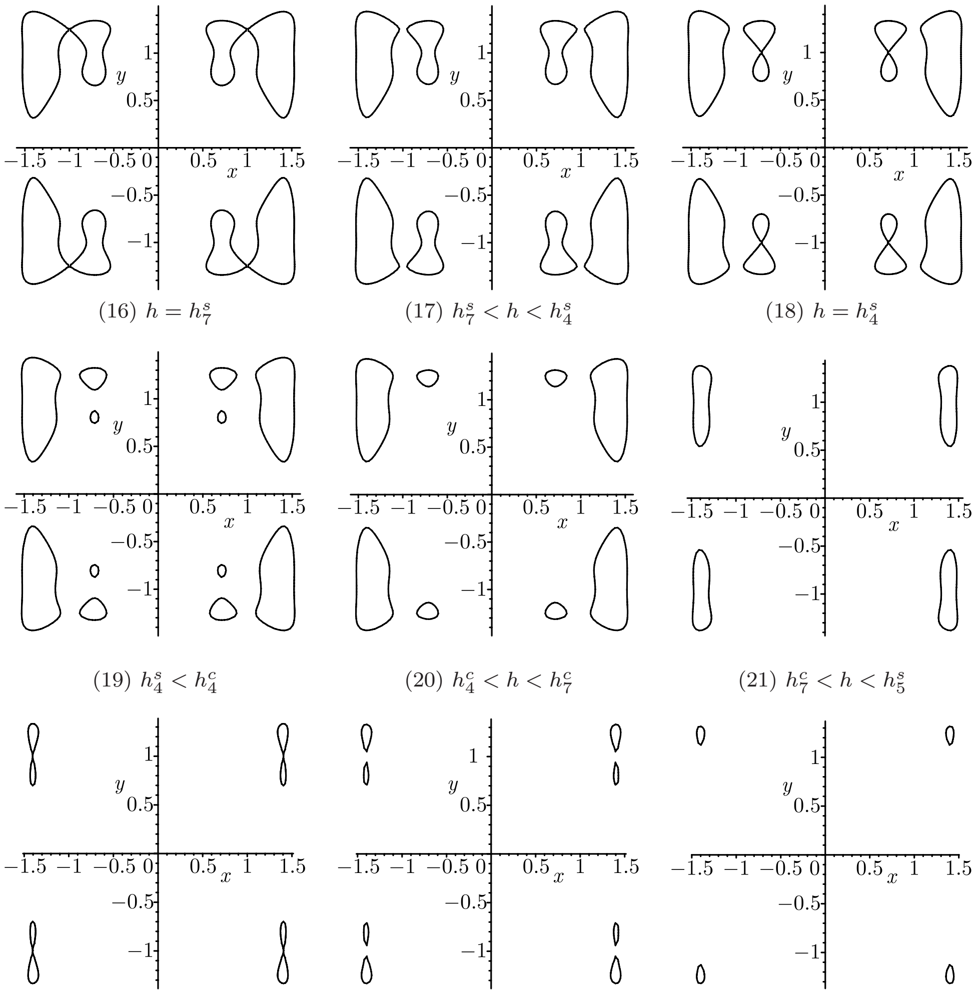
(13) $h_3^c < h < h_3^s$



(14) $h = h_3^s$



(15) $h_3^s < h < h_7^s$



(16) $h = h_7^s$ (17) $h_7^s < h < h_4^s$ (18) $h = h_4^s$

(19) $h_4^s < h_4^c$ (20) $h_4^c < h < h_7^c$ (21) $h_7^c < h < h_5^s$

(22) $h = h_5^s$ (23) $h_5^s < h < h_6^c$ (24) $h_6^c < h < h_5^c$

Fig.8.1.2 Different schemes of ovals defined by (8.1.6)

We see from Fig.8.1.2 that as h increases from $-\infty$ to h_5^c the schemes of ovals of the eighth algebraic curves defined by $H(x, y) = h$ are varied as follows.

(1) $h \in (-\infty, 0)$, there is a global periodic orbit family $\{\Gamma_8^h\}$, enclosing 49 singular points.

(2) $h \in (0, h_1^c)$, there exists a periodic orbit family $\{\Gamma_0^h\}$, enclosing the center $(0, 0)$. Together with $\{\Gamma_8^h\}$ there exist two families of periodic orbits.

(3) $h \in (h_1^c, h_1^s)$, there exist 2 families of periodic orbits $\{\Gamma_{1i}^h\},\ i = 1, 2$, enclosing the centers $(1, 0)$ and $(-1, 0)$, respectively. Together whit $\{\Gamma_0^h\}$ and $\{\Gamma_8^h\}$ there exist 4 families periodic orbits.

(4) $h = h_1^s$, there are two heteroclinic orbits $\{\Gamma_{0i}^{h_1^s}\},\ i = 1, 2$ and two homoclinic orbits $\{\Gamma_{1i}^{h_1^s}\},\ i = 1, 2$, and a periodic orbit $\{\Gamma_8^{h_1^s}\}$.

(5) $h_1^s < h < h_2^c$, there exist two families of periodic orbits: $\{\Gamma_{(1-2)}^h\}$, enclosing 5 finite singular points, and $\{\Gamma_8^h\}$.

(6) $h \in (h_2^c, h_2^s)$, there exist 4 families of periodic orbits: $\{\Gamma_{(1-2)}^h\}$, enclosing 5 finite singular points, and $\{\Gamma_{2i}^h\},\ i = 1, 2$, enclosing the singular point $(0, 1)$ and $(0, -1)$, respectively, and a global periodic orbit $\{\Gamma_8^h\}$, enclosing all 49 singular points.

(7) $h = h_2^s$, there are two heteroclinic orbits $\{\Gamma_{(1-2)}^{h_2^s}\}$ and two homoclinic orbits $\{\Gamma_{2i}^{h_1^s}\},\ i = 1, 2$ and a global periodic orbit $\{\Gamma_8^{h_2^s}\}$, enclosing all 49 singular points.

(8) $h \in (h_2^s, h_8^s)$, there exist two families of periodic orbits $\{\Gamma_{(2-8)}^h\}$, enclosing 9 finite singular points, and a periodic orbit family $\{\Gamma_8^h\}$, enclosing all 49 singular points.

(9) $h = h_8^s$, there are 2 heteroclinic orbits $\{\Gamma_{(2-8)i}^{h_8^s}\},\ i = 1, 2$ and 2 heteroclinic orbits $\{\Gamma_{(8-6)i}^{h_8^s}\}, \imath = 1, 2$.

(10) $h \in (h_8^s, h_6^s)$, there exist two families of periodic orbits $\{\Gamma_{(8-6)}^h\},\ i = 1, 2$, enclosing 19 singular points.

(11) $h = h_6^s$, there are 4 homoclinic orbits $\{\Gamma_{(6-3)i}^{h_6^s}\},\ i = 1, 2, 3, 4$.

(12) $h \in (h_6^s, h_3^c)$, there exist 4 families of periodic orbits $\{\Gamma_{(6-3)}^h\}$, enclosing 9 finite singular points.

(13) $h_3^c < h < h_3^s$, there are 4 families of periodic orbits $\{\Gamma_{(6-3)i}^h\}$, and $\{\Gamma_{3i}^h\},\ i = 1, 2, 3, 4$.

(14) $h = h_3^s$, there exist 4 homoclinic orbits $\{\Gamma_{3i}^{h_3^s}\},\ i = 1, 2, 3, 4$, enclosing one singular point $(1, 1)$ and its z_2-symmetric points, respectively, and there exist 4 homoclinic orbits $\{\Gamma_{(6-3)i}^{h_3^s}\},\ i = 1, 2, 3, 4$, enclosing 9 singular points, respectively.

(15) $h \in (h_3^s, h_7^s)$, there exist 4 families of periodic orbits $\{\Gamma_{(3-7)}^h\}$, enclosing 7 finite singular points,respectively.

(16) $h = h_7^s$, there are 8 homoclinic orbits $\{\Gamma_{(7-4)i}^{h_7^s}\},\ i = 1, 2, 3, 4$ and $\{\Gamma_{(7-5)i}^{h_7^s}\}$, $i = 1, 2, 3, 4$.

(17) $h \in (h_7^s, h_4^s)$, there exist 8 families of periodic orbits $\{\Gamma_{(7-4)i}^h\},\ i = 1, 2, 3, 4$ and $\{\Gamma_{(7-5)i}^h\},\ i = 1, 2, 3, 4$, enclosing 3 finite singular points, respectively.

(18) $h = h_4^s$, there are 8 homoclinic orbits $\{\Gamma_{7i}^{h_4^s}\},\ i = 1, 2, 3, 4$, $\{\Gamma_{4i}^{h_4^s}\},\ i = 1, 2, 3, 4$ and 4 periodic orbits $\{\Gamma_{(7-5)i}^{h_4^s}\}$.

(19) $h \in (h_4^s, h_4^c)$, there exist 12 families of periodic orbits: $\{\Gamma_{7i}^h\},\ i = 1, 2, 3, 4$, enclosing the singular point (x_1, y_3) and its Z_2-equivariant symmetry points, and

$\{\Gamma^h_{4i}\},\ i=1,2,3,4$, enclosing the singular point (x_1,y_1) and its Z_2-equivariant symmetry points, and $\{\Gamma^h_{(7-5)i}\},\ i=1,2,3,4$, enclosing 3 finite singular points, respectively.

(20) $h\in(h^c_4,h^c_7)$, there exist 8 families of periodic orbits:$\{\Gamma^h_{7i}\},\ i=1,2,3,4$, enclosing the singular point (x_1,y_3) and its Z_2-equivariant symmetry points, and $\{\Gamma^h_{(7-5)i}\},\ i=1,2,3,4$, enclosing 3 finite singular points, respectively.

(21) $h\in(h^c_7,h^s_5)$, there exist 4 families of periodic orbits $\{\Gamma^h_{(7-5)i}\},\ i=1,2,3,4$, enclosing 3 finite singular points, respectively.

(22) $h=h^s_5$, there are 8 homoclinic orbits $\Gamma^{h^s_5}_{5i},\imath=1,2,3,4$ and $\{\Gamma^{h^s_5}_{6i}\},\ i=1,2,3,4$.

(23) $h\in(h^s_5,h^c_6)$, there exist 8 families of periodic orbits: $\{\Gamma^h_{5i}\},\ i=1,2,3,4$ and $\{\Gamma^h_{6i}\},\ i=1,2,3,4$.

(24) $h\in(h^c_6,h^c_5)$, there exist 4 families of periodic orbits $\{\Gamma^h_{5i}\},\ i=1,2,3,4$.

Notice that as h increases the periodic orbits $\Gamma^h_0,\ \Gamma^h_{1i},\ \Gamma^h_{(1-2)},\ \Gamma^h_2,\ \Gamma^h_3$ expand outwards, all other periodic orbits contract inwards.

8.2 The Method of Detection Functions: Rough Perturbations of Z_q-Equivariant Hamiltonian Vector Fields

Consider the following perturbed planar Hamiltonian system

$$\begin{aligned}\frac{dx}{dt}&=\frac{\partial H}{\partial y}-\varepsilon x\big[p(x,y)-\lambda\big],\\ \frac{dy}{dt}&=-\frac{\partial H}{\partial x}-\varepsilon y\big[p(x,y)-\lambda\big],\end{aligned}\tag{8.2.1}$$

where $H(x,y)$ is the Hamiltonian, $p(0,0)=0, 0<\varepsilon\ll 1,\lambda\in R$. Because the perturbations in the right hand of (8.2.1) always have two linear terms λx and λy, so that it is called rough perturbations.

Suppose that the origin in the phase plane is a singular point of (8.2.1) and the following conditions hold:

($\mathbf{A}_1$) The unperturbed system $(8.2.1)_{\varepsilon=0}$ is a Z_q-equivariant Hamiltonian vector field. For $h\in(h_1,h_2)$ one branch family of the curves $\{\Gamma^h\}$ defined by the Hamiltonian function $H(x,y)=h$ lies in a period annulus enclosing at least one singular point. As h increases, Γ^h expands outwards. When $h\to h_1,\Gamma^h$ approaches a singular point or an inner boundary of the period annulus consisting of a heteroclinic (or homoclinic) loop.

($\mathbf{A}_2$) Surrounding the period annulus, there exists a heteroclinic (or homoclinic) loop Γ^{h_2} at $h=h_2$ connecting some hyperbolic saddle points $(\alpha_i,\beta_i), 1\leqslant i\leqslant q$.

($\mathbf{A}_3$) The divergence $2\varepsilon[\lambda-F(x,y)]\equiv 2\varepsilon\left[\lambda-\dfrac{x}{2}\dfrac{\partial p}{\partial x}-\dfrac{y}{2}\dfrac{\partial p}{\partial y}-p(x,y)\right]$ of the perturbed vector field is a Z_q-invariant function.

We define the function

$$\lambda = \lambda(h) = \frac{\displaystyle\iint_{D^h} F(x,y)dxdy}{\displaystyle\iint_{D^h} dxdy} = \frac{\psi(h)}{\phi(h)}, \tag{8.2.2}$$

which is called a detection function corresponding to the periodic family $\{\Gamma^h\}$. The graph of $\lambda = \lambda(h)$ in the plane (h, λ) is called a detection curve, where D^h is the area inside Γ^h.

Clearly, if $H(x,y) = h$ is a polynomial, then $\lambda(h)$ is a ratio between two Abelian integrals. In this case, $\lambda(h)$ is a differentiable function with respect to h. Of course, when the degree of $H(x,y)$ is more than 4, classical mathematical analysis cannot provide the calculating method for $\lambda(h)$ in general. We must use a numerical technique to compute these Abelian integrals. Our following approach is computational. It is satisfying that for finding much more limit cycles and their complicated patterns.

On the basis of the Poincaré-Pontrjagin-Andronov theorem on the global center bifurcation and Melnikov method (see [Melnikov, 1963]), we have the following two conclusions (as in [Li Jibin etc, 1985, 1992]):

Theorem 8.2.1 (Bifurcation of limit cycles). *Suppose that the conditions* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_3)$ *hold. For a given* $\lambda = \lambda_0$, *considering the set* S *of the intersection points of the straight line* $\lambda = \lambda_0$ *and the curve* $\lambda = \lambda(h)$ *in the* (h, λ)*-plane, we have*

(1) *If* S *consists of exactly one point* (h_0, λ_0) *and* $\lambda'(h) > 0 (< 0)$, *then there exists a stable(unstable) limit cycle of* (8.2.1) *near* Γ^{h_0};

(2) *If* S *consists of two points* (h_0, λ_0) *and* $(\tilde{h}_0, \lambda_0)$ *having* $\tilde{h}_0 > h_0$ *and* $\lambda'(\tilde{h}_0) > 0, \lambda'(h_0) < 0$, *then there exist two limit cycles near* $\Gamma^{\tilde{h}_0}$ *and* Γ^{h_0} *respectively, the former is stable and the latter is unstable*;

(3) *If* S *contains a point* (h_0, λ_0) *and* $\lambda'(h_0) = \lambda''(h_0) = \cdots = \lambda^{(k-1)}(h_0) = 0$, *but* $\lambda^{(k)}(h_0) \neq 0$, *then* (8.2.1) *has at most* k *limit cycles near* Γ^{h_0};

(4) *If* S *is empty, then* (8.2.1) *has no limit cycle.*

Theorem 8.2.2 (Bifurcation parameter created by a heteroclinic or homoclinic loop). *Suppose that the conditions* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *and* $(\mathbf{A}_3)$ *hold. Then for* $0 < \varepsilon \ll 1$, *when* $\lambda = \lambda(h_2) + O(\varepsilon)$, *system* (8.2.1) *has a heteroclinic (or homoclinic) loop having* Z_q*-equivariance.*

The following two propositions describe the properties of the detection function at the boundary values of h.

Proposition 8.2.1 (The parameter value of Hopf bifurcation). *Suppose that as* $h \to h_1$, *the periodic orbit* Γ^h *of* (8.2.1) *approaches a singular point* (ξ, η), *then at this point the Hopf bifurcation parameter value is given by*

$$b_H = \lambda(h_1) + O(\varepsilon) = \lim_{h \to h_1} \lambda(h) + O(\varepsilon) = F(\xi, \eta) + O(\varepsilon). \tag{8.2.3}$$

Proposition 8.2.2 (Bifurcation direction of heteroclinic or homoclinic loop). *Suppose that as $h \to h_2$, the periodic orbit Γ^h of* (8.2.1) *approaches a heteroclinic* (*or homoclinic*) *loop connecting a hyperbolic saddle point* (α, β), *where the saddle point value satisfies*

$$SQ(\alpha, \beta) = 2\varepsilon\sigma(\alpha, \beta) \equiv 2\varepsilon\big[\lambda(h_2) - F(\alpha, \beta)\big] > 0 \ (< 0),$$

then we have

$$\lambda'(h_2) = \lim_{h \to h_2} \lambda'(h) = -\infty \ (+\infty). \tag{8.2.4}$$

Remark 8.2.1. (1) *If Γ^h contracts inwards as h increases, then the stability of limit cycles mentioned in Theorem* 8.2.1 *and the sign of $\lambda'(h_2)$ in* (8.2.4) *have the opposite conclusion.*

(2) *If the curve Γ^h defined by $H(x, y) = h$ ($h \in (h_1, h_2)$) consists of m components of oval families having Z_q-equivariance, then Theorem* 8.2.1 *gives rise to simultaneous global bifurcations of limit cycles from all these m oval families.*

(3) *If* $(8.2.1)_{\varepsilon=0}$ *has several different period annuluses filled by periodic orbit families $\{\Gamma_i^h\}$, then by calculating detection functions for every oval families, the global information of bifurcations of system* (8.2.1) *can be obtained.*

We notice that the hypothesis (A_3) is very important symmetry condition to guarantee simultaneous global bifurcations of limit cycles from all symmetric ovals of unperturbed systems. There were some errors which appeared in some literatures (the authors used asymmetric perturbations to give symmetric bifurcations of limit cycles). Work in this area requires a great deal of care.

8.3 Bifurcations of Limit Cycles of a Z_2-Equivariant Perturbed Hamiltonian Vector Fields

In this section, we use the method of detection functions to study the following Z_2-equivariant perturbed polynomial Hamiltonian vector field of degree 7:

$$\begin{aligned} \frac{dx}{dt} =& -y(-1 + ay^2 - ay^4 + y^6) \\ & -\varepsilon x(x^6 + py^6 + qx^4y^2 + mx^2y^4 + nx^4 + ly^4 + ex^2y^2 + gx^2 + ky^2 - \lambda), \\ \frac{dy}{dt} =& x(-1 + bx^2 - bx^4 + x^6) \\ & -\varepsilon y(x^6 + py^6 + qx^4y^2 + mx^2y^4 + nx^4 + ly^4 + ex^2y^2 + gx^2 + ky^2 - \lambda). \end{aligned} \tag{8.3.1}$$

Our main result of this section is the following theorem.

Theorem 8.3.1. *For all small* $\varepsilon > 0$, *there is a parameter group* $G = (a, b) = (3.2,\ 3.48)$, $PG = (p,\ q,\ m,\ n,\ l,\ e,\ g,\ k)$ *and* $\lambda = \widetilde{\lambda} \in (\lambda_{44}(h_4^s),\ max(\lambda_{56}(h)) = (3.931785676,\ 3.931785952)$, *where*

$$p = -0.680337, \quad q = 0.728, \quad m = -0.01983226, \quad n = -5.5693079,$$
$$l = 2.971005, \quad e = -2.3624684728, \quad g = 10.3882075, \quad k = -3.067249,$$

such that system (8.3.1) *has at least* 50 *limit cycles with the configuration as Fig.*8.3.1. *It implies that* $H(7) \geqslant 50$.

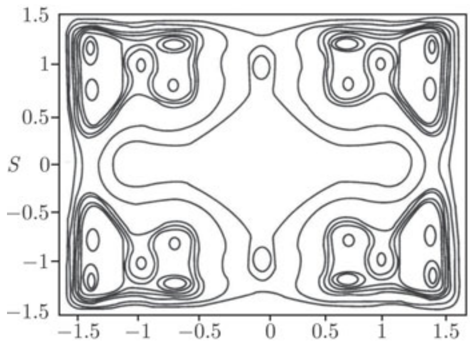


Fig.8.3.1 The configuration of 50 limit cycles of (8.3.1)

Corresponding to (8.3.1), the function F(x,y) defined in Section 8.2, (A_3) for the divergence of the perturbed vector field has the form

$$F(x, y) = 4x^6 + 4py^6 + 4qx^4y^2 + 4mx^2y^4 + 3nx^4 + 3ly^4 + 3ex^2y^2 + 2gx^2 + 2ky^2. \tag{8.3.2}$$

We consider the case $G = (a, b) = (3.2,\ 3.48)$.

We compute 16 detection functions defined by (8.2.2) which correspond to the above 16 types of period annuluses laid by $\{\Gamma_i^h\}$, $i = 0 \sim 8$ and $\{\Gamma_{(i-j)}^h\}$, $i, j = 1 \sim 8$.

$$\begin{aligned}\lambda_i(h) &= \frac{\displaystyle\iint_{D_i^h} F(x,y)dxdy}{\displaystyle\iint_{D_i^h} dxdy} = \frac{\psi_i}{\phi_i} \\ &= \frac{1}{\phi_i}\,[4I_{i1}(h) + 4pI_{i2}(h) + 4qI_{i3}(h) + 4mI_{i4}(h) + 3nI_{i5}(h) \\ &\quad +3lI_{i6}(h) + 3eI_{i7}(h) + 2gI_{i8}(h) + 2kI_{i9}(h)]\end{aligned}$$

$$
\begin{aligned}
&= 4J_{i1}(h) + 4pJ_{i2}(h) + 4qJ_{i3}(h) + 4mJ_{i4}(h) + 3nJ_{i5}(h) \\
&\quad +3lJ_{i6}(h) + 3eJ_{i7}(h) + 2gJ_{i8}(h) + 2kJ_{i9}(h),
\end{aligned} \tag{8.3.3}
$$

where $i = 0, \cdots, 15$, $J_{ij} = \dfrac{I_{ij}}{\phi_i(h)}$, $j = 1 \sim 9$ and D_i^h is the area inside Γ_i^h, $i = 0 \sim 8$ and $\{\Gamma_{(i-j)}^h\}$, $i, j = 1 \sim 8$.

$$
I_{i1}(h) = \iint_{D_i^h} x^6 dxdy, \quad I_{i2}(h) = \iint_{D_i^h} y^6 dxdy, \quad I_{i3}(h) = \iint_{D_i^h} x^4y^2 dxdy,
$$

$$
I_{i4}(h) = \iint_{D_i^h} x^2y^4 dxdy, \quad I_{i5}(h) = \iint_{D_i^h} x^4 dxdy, \quad I_{i6}(h) = \iint y^4 dxdy,
$$

$$
I_{i7}(h) = \iint_{D_i^h} x^2y^2 dxdy, \quad I_{i8}(h) = \iint_{D_i^h} x^2 dxdy, \quad I_{i9}(h) = \iint_{D_i^h} y^2 dxdy.
$$

For the given parameter group G, the functions $J_{ij}(h)$ can be numerically calculated to a given degree of accuracy (in this section, up to 8 digits accuracy after the decimal point).

By using the theory given in Section 8.2, we immediately obtain the following values of bifurcation parameters and bifurcation direction detection.

8.3.1 Hopf Bifurcation Parameter Values

(1) Bifurcation from the origin (0,0):

$$
b_0^H = F(0,0) + O(\varepsilon) = \lambda_0(h_0^c) + O(\varepsilon) = 0 + O(\varepsilon);
$$

(2) Simultaneous bifurcations from the center (1,0) and its Z_2-equivariant symmetry point:

$$
b_1^H = F(1,0) + O(\varepsilon) = \lambda_1(h_1^c) + O(\varepsilon) = 4 + 3n + 2g + O(\varepsilon);
$$

(3) Simultaneous bifurcations from the center (0,1) and its Z_2-equivariant symmetry point:

$$
b_2^H = F(0,1) + O(\varepsilon) = \lambda_2(h_2^c) + O(\varepsilon) = 4p + 3l + 2k + O(\varepsilon);
$$

(4) Simultaneous bifurcations from the center (1,1) and its Z_2-equivariant symmetry points:

$$
\begin{aligned}
b_3^H =& F(1,1) + O(\varepsilon) = \lambda_3(h_3^c) + O(\varepsilon) \\
=& 4 + 4p + 4q + 4m + 3n + 3l + 3e + 2g + 2k + O(\varepsilon);
\end{aligned}
$$

(5) Simultaneous bifurcations from the center $(x_1, y_1)$and its Z_2-equivariant symmetry points:

$$\begin{aligned} b_4^H =& F(\xi_2, \eta_2) + O(\varepsilon) = \lambda_7(h_4^c) + O(\varepsilon) \\ =& 0.5206413704 + 1.057163732p + 0.659285010q + 0.8348486100m \\ &+ 0.7705018932n + 1.235500041l + 0.9756818745e \\ &+ 1.013575778g + 1.283484861k + O(\varepsilon); \end{aligned}$$

(6) Simultaneous bifurcations from the center $(x_3, y_3)$and its Z_2-equivariant symmetry points:

$$\begin{aligned} b_5^H =& F(x3, y_3) + O(\varepsilon) = \lambda_5(h_5^c) + O(\varepsilon) \\ =& 30.73132664 + 15.13483632p + 24.26871502q + 19.16515143m \\ &+ 11.68069811n + 7.284499974l + 9.224318133e \\ &+ 3.946424222g + 3.116515142k + O(\varepsilon); \end{aligned}$$

(7) Simultaneous bifurcations from the center (x_3, y_1) and its Z_2-equivariant symmetry points:

$$\begin{aligned} b_6^H =& F(x_3, y_1) + O(\varepsilon) = \lambda_6(h_6^c) + O(\varepsilon) \\ =& 30.73132664 + 1.057163732p + 9.994666124q + 3.250538192m \\ &+ 11.68069811n + 1.235500041l + 3.798881808e + 3.946424222g \\ &+ 1.283484861k + O(\varepsilon); \end{aligned}$$

(8) Simultaneous bifurcations from the center(x_1, y_3) and its Z_2-equivariant symmetry points:

$$\begin{aligned} b_7^H =& F(x_1, y_3) + O(\varepsilon) = \lambda_5(h_7^c) + O(\varepsilon) \\ =& 0.5206413704 + 15.13483632p + 1.600853878q + 4.922261820m \\ &+ 0.7705018932n + 7.284499974l + 2.369118194e \\ &+ 1.013575778g + 3.116515142k + O(\varepsilon). \end{aligned}$$

8.3.2 Bifurcations from Heteroclinic or Homoclinic Loops

(1) The heteroclinic bifurcation value from $\Gamma_{0i}^{h_1^s}$:

$$\begin{aligned} \lambda_0(h_1^s) =& 4\Big[J_{01}(h_1^s) + pJ_{02}(h_1^s) + qJ_{03}(h_1^s) + mJ_{04}(h_1^s)\Big] \\ &+ 3\Big[nJ_{05}(h_1^s) + lJ_{06}(h_1^s) + eJ_{07}(h_1^s)\Big] + 2\Big[gJ_{08}(h_1^s) + kJ_{09}(h_1^s)\Big] \\ =& 0.01847349678 + 0.008628383552p + 0.001096236119q \\ &+ 0.0009416294268m + 0.05147676987n + 0.3177776748l \\ &+ 2.849669323e + 0.1708886847g + 0.1357139172k; \end{aligned}$$

(2) The homoclinic bifurcation value from $\Gamma_{1i}^{h_1^s}$:

$$\begin{aligned}\lambda_1(h_1^s) =&4\Big[J_{11}(h_1^s)+pJ_{12}(h_1^s)+qJ_{13}(h_1^s)+mJ_{14}(h_1^s)\Big]\\&+3\Big[nJ_{15}(h_1^s)+lJ_{16}(h_1^s)+eJ_{17}(h_1^s)\Big]+2\Big[gJ_{18}(h_1^s)+kJ_{19}(h_1^s)\Big]\\=&3.611784974+0.000004518102844p+0.0222157829q\\&+0.0002817271755m+2.701412176n+0.0002165886604l\\&+0.01691288606e+1.860157372g+0.01170095915k;\end{aligned}$$

(3) The homoclinic loop bifurcation value from $\Gamma_{(1-2)i}^{h_1^s}$:

$$\begin{aligned}\lambda_{12}(h_1^s) =&4\Big[J_{12a_1}(h_1^s)+pJ_{12a_2}(h_1^s)+qJ_{12a_3}(h_1^s)+mJ_{12a_4}(h_1^s)\Big]\\&+3\Big[nJ_{12a_5}(h_1^s)+lJ_{12a_6}(h_1^s)+eJ_{12a_7}(h_1^s)\Big]+2\Big[gJ_{12a_8}(h_1^s)+kJ_{12a_9}(h_1^s)\Big]\\=&0.619127664+0.007186827272p+0.004626557344q\\&+0.0008313208604m+0.4944371853n+0.02650203403l\\&+0.00914146392e+0.4532650618g+0.1149840416k;\end{aligned}$$

(4) The homoclinic loop bifurcation value from $\Gamma_{(1-2)i}^{h_2^s}$:

$$\begin{aligned}\lambda_{12}(h_2^s) =&4\Big[J_{12b_1}(h_2^s)+pJ_{12b_2}(h_2^s)+qJ_{12b_3}(h_2^s)+mJ_{12b_4}(h_2^s)\Big]\\&+3\Big[nJ_{12b_5}(h_2^s)+lJ_{12b_6}(h_2^s)+eJ_{12b_7}(h_2^s)\Big]+2\Big[gJ_{12b_8}(h_2^s)+kJ_{12b_9}(h_2^s)\Big]\\=&1.634387781+0.0397726711p+0.02472219963q\\&+0.002861286998m+1.239285198n+0.09018424998l\\&+0.03143078892e+1.035484067g+0.2586575882k;\end{aligned}$$

(5) The homoclinic bifurcation value from $\Gamma_{2i}^{h_2^s}$:

$$\begin{aligned}\lambda_2(h_2^s) =&4\Big[J_{21}(h_2^s)+p(J_{22}h_2^s)+qJ_{23}(h_2^s)+mJ_{24}(h_2^s)\Big]\\&+3\Big[nJ_{25}(h_2^s)+lJ_{26}(h_2^s)+eJ_{27}(h_2^s)\Big]+2\Big[gJ_{28}(h_2^s)+kJ_{29}(h_2^s)\Big]\\=&0.00000005221480448+3.743975326p+0.0000146116802q\\&+0.005160125348m+0.00001111664026n+2.813468786l\\&+0.003907110966e+0.00266141767g+1.916502045k;\end{aligned}$$

(6) The homoclinic loop bifurcation value from $\Gamma_{(2-8)i}^{h_2^s}$:

$$\begin{aligned}\lambda_{28}(h_2^s) =&4\Big[J_{28a_1}(h_2^s)+pJ_{28a_2}(h_2^s)+qJ_{28a_3}(h_2^s)+mJ_{28a_4}(h_2^s)\Big]\\&+3\Big[nJ_{28a_5}(h_2^s)+lJ_{28a_6}(h_2^s)+eJ_{28a_7}(h_2^s)\Big]+2\Big[gJ_{28a_8}(h_2^s)+kJ_{28a_9}(h_2^s)\Big]\end{aligned}$$

$$=1.528568166+0.2796039501p+0.02312248899q$$
$$+0.003010126949m+1.159047502n+0.2665052762l$$
$$+0.02964874849e+0.9686132178g+0.3659959262k,$$

(7) The homoclinic loop bifurcation value from $\Gamma_{(2-8)i}^{h_8^s}$:

$$\begin{aligned}\lambda_{28}(h_8^s)=&4\Big[J_{28b_1}(h_8^s)+pJ_{28b_2}(h_8^s)+qJ_{28b_3}(h_8^s)+mJ_{28b_4}(h_8^s)\Big]\\&+3\Big[nJ_{28b_5}(h_8^s)+lJ_{28b_6}(h_8^s)+eJ_{28b_7}(h_8^s)\Big]+2\Big[gJ_{28b_8}(h_8^s)+kJ_{28b_9}(h_8^s)\Big]\\=&1.372838402+0.7130790244p+0.02496369812q\\&+0.005925672356m+0.9688062621n+0.5469176559l\\&+0.03004169472e+0.6924383026g+0.5056410860k;\end{aligned}$$

(8) The homoclinic bifurcation value from $\Gamma_8^{h_8^s}$:

$$\begin{aligned}\lambda_8(h_8^s)=&4\Big[J_{81}(h_8^s)+pJ_{82}(h_8^s)+qJ_{83}(h_8^s)+mJ_{84}(h_8^s)\Big]\\&+3\Big[nJ_{85}(h_8^s)+lJ_{86}(h_8^s)+eJ_{87}(h_8^s)\Big]+2\Big[gJ_{88}(h_8^s)+kJ_{89}(h_8^s)\Big]\\=&8.450862632+5.105966132p+3.636837927q\\&+3.134050664m+3.641439873n+3.336393348l\\&+1.823126874e+1.657717122g+1.384773977k;\end{aligned}$$

(9) The heteroclinic loop bifurcation value from $\Gamma_{(8-6)i}^{h_8^s}$:

$$\begin{aligned}\lambda_{86}(h_8^s)=&4\Big[J_{86a_1}(h_8^s)+pJ_{86a_2}(h_8^s)+qJ_{86a_3}(h_8^s)+mJ_{86a_4}(h_8^s)\Big]\\&+3\Big[nJ_{86a_5}(h_8^s)+lJ_{86a_6}(h_8^s)+eJ_{86a_7}(h_8^s)\Big]+2\Big[gJ_{86a_8}(h_8^s)+kJ_{86a_9}(h_8^s)\Big]\\=&9.754388324+5.914983032p+4.30201944q\\&+3.710142403m+4.133645955n+3.850117674l\\&+2.153350750e+1.835487881g+1.546679654k;\end{aligned}$$

(10) The homoclinic bifurcation value from $\Gamma_6^{h_6^s}$:

$$\begin{aligned}\lambda_6(h_6^s)=&4\Big[J_{61}(h_6^s)+pJ_{62}(h_6^s)+qJ_{63}(h_6^s)+mJ_{64}(h_6^s)\Big]\\&+3\Big[nJ_{65}(h_6^s)+lJ_{66}(h_6^s)+eJ_{67}(h_6^s)\Big]+2\Big[gJ_{68}(h_6^s)+kJ_{69}(h_6^s)\Big]\\=&7.155564872+6.199982264p+3.530078178q\\&+3.339413539m+3.343305972n+3.066794199l\\&+1.962417754e+1.646792689g+1.590722154k;\end{aligned}$$

(11) The homoclinic loop bifurcation value from $\Gamma^{h_3^s}_{(6-3)i}$:

$$\begin{aligned}\lambda_{63}(h_3^s) =&4\Big[J_{361}(h_3^s) + pJ_{362}(h_3^s) + qJ_{363}(h_3^s) + mJ_{364}(h_3^s)\Big]\\&+ 3\Big[nJ_{365}(h_3^s) + lJ_{366}(h_3^s) + eJ_{367}(h_3^s)\Big] + 2\Big[gJ_{368}(h_3^s) + kJ_{369}(h_3^s)\Big]\\=&11.76423828 + 4.48408094p + 5.12115052q\\&+ 3.79492621m + 5.254572348n + 2.572807483l\\&+ 2.640530609e + 2.37684288g + 1.566411436k;\end{aligned}$$

(12) The homoclinic bifurcation value from $\Gamma^{h_3^s}_{3i}$:

$$\begin{aligned}\lambda_{3}(h_3^s) =&4\Big[J_{31}(h_3^s) + pJ_{32}(h_3^s) + qJ_{33}(h_3^s) + mJ_{34}(h_3^s)\Big]\\&+ 3\Big[nJ_{35}(h_3^s) + lJ_{36}(h_3^s) + eJ_{37}(h_3^s)\Big] + 2\Big[gJ_{38}(h_3^s) + kJ_{39}(h_3^s)\Big]\\=&3.962380738 + 3.746263574p + 3.787478389q\\&+ 3.718889686m + 2.9641166n + 2.815010396l\\&+ 2.849669323e + 1.982244017g + 1.917167834k;\end{aligned}$$

(13) The homoclinic loop bifurcation value from $\Gamma^{h_3^s}_{(3-7)i}$:

$$\begin{aligned}\lambda_{37}(h_3^s) =&4\Big[J_{371}(h_3^s) + pJ_{372}(h_3^s) + qJ_{373}(h_3^s) + mJ_{374}(h_3^s)\Big]\\&+ 3\Big[nJ_{375}(h_3^s) + lJ_{376}(h_3^s) + eJ_{377}(h_3^s)\Big] + 2\Big[gJ_{378}(h_3^s) + kJ_{379}(h_3^s)\Big]\\=&12.11002223 + 4.516781536p + 5.180259832q\\&+ 3.798296204m + 5.356087002n + 2.562072875l\\&+ 2.63126143e + 2.394331786g + 1.550865659k;\end{aligned}$$

(14) The homoclinic bifurcation value from $\Gamma^{h_7^s}_{7i}$:

$$\begin{aligned}\lambda_{7}(h_7^s) =&4\Big[J_{71}(h_7^s) + pJ_{72}(h_7^s) + qJ_{73}(h_7^s) + mJ_{74}(h_7^s)\Big]\\&+ 3\Big[nJ_{75}(h_7^s) + lJ_{76}(h_7^s) + eJ_{77}(h_7^s)\Big] + 2\Big[gJ_{78}(h_7^s) + kJ_{79}(h_7^s)\Big]\\=&13.11694704 + 7.1741212p + 8.364053504q\\&+ 6.711148228m + 5.590494093n + 3.812008314l\\&+ 3.908296332e + 2.405761824g + 2.02425938k;\end{aligned}$$

(15) The homoclinic loop bifurcation value from $\Gamma^{h_7^s}_{(7-4)i}$:

$$\begin{aligned}\lambda_{741}(h_7^s) =&4\Big[J_{741}(h_7^s) + pJ_{742}(h_7^s) + qJ_{743}(h_7^s) + mJ_{744}(h_7^s)\Big]\\&+ 3\Big[nJ_{745}(h_7^s) + lJ_{746}(h_7^s) + eJ_{747}(h_7^s)\Big] + 2\Big[gJ_{748}(h_7^s) + kJ_{749}(h_7^s)\Big]\end{aligned}$$

$$=0.7858417652+8.108441212p+1.50483635q$$
$$+3.278742351m+0.9616798872n+4.417631181l$$
$$+1.916102631e+1.102333304g+2.313089952k;$$

(16) The homoclinic loop bifurcation value from $\Gamma^{h_7^s}_{(7-5)i}$:

$$\begin{aligned}\lambda_{751}(h_7^s) =&4\Big[J_{751}(h_7^s)+pJ_{752}(h_7^s)+qJ_{753}(h_7^s)+mJ_{754}(h_7^s)\Big]\\&+3\Big[nJ_{755}(h_7^s)+lJ_{756}(h_7^s)+eJ_{757}(h_7^s)\Big]+2\Big[gJ_{758}(h_7^s)+kJ_{759}(h_7^s)\Big]\\=&18.40792832+6.773226916p+11.30717893q\\&+8.183911196m+7.576607154n+3.552150081l\\&+4.763098827e+2.965031712g+1.900329115k;\end{aligned}$$

(17) The homoclinic loop bifurcation value from $\Gamma^{h_4^s}_{(4-7)i}$:

$$\begin{aligned}\lambda_{47}(h_4^s) =& 4\Big[J_{471}(h_4^s)+pJ_{472}(h_4^s)+qJ_{473}(h_4^s)+mJ_{474}(h_4^s)\Big]\\&+3\Big[nJ_{475}(h_4^s)+lJ_{476}(h_4^s)+eJ_{477}(h_4^s)\Big]+2\Big[gJ_{478}(h_4^s)+kJ_{479}(h_4^s)\Big]\\=&0.7043393632+12.87612256p+1.754596815q\\&+4.609943068m+0.8994385899n+6.448646343l\\&+2.337070265e+1.068560984g+2.910511176k;\end{aligned}$$

(18) The homoclinic bifurcation value from $\Gamma^{h_4^s}_{(4-4)i}$:

$$\begin{aligned}\lambda_{44}(h_4^s) =&4\Big[J_{441}(h_4^s)+pJ_{442}(h_4^s)+qJ_{443}(h_4^s)+mJ_{444}(h_4^s)\Big]\\&+3\Big[nJ_{445}(h_4^s)+lJ_{446}(h_4^s)+eJ_{447}(h_4^s)\Big]+2\Big[gJ_{448}(h_4^s)+kJ_{449}(h_4^s)\Big]\\=&0.56571565+1.42863293p+0.7373783236q\\&+1.004593847m+0.8028564387n+1.468520833l\\&+1.062299310e+1.027178501g+1.379538797k;\end{aligned}$$

(19) The heteroclinic bifurcation value from $\Gamma^{h_4^s}_{4i}$:

$$\begin{aligned}\lambda_{4}(h_4^s) =&4\Big[J_{41}(h_4^s)+pJ_{42}(h_4^s)+qJ_{43}(h_4^s)+mJ_{44}(h_4^s)\Big]\\&+3\Big[nJ_{45}(h_4^s)+lJ_{46}(h_4^s)+eJ_{47}(h_4^s)\Big]+2\Big[gJ_{48}(h_4^s)+kJ_{49}(h_4^s)\Big]\\=&0.6570780724+8.97330446p+1.407794252q\\&+3.38076334m+0.866510622n+4.750761078l\\&+1.902459768e+1.054452362g+2.388553358k;\end{aligned}$$

(20) The heteroclinic bifurcation value from $\Gamma^{h_5^s}_{(5-5)i}$:

$$\begin{aligned}\lambda_{55}(h_5^s) =&4\Big[J_{551}(h_5^s)+pJ_{552}(h_5^s)+qJ_{553}(h_5^s)+mJ_{554}(h_5^s)\Big]\\&+3\Big[nJ_{555}(h_5^s)+lJ_{556}(h_5^s)+eJ_{557}(h_5^s)\Big]+2\Big[gJ_{558}(h_5^s)+kJ_{559}(h_5^s)\Big]\\=&30.32576921+12.81935486p+22.3809865q\\&+16.80312911m+11.55980965n+6.427436025l\\&+8.545855461e+3.922899756g+2.90521144k;\end{aligned}$$

(21) The heteroclinic bifurcation value from $\Gamma^{h_5^s}_{(5-6)i}$:

$$\begin{aligned}\lambda_{56}(h_5^s) =&4\Big[J_{561}(h_5^s)+pJ_{562}(h_5^s)+qJ_{563}(h_5^s)+mJ_{564}(h_5^s)\Big]\\&+3\Big[nJ_{565}(h_5^s)+lJ_{566}(h_5^s)+eJ_{567}(h_5^s)\Big]+2\Big[gJ_{568}(h_5^s)+kJ_{569}(h_5^s)\Big]\\=&30.6043408+1.430604303p+10.71292984q\\&+3.860168266m+11.6428985n+1.469740995l\\&+4.077323181e+3.939085054g+1.380040031k;\end{aligned}$$

(22) The heteroclinic bifurcation value from $\Gamma^{h_5^s}_{5i}$:

$$\begin{aligned}\lambda_{5}(h_5^s) =&4\Big[J_{51}(h_5^s)+pJ_{52}(h_5^s)+qJ_{53}(h_5^s)+mJ_{54}(h_5^s)\Big]\\&+3\Big[nJ_{55}(h_5^s)+lJ_{56}(h_5^s)+eJ_{57}(h_5^s)\Big]+2\Big[gJ_{58}(h_5^s)+kJ_{59}(h_5^s)\Big]\\=&30.42297952+8.845137676p+18.30930269q\\&+12.28655488m+11.58880433n+4.697399295l\\&+6.986516913e+3.928547774g+2.372987788k.\end{aligned}$$

8.3.3 The Values of Bifurcation Directions of Heteroclinic and Homoclinic Loops

$$\begin{aligned}(1)\quad \sigma_0 =&\lambda_0(h_1^s)-F(x_1,0)\\=&-0.5021678736+0.008628383552p+0.001096236119q\\&+0.0009416294268m-0.7190251233n+0.03177776748l\\&+2.849669323e-0.8426870933g+0.1357139172k;\end{aligned}$$

$$\begin{aligned}(2)\quad \sigma_1 =&\lambda_1(h_1^s)-F(x_1,0)\\=&3.091143604+0.000004518102844p+0.0222157829q\\&+0.0002817271755m+1.930910283n+0.0002165886604l\\&+0.01691288606e+0.846581594g+0.01170095915k;\end{aligned}$$

(3) $$\begin{aligned}\sigma_{121} =&\lambda_1(h_1^s) - F(x_1, 0)\\ =&0.00000005221480448 + 2.686811594p + 0.0000146116802q\\ &+ 0.005160125348m + 0.00001111664026n + 1.577968745l\\ &+ 0.003907110966e + 0.00266141767g + 0.633017184k;\end{aligned}$$

(4) $$\begin{aligned}\sigma_2 =&\lambda_2(h_2^s) - F(0, y_1)\\ =&0.0984862936 + 0.007186827272p + 0.004626557344q\\ &+ 0.0008313208604m - 0.2760647079n + 0.02650203403l\\ &+ 0.00914146392e - 0.5603107162g + 0.1149840416k;\end{aligned}$$

(5) $$\begin{aligned}\sigma_{122} =&\lambda_2(h_2^s) - F(0, y_1)\\ =&0.00000005221480448 + 2.686811594p + 0.0000146116802q\\ &+ 0.005160125348m + 0.00001111664026n + 1.577968745l\\ &+ 0.003907110966e + 0.00266141767g + 0.633017184k;\end{aligned}$$

(6) $$\begin{aligned}\sigma_8 =&\lambda_8(h_8^s) - F(0, y_3)\\ =&8.450862632 - 10.02887019p + 3.636837927q\\ &+ 3.134050664m + 3.641439873n - 3.948106626l\\ &+ 1.823126874e + 1.657717122g - 1.731741165k;\end{aligned}$$

(7) $$\begin{aligned}\sigma_{281} =&\lambda_{28}(h_2^s) - F(0, y_1)\\ =&1.528568166 - 0.7775597819p + 0.02312248899q\\ &+ 0.003010126949m + 1.159047502n - 0.9689947648l\\ &+ 0.02964874849e + 0.9686132178g - 0.9174889348k;\end{aligned}$$

(8) $$\begin{aligned}\sigma_{282} =&\lambda_{28}(h_8^s) - F(0, y_3)\\ =&1.372838402 - 14.4217573p + 0.02496369812q\\ &+ 0.005925672356m + 0.9688062621n - 6.737582318l\\ &+ 0.03004169472e + 0.6924383026g - 2.610874056k;\end{aligned}$$

(9) $$\begin{aligned}\sigma_{86} =&\lambda_8(h_8^s) - F(0, y_3)\\ =&9.754388324 - 9.219853288p + 4.30201944q\\ &+ 3.710142403m + 4.133645955n - 3.434382300l\\ &+ 2.15335075e + 1.835487881g - 1.569835488k;\end{aligned}$$

(10) $$\begin{aligned}\sigma_6 =&\lambda_6(h_6^s) - F(x_3, 0)\\ =&- 23.57576177 + 6.199982264p + 3.530078178q\\ &+ 3.339413539m - 8.337392138n + 3.066794199l\\ &+ 1.962417754e - 2.299631533g + 1.590722154k;\end{aligned}$$

$$\begin{aligned}
(11)\quad \sigma_{63} =&\lambda_{63}(h_3^s) - F(1, y_1)\\
&= 7.76423828 + 3.426917208p + 2.554180798q\\
&\quad + 2.147592822m + 2.254572348n + 1.337307442l\\
&\quad + 0.715303318e + 0.37684288g + 0.282926575k;\\
(12)\quad \sigma_{3} =&\lambda_{3}(h_3^s) - F(1, y_1)\\
=& -0.037619262 + 2.689099842p + 1.220508667q\\
&\quad + 2.071556298m - 0.0358834n + 1.579510355l\\
&\quad + 0.924442032e - 0.017755983g + 0.633682973k;\\
(13)\quad \sigma_{37} =&\lambda_{37}(h_3^s) - F(1, y_1)\\
=&8.11002223 + 3.459617804p + 2.61329011q\\
&\quad + 2.150962816m + 2.356087002n + 1.326572834l\\
&\quad + 0.706034139e + 0.394331786g + 0.267380798k;\\
(14)\quad \sigma_{7} =&\lambda_{7}(h_7^s) - F(1, y_3)\\
=&9.11694704 - 7.96071512p + 2.13102322q\\
&\quad - 3.001518404m + 2.590494093n - 3.47249166l\\
&\quad - 0.766476381e + 0.405761824g - 1.092255762k;\\
(15)\quad \sigma_{741} =&\lambda_{741}(h_7^s) - F(1, y_3)\\
=& -3.214158235 - 7.026395108p - 4.728193934q\\
&\quad - 6.433924281m - 2.038320113n - 2.866868793l\\
&\quad - 2.758670082e - 0.897666696g - 0.80342519k;\\
(16)\quad \sigma_{751} =&\lambda_{751}(h_7^s) - F(1, y_3)\\
=&14.40792832 - 8.361609404p + 5.074148646q\\
&\quad - 1.528755436m + 4.576607154n - 3.732349893l\\
&\quad + 0.088326114e + 0.965031712g - 1.216186027k;\\
(17)\quad \sigma_{47} =&\lambda_{47}(h_4^s) - F(x_1, 1)\\
=&0.1836979928 + 8.87612256p + 0.727260957q\\
&\quad + 2.582791512m + 0.1289366967n + 3.448646343l\\
&\quad + 0.816706598e + 0.054985206g + 0.910511176k;\\
(18)\quad \sigma_{44} =&\lambda_{44}(h_4^s) - F(x_1, 1)\\
=&0.0450742796 + 10.52043547p - 0.2899575344q\\
&\quad - 1.022557709m + 0.0323545455n - 1.531479167l\\
&\quad - 0.458064357e + 0.013602723g - 0.620461203k;
\end{aligned}$$

$$
\begin{aligned}
(19)\quad \sigma_4 =& \lambda_4(h_4^s) - F(x_1, 1)\\
=& 0.136436702 + 9.43672181p + 0.380458394q\\
&+ 1.353611784m + 0.0960087288n + 1.750761078l\\
&+ 0.382096101e + 0.040876584g + 0.388553358k;
\end{aligned}
$$

$$
\begin{aligned}
(20)\quad \sigma_{55} =& \lambda_{55}(h_5^s) - F(x_3, 1)\\
=& -0.40555743 + 8.81935486p + 6.80672236q\\
&+ 8.910280666m - 0.12088846n + 3.427436025l\\
&+ 2.626219128e - 0.023524466g + 0.90521144k;
\end{aligned}
$$

$$
\begin{aligned}
(21)\quad \sigma_{56} =& \lambda_{56}(h_5^s) - F(x_3, 1)\\
=& -0.12698584 - 2.569395697p - 4.8613343q\\
&- 4.032680178m - 0.03779961n - 1.530259005l\\
&- 1.842313152e - 0.007339168g - 0.619959969k;
\end{aligned}
$$

$$
\begin{aligned}
(22)\quad \sigma_5 =& \lambda_5(h_5^s) - F(x_3, 1)\\
=& -0.30834712 + 4.845137676p + 2.73503855q\\
&+ 4.393706436m - 0.09189378n + 1.697399295l\\
&+ 1.06688058e - 0.017876448g + 0.372987788k
\end{aligned}
$$

8.3.4 Analysis and Conclusions

We notice that the above bifurcation parameter values $\lambda_i(h_j^c)$, $\lambda_l(h_k^s)$ and the values of bifurcation directions of heteroclinic and homoclinic loops σ_i are 52 linear combinations of the perturbed parameter group $GP = (p, q, m, n, l, e, g, k)$ in which all coefficients are determined by the unperturbed parameter group $G = (a, b)$.

Our main idea is to control the perturbed parameter group GP such that system (8.3.1) has more limit cycles and interesting configurations of limit cycles. We now assume that the following 9 conditions hold.

(A$_1$) $\lambda_3(h_3^s) - \lambda_4(h_4^s) - 0.000004 = 0$, i.e.

$$
\begin{aligned}
&3.305294126 - 9.690458236p + 2.379684137q\\
&+0.338126346m + 2.097605978n - 1.935750682l\\
&+0.947209555e + 0.927791655g - 0.471385524k = 0;
\end{aligned}
$$

(A$_2$) $\lambda_{47}(h_4^s) - \lambda_{44}(h_4^s) = 0$, i.e.

$$
\begin{aligned}
&-0.1386317132 - 11.44748963p - 1.017218491q\\
&-3.605349221m - 0.0965821512n - 4.980125510l\\
&-1.274770955e - 0.041382483g - 1.530972379k = 0;
\end{aligned}
$$

(A$_3$) $\lambda_5(h_5^s) - \lambda_{55}(h_5^s) = 0$, i.e.

$$-0.09721031 + 3.974217184p + 4.07168381q + 4.51657423m - 0.02899468n + 1.730036730l + 1.559338548e - 0.005648018g + 0.532223652k = 0;$$

(A$_4$) $\lambda_7(h_7^s) - 0.0000074806 - \lambda_4(h_4^s) = 0$, i.e.

$$12.45986133 - 6.262600610p + 6.956259252q + 3.330384888m + 4.723983471n - 0.938752764l + 2.005836564e + 1.351309462g - 0.364293978k = 0;$$

(A$_5$) $\lambda_{44}(h_4^s) - \lambda_{55}(h_5^s) = 0$, i.e.

$$-29.76005356 + 1.70108061p - 21.64360818q - 15.79853526m - 10.75695321n - 4.958915192l - 7.483556151e - 2.895721255g - 1.525672643k = 0;$$

(A$_6$) $\sigma 4 > 0$, i.e.

$$0.1364367020 + 9.43672181p + 0.380458394q + 1.353611784m + 0.0960087288n + 1.750761078l + 0.382096101e + 0.040876584g + 0.388553358k > 0;$$

(A$_7$) $\sigma 5 > 0$, i.e.

$$-0.30834712 + 4.845137676p + 2.73503855q + 4.393706436m - 0.09189378n + 1.697399295l + 1.066880580e - 0.017876448g + 0.372987788k > 0;$$

(A$_8$) $\lambda_{44}(h_4^s) - \lambda_4(h_4^c) > 0$, i.e.

$$9.11694704 - 7.960715120p + 2.131023220q - 3.001518404m + 2.590494093n - 3.472491660l - 0.766476381e + 0.405761824g - 1.092255762k > 0;$$

(A$_9$) $\sigma_{282} > 0$, i.e.

$$0.0450742796 + 13.46327174p + 0.0780933136q + 0.1697452370m + 0.0323545455n + 0.233020792l + 0.0866174355e + 0.013602723g + 0.096053936k > 0.$$

This condition group implies that

$$n = -4.959044757 - 218.7859884q - 189.4342184m - 65.57083389e,$$
$$g = 8.145249994 + 811.1774884q + 704.3383284m + 243.1040413e,$$
$$p = 2.735300854 + 2189.269964q + 1906.335655m + 660.0711271e,$$
$$l = -10.43117841 - 8516.324523q - 7416.925354m - 2567.734602e,$$
$$k = 13.48130813 + 11324.34856q + 9863.015869m + 3413.831129e,$$

$$3410.376340q + 2967.465567m + 1027.306312e < -2.156351598,$$
$$-390.5034948q - 339.7885394m - 117.7244154e < 0.5724654325,$$
$$79.45248062q + 69.14974198m + 23.95556181e < -0.1242491626,$$

$$-3.493693868 < e < -2.341298974,$$
$$-0.04501413607 < m < 0.2064423284,$$
$$0.7251234415 < q < 1.075899527.$$

Let

$$m = -0.01983226, \quad q = 0.728, \quad e = -2.3624684728.$$

Then, we have

$$n = -5.5693079, \quad g = 10.3882075, \quad l = 2.971005,$$
$$p = -0.680337, \quad k = -3.067249.$$

Write $GP = (p, q, m, n, l, e, g, k)$, where

$$p = -0.680337, \quad q = 0.728, \quad m = -0.01983226, \quad n = -5.5693079,$$
$$l = 2.971005, \quad e = -2.3624684728, \quad g = 10.3882075, \quad k = -3.067249.$$

Under this parameter group we obtain the following results.

$$\begin{array}{lll}
\lambda_1(h_1^c) = 8.0684913, & \lambda_2(h_2^c) = 0.057169, & \lambda_3(h_3^c) = 3.87092584, \\
\lambda_4(h_4^c) = 3.931781015, & \lambda_5(h_5^c) = 3.955940327, & \lambda_6(h_6^c) = 3.925790544, \\
\lambda_7(h_7^c) = 4.015918357, & \lambda_0(h_1^s) = -5.552190597, & \lambda_1(h_1^s) = 7.831451565, \\
\lambda_{12}(h_1^s) = 2.276985007, & \lambda_{12}(h_2^s) = 4.880448863, & \lambda_2(h_2^s) = -0.048460718, \\
\lambda_{28}(h_2^s) = 4.561323043, & \lambda_{28}(h_8^s) = 2.706368228, & \lambda_{86}(h_8^s) = 6.441822038, \\
\lambda_8(h_8^s) = 5.860872204, & \lambda_6(h_6^s) = 3.524650018, & \lambda_{63}(h_3^s) = 4.394373129, \\
\lambda_3(h_3^s) = 3.931798368, & \lambda_{37}(h_3^s) = 4.414874768, & \lambda_7(h_7^s) = 3.931792091, \\
\lambda_{741}(h_7^s) = 3.898490328, & \lambda_{751}(h_7^s) = 3.946081102, & \lambda_{47}(h_4^s) = 3.931783666, \\
\lambda_{44}(h_4^s) = 3.931785676, & \lambda_4(h_4^s) = 3.931784361, & \lambda_{55}(h_5^s) = 3.931782966, \\
\lambda_{56}(h_5^s) = 3.931785105, & \lambda_5(h_5^s) = 3.931783670, &
\end{array}$$

$$\max(\lambda_{44}(h)) = 3.948799772, \quad \max(\lambda_{56}(h)) = 3.931785952,$$

and

$$\sigma_0 < 0,\ \sigma_1 > 0,\ \sigma_2 > 0,\ \sigma_{121} < 0,\ \sigma_{122} > 0,\ \sigma_8 > 0,\ \sigma_{281} > 0,\ \sigma_{282} > 0,$$

$$\sigma_{86} > 0,\ \sigma_8 > 0,\ \sigma_6 < 0,\ \sigma_{63} > 0,\ \sigma_3 < 0,\ \sigma_{37} > 0, \sigma_7 > 0,\ \ \sigma_{741} > 0,$$

$$\sigma_{751} > 0,\ \sigma_{44} > 0,\ \sigma_{47} > 0,\ \sigma_4 > 0,\ \sigma_{55} > 0,\ \sigma_{56} > 0,\ \sigma_5 > 0.$$

It follows that under the parameter conditions of G and PG, system (8.3.1) has the graphs of detection curves shown in Fig.8.3.2.

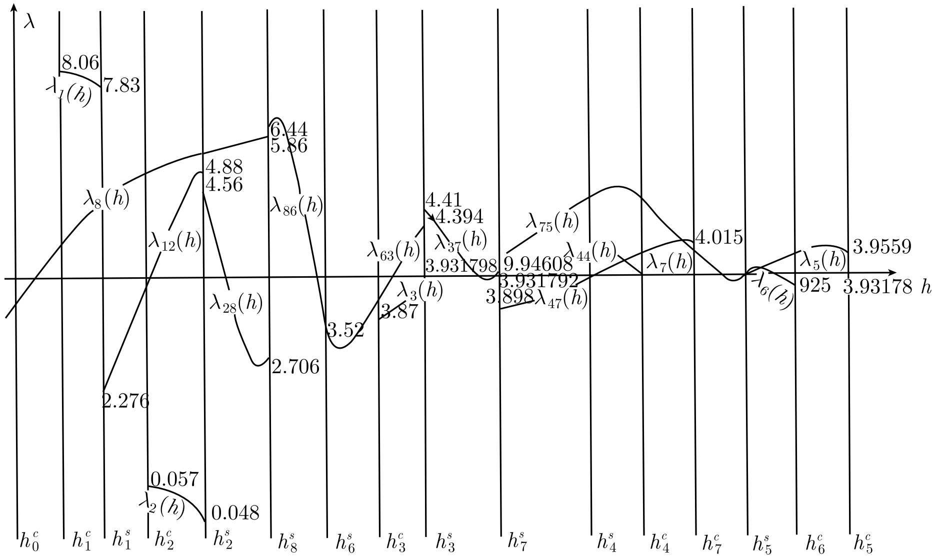


Fig.8.3.2 Graphs of detection curves of (8.3.1) with parameters G and PG

We see from Fig.8.3.2 that when

$$\widetilde{\lambda} \in \big(\lambda_{44}(h_4^s),\ \max(\lambda_{56}(h)\big) = (3.931785676,\ 3.931785952), \tag{8.3.4}$$

in the straight line $\lambda = \widetilde{\lambda}$ intersects the curves $\lambda = \lambda_{37}(h),\ \lambda = \lambda_{44}(h),\ \lambda = \lambda_6(h)$ at two points, and also intersects the curves $\lambda = \lambda_3(h),\ \lambda = \lambda_{63}(h),\ \ \lambda = \lambda_{86}(h),$ $\lambda = \lambda_{28}(h),\ \ \lambda = \lambda_{12}(h),\ \ \lambda = \lambda_7(h),\ \ \lambda = \lambda_5(h)$ at one point, respectively.

Hence, by using the Z_2-equivariance of (8.3.1) and the results in Section 8.2, we obtain the conclusion of Theorem 8.3.1.

8.4 The Rate of Growth of Hilbert Number $H(n)$ with n

While it has not been possible to obtain uniform upper bounds for $H(n)$ in the near future, there has been success in finding lower bounds (see Bibliographical notes below). In this section, we shall use idea stated in §8.4.1 and the method posed by [Christopher and Lloyd, 1995] to investigate some perturbed Z_2-(or Z_4-) equivariant planar Hamiltonian vector field sequences of degree $n(n = 2^k - 1$ and $n = 3 \times 2^{k-1} - 1$). We obtain some new lower bounds for $H(n)$ in Hilbert's 16th problem and configurations of compound eyes of limit cycles. In addition, we give some correct rates of growth of Hilbert number $H(n)$ with n are obtained.

8.4.1 Preliminary Lemmas

We consider the following perturbed planar polynomial Hamiltonian system

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\partial H}{\partial y} + \varepsilon R_1(x,y) = f_1(x,y) + \varepsilon R_1(x,y), \\ \frac{dy}{dt} &= \frac{\partial H}{\partial x} + \varepsilon R_2(x,y) = f_2(x,y) + \varepsilon R_2(x,y), \end{aligned} \tag{8.4.1}$$

where $H(x,y)$ is the Hamiltonian, $0 < \varepsilon \ll 1$.

The following lemma is given by [Christopher and Lloyd, 1995].

Lemma 8.4.1. (1) *Suppose that* $R_2(x,y) = 0$, $p = (x_c, y_c)$ *is a non-degenerate center of the unperturbed Hamiltonian system of* (8.4.1) *and let* U *be a neighborhood of* p. *For* $n \in Z$, *there is* ε_0 *and a polynomial* $R_1(x,y)$ *of degree* $2n+1$ *such that the perturbed system* (8.4.1) *has at least* n *limit cycles in* U *for* $0 < \varepsilon < \varepsilon_0$. *Without loss of generality, suppose that* $p = (0, y_c)$ *is on the* y*-axis. Then, the perturbation term* $R_1(x,y)$ *can have the form*

$$R(x) = \sum_{k=0}^{n} (-1)^k \eta_k x^{2(n-k)+1}, \tag{8.4.2}$$

where $\eta_0 = 1$ *and* $\eta_k \ll \eta_{k-1}$ $(k = 1, \cdots, n)$.

(2) *Suppose that* (8.4.1) *has* N *collinear non-degenerate centers and* $R_2(x,y) = 0$. *Then the* η_k *of* (8.4.2) *can be so chosen that* n *limit cycles appear around each of the centers simultaneously.*

Suppose the following conditions hold:

(A_1) The unperturbed system $(8.4.1)_{\varepsilon=0}$ defines a Z_q-equivariant Hamiltonian vector field ($q \geqslant 2$) for which all centers are non-degenerate and all saddle points are hyperbolic.

(A_2) When $h \in (-\infty, h_1)$(or $h \in (h_1, \infty)$), one branch family of the curves $\{\Gamma^h\}$ defined by the Hamiltonian function $H(x,y) = h$ lies in a global period annulus enclosing all finite singular points of $(8.4.1)_{\varepsilon=0}$. As $h \to h_1, \Gamma^h$ approaches an inner boundary of the period annulus consisting of a heteroclinic (or homoclinic) loop.

We know from [Li Jibin and Li Cunfu, 1995] that the condition (A2) holds if and only if the Hamiltonian $H(x,y)$ of $(8.4.1)_{\varepsilon=0}$ is positive (or negative) definite at infinity. Let d_0 be the maximal diameter of the area inside the inner boundary and $A > d_0$. For the "quadruple transformation" defined by [Christopher and Lloyd, 1995] (p222), we have the following generalized result.

Lemma 8.4.2. *Suppose that* (A_1) *and* (A_2) *hold. Then the map*

$$(x, y) \to (X^2 - A, Y^2 - A) \tag{8.4.3}$$

transforms (8.4.1) *into a new system which has the same orbits as*

$$\begin{aligned}\frac{dX}{dt} &= -\frac{\partial H_d}{\partial Y} + \varepsilon Y R_1(X^2 - A, Y^2 - A),\\ \frac{dY}{dt} &= \frac{\partial H_d}{\partial X} + \varepsilon X R_2(X^2 - A, Y^2 - A),\end{aligned} \tag{8.4.4}$$

where $H_d(X, Y) = H(X^2 - A, Y^2 - A)$ *is the new Hamiltonian of the unperturbed system* $(8.4.4)_{\varepsilon=0}$*. Furthermore, we have*

(1) *For the unperturbed system* $(8.4.4)_{\varepsilon=0}$*, it has four times as many period annuluses as* $(8.4.1)_{\varepsilon=0}$ *which lie in each quadrant and do not intersect the* X*-axis and* Y*-axis. At all image points except the origin of the singular points of* $(8.4.1)_{\varepsilon=0}$*, their Hamiltonian values are preserved. There exist new singular points* $(X_i, 0)$ *and* $(0, Y_j)$ *on the axes where* X_i *and* Y_j *satisfy* $f_1(X_i^2 - A, -A) = 0$ *and* $f_2(-A, Y_j^2 - A) = 0$*, respectively. There is a global period annulus surrounding all finite singular points of* $(8.4.4)_{\varepsilon=0}$*.*

(2) *For the perturbed system* (8.4.4)*, it has four copies of the existing limit cycles of* (8.4.1)*. These limit cycles do not intersect the* X *and* Y *axes, if the "shift constant"* A *is moderately large.*

As an example to understand Lemma 8.4.2, we consider a Z_6-equivariant Hamiltonian system of degree 5:

$$\begin{aligned}\frac{dx}{dt} =& -y + 2\delta(x^2 + y^2)y - \alpha(x^2 + y^2)^2 y\\ &+\beta[5(x^2 + y^2)^2 y - 20(x^2 + y^2)y^3 + 16y^5],\\ \frac{dy}{dt} =& \, x - 2\delta(x^2 + y^2)x + \alpha(x^2 + y^2)^2 x\\ &+\beta[5(x^2 + y^2)^2 x - 20(x^2 + y^2)x^3 + 16x^5],\end{aligned} \tag{8.4.5}$$

or in the polar coordinate form

$$\begin{aligned}\frac{dr}{dt} &= \beta r^5 \sin 6\theta,\\ \frac{d\theta}{dt} &= 1 - 2\delta r^2 + (\alpha + \beta \cos 6\theta)r^4,\end{aligned}$$

which has the Hamiltonian

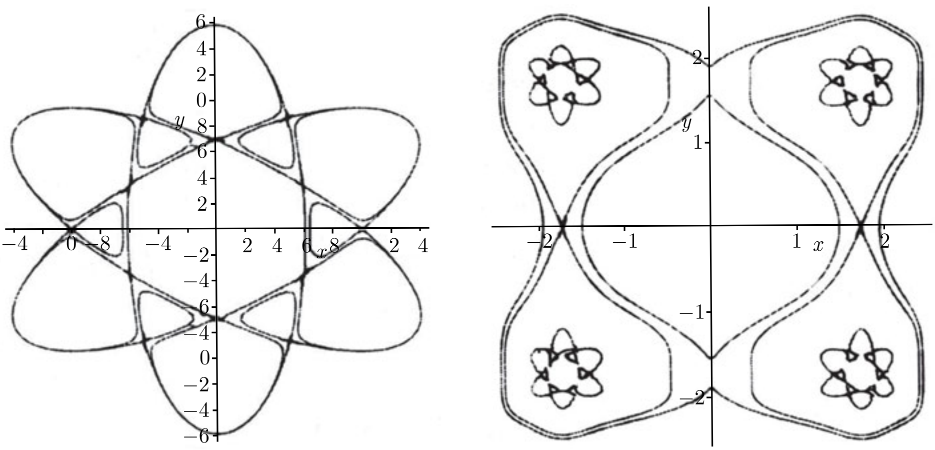
$$H(r, \theta) = -\frac{1}{2}r^2 + \frac{1}{2}\delta r^4 - \frac{1}{6}(\alpha + \beta \cos 6\theta)r^6.$$

Suppose that $\alpha > \beta > 0, \alpha + \beta > 1$ and $\delta = \dfrac{1}{2}(\alpha + \beta + 1)$. We see that the system (8.4.5) has 25 finite singular points at $(0, 0)$ and $(z_1, 0), (z_2, 0), \left(z_3, \dfrac{1}{6}\pi\right), \left(z_4, \dfrac{1}{6}\pi\right)$ and their Z_6-equivariant symmetric points.

Let $G = (\alpha, \beta, \delta) = (1.4,\ 0.25,\ 1.325)$. We have $z_1 = 0.7784989442,\ z_2 = 1,\ z_3 = 0.6895372608,\ z_4 = 1.352363188$ and

$$\begin{aligned} h_1 &= H(z_1,\ 0) = -0.12090603, \\ h_2 &= H(z_2,\ 0) = -0.1125, \\ h_3 &= H\left(z_3,\ \frac{1}{6}\pi\right) = -0.1085647965, \\ h_4 &= H\left(z_4,\ \frac{1}{6}\pi\right) = 0.1290200579. \end{aligned}$$

In this case, the phase portrait of (8.4.5) is shown in figure (1) of Fig.8.4.1 (only homoclinic and heteroclinic orbits are drawn in all phase portraits of this paper). Under the map $(x, y) \to (x^2-3, y^2-3)$, the new system of degree 11 is Z_2-equivariant. It has 109 finite simple singular points and the phase portrait shown in figure (2) of Fig.8.4.1.



(1) Phase portrait of a fifth system (2) Four copies of (1)

Fig.8.4.1 Four copies of a Z_6-equivariant Hamiltonian system

We also need to use the following obvious conclusion.

Lemma 8.4.3. *Suppose that the Hamiltonian function $H(x, y)$ of $(8.4.1)_{\varepsilon=0}$ is Z_q-invariant, then the Hamiltonian function $H_d(X, Y) = H(X^2 - A, Y^2 - A)$ of $(8.4.4)_{\varepsilon=0}$ is Z_2-invariant. In other words, the orbits of $(8.4.4)_{\varepsilon=0}$ have Z_2-equivariant symmetry. Thus, if Γ_i^h is a closed orbit around a center C_i of $(8.4.4)_{\varepsilon=0}$*

on an axis for any $h \in (h_c, h_s)$*, then*

$$
\begin{aligned}
I(h) &= \oint_{\Gamma_i^h} (YR_1(X^2 - A, Y^2 - A)dY - XR_2(X^2 - A, Y^2 - A)dX) \\
&= \iint_{\mathrm{int}\Gamma_i^h} 2XY \left[\frac{\partial R_1(X^2 - A, Y^2 - A)}{\partial (X^2 - A)} + \frac{\partial R_2(X^2 - A, Y^2 - A)}{\partial (Y^2 - A)} \right] dXdY \\
&= 0.
\end{aligned}
\tag{8.4.6}
$$

This lemma implies that the perturbation terms of the right hand of (8.4.4) do not create any limit cycle around the neighborhood of a center on an axis.

In the following subsections, we shall consider the following perturbed Hamiltonian system sequence:

$$
\frac{dx}{dt} = -\frac{\partial H_k}{\partial y} + \varepsilon P_k(x, y), \quad \frac{dy}{dt} = \frac{\partial H_k}{\partial x} + \varepsilon Q_k(x, y), \tag{PH_k}
$$

for $k = 2, 3, \cdots$, where

$$
\begin{aligned}
H_{k+1}(x, y) &= H_k(x^2 - A^{k-1}, y^2 - A^{k-1}), \\
P_{k+1}(x, y) &= P_k(x^2 - A^{k-1}, y^2 - A^{k-1}), \\
Q_{k+1}(x, y) &= Q_k(x^2 - A^{k-1}, y^2 - A^{k-1}).
\end{aligned}
$$

8.4.2 A Correction to the Lower Bounds of $H(2^k - 1)$ Given in [Christopher and Lloyd, 1995]

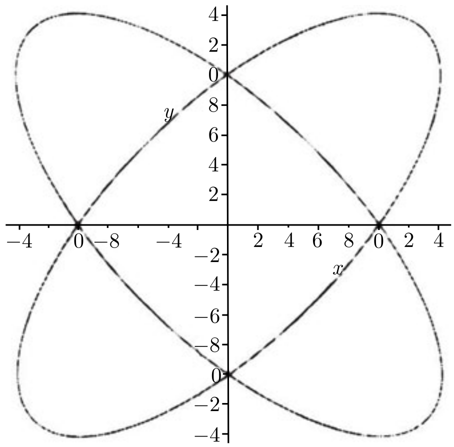
We first discuss the system given in [Christopher and Lloyd, 1995]. Suppose that $H_2(x, y) = (x^2 - 1)^2 + (y^2 - 1)^2$, i.e., we consider the cubic system

$$
\begin{aligned}
\frac{dx}{dt} &= -4y(y^2 - 1) + \varepsilon \left[\frac{1}{3}(x - y)^3 - \varepsilon(x - y) \right], \\
\frac{dy}{dt} &= 4x(x^2 - 1).
\end{aligned}
\tag{8.4.7}
$$

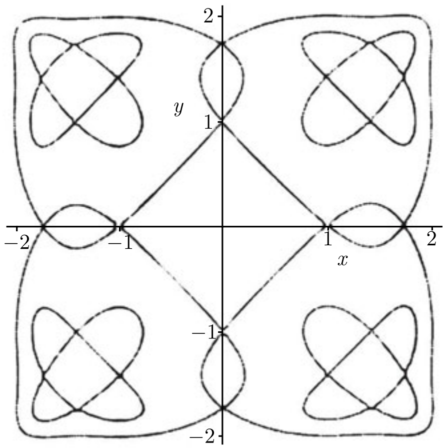
Let (8.4.7) be the system (PH_2). Then $(PH_2)_{\varepsilon=0}$ is a Z_4-equivariant system which has the phase portrait shown in figure(1) of Fig.8.4.2. Since $P_2(x, y) = \frac{1}{3}(x - y)^3 - \varepsilon(x - y)$ and $Q_2(x, y) = 0$. By using Lemma 8.4.1, it follows that there exist at least 3 limit cycles around 3 centers $(-1, -1), (0, 0)$ and $(1, 1)$ of $(8.4.7)_{\varepsilon=0}$, respectively.

We now consider the map: $(x, y) \to (x^2 - 1, y^2 - 1)$. By Lemma 8.4.2, under this map, the unperturbed system $(PH_3)_{\varepsilon=0}$ has the phase portrait shown in figure (2) in Fig.8.4.2 . For the perturbed system (PH_3), the perturbed terms become $P_3^{(1)}(x, y) = yP_2(x^2 - 1, y^2 - 1)$. As the first step, the map creates a new system having at least $12 = 4 \times 3$ limit cycles surrounding the image points of $(-1, -1)$, $(0, 0)$ and $(1, 1)$, respectively.

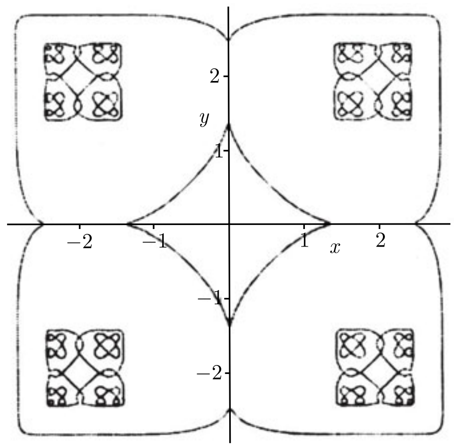
As the second step, by using Lemma 8.4.1, we take $P_3^{(2)}(x) = \eta_0 x^7 - \eta_1 x^5 + \eta_2 x^3 - \eta_3 x$. Thus, around $3 = 2^2 - 1$ centers on the y-axis of $(PH_3)_{\varepsilon=0}$, at least $9 = 3 \times 3$ limit cycles are created.



(1) A cubic system



(2) A system of degree 7



(3) A system of degree 15

Fig.8.4.2 Copies of a Z_4-equivariant polynomial vector fields

Let $P_3(x, y) = P_3^{(1)}(x, y) + P_3^{(2)}(x)$, then the system (PH_3) has at least $S_3 = 4 \times 3 + 3 \times 3 = 21$ limit cycles.

We next consider the map:$(x, y) \to (x^2 - 2, y^2 - 2)$. By Lemma 8.4.2, under this map, the unperturbed system $(PH_4)_{\varepsilon=0}$ has the phase portrait shown in figure (3) in Fig.8.4.2. The same two-step method as the above shows that the system (PH_4) has at least $S_4 = 4 \times 21 + 7 \times 7 = 133$ limit cycles.

By using inductive method for the system (PH_k), first, taking the map: $(x, y) \to (x^2 - 2^{k-2}, y^2 - 2^{k-2})$, we have the perturbed terms $P^{(1)}_{k+1}(x, y) = yP_k(x^2 - 2^{k-2}, y^2 - 2^{k-2})$. Second, by using Lemma 8.4.1 to perturb the $2^k - 1$ centers on the y-axis of $(PH_{k+1})_{\varepsilon=0}$, we obtain the perturbed terms $P^{(2)}_{k+1}(x)$ as (8.4.2). It gives $2^k - 1$ limit cycles. Hence, by using $P_{k+1}(x, y) = P^{(1)}_{k+1}(x, y) + P^{(2)}_{k+1}(x, y)$ as the perturbation for (PH_{k+1}), we have

$$S_{k+1} = 4 \times S_k + (2^k - 1)^2.$$

Let $S_k = 4^k \sigma_k$. Then

$$\begin{aligned} \sigma_{k+1} &= \sigma_k + \frac{1}{4} - \frac{1}{2^{k+1}} + \frac{1}{4^{k+1}}. \\ \sigma_k &= \sigma_{k-1} + \frac{1}{4} - \frac{1}{2^k} + \frac{1}{4^k} \\ &= \sigma_2 + \frac{1}{4}(k-2) - \left(\frac{1}{2^3} + \cdots + \frac{1}{2^k}\right) + \left(\frac{1}{4^3} + \cdots + \frac{1}{4^k}\right) \\ &= \sigma_2 + \frac{1}{4}k - \frac{35}{48} + \frac{1}{2^k} - \frac{1}{3 \times 4^k}. \end{aligned} \tag{8.4.8}$$

Note that $\sigma_2 = \dfrac{3}{16}$. Thus,

$$S_k = 4^{k-1}\left(k - \frac{13}{6}\right) + 2^k - \frac{1}{3}. \tag{8.4.9}$$

Remark 8.4.1. *It was stated in Ref.* [*Christopher and Lloyd,* 1995] (p223) *that "We take $R(x, y)$ to be of the form $yR_1(x) + R_2(y)$, $\cdots$. We then construct $R_2(y)$, $\cdots$, to be a polynomial of degree $2^{k+1} - 1$ so that $2^k - 1$ limit cycles appear near each of the $2^k - 1$ centers on the x-axis." The last conclusion is incorrect! Because $\dfrac{\partial}{\partial x}R_2(y) \equiv 0$, under the perturbed terms given in* [*Christopher and Lloyd,* 1995] *(for which $R_2(x, y) \equiv 0$ in* (8.4.1)*), it has no contribution to the divergence of the vector field. Therefore, the term $R_2(y)$ cannot create any limit cycle from the centers on the x-axis.*

Write that $n = 2^k - 1$. Then $k = \log_2(n + 1) = \dfrac{\ln(n + 1)}{\ln 2}$. From (8.4.9), we obtain

Proposition 8.4.1. *By using the Z_4-equivariant systems (PH_k) to create limit cycles, where $Q_k(x, y) = 0$ and (PH_2) is* (8.4.7)*, we have*

$$H(n) \geqslant \frac{1}{4}(n + 1)^2\left[\frac{\ln(n + 1)}{\ln 2} - \frac{13}{6}\right] + n + \frac{2}{3}. \tag{8.4.10}$$

This proposition is the correction of Theorem 3.4 of Ref. [Christopher and Lloyd, 1995].

8.4.3 A New Lower Bound for $H(2^k - 1)$

In this subsection, we consider the perturbed Z_2-equivariant vector field (see [Li Jibin etc, 1987]):

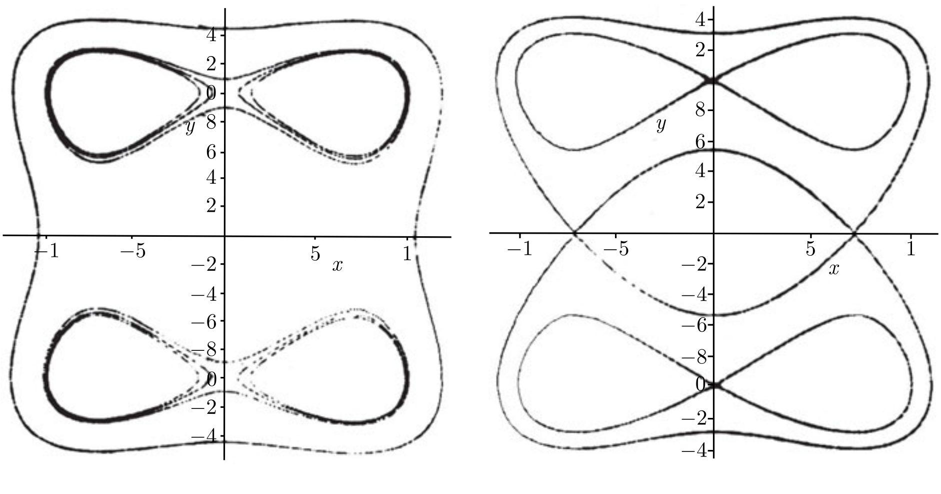
$$\frac{dx}{dt} = y(1-y^2) + \varepsilon x(y^2 - x^2 - \lambda),$$
$$\frac{dy}{dt} = -x(1-2x^2) + \varepsilon y(y^2 - x^2 - \lambda), \tag{8.4.11}$$

where $0 < \varepsilon \ll 1$. The system $(8.4.11)_{\varepsilon=0}$ has Hamiltonian

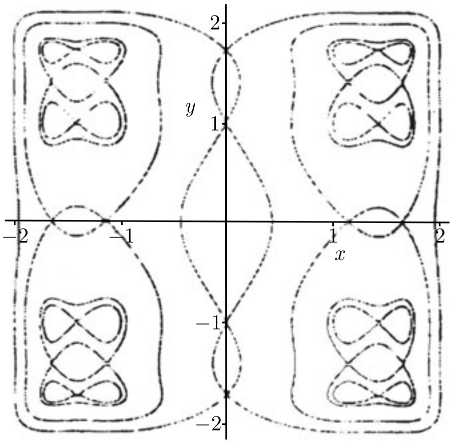
$$H_2(x, y) = -2x^4 - y^4 + 2(x^2 + y^2). \tag{8.4.12}$$

There exist 9 finite singular points of $(8.4.11)_{\varepsilon=0}$ which are the intersection points of the straight lines $x = 0, x = \pm\dfrac{1}{\sqrt{2}}$ and $y = 0, y = \pm 1$. The phase portrait of $(8.4.11)_{\varepsilon=0}$ is shown in figure (2) of Fig.8.4.3.

Let (8.4.10) be the system (PH_2) and suppose that $-4.80305 + O(\varepsilon) < \lambda < -4.79418 + O(\varepsilon)$. We know from [Li Jibin etc, 1987] that the system (PH_2) has at least 11 limit cycles having the configuration shown in figure (1) of Fig. 8.4.3. By taking the map: $(x, y) \to (x^2 - 3, y^2 - 3)$, the new unperturbed system $(PH_3)_{\varepsilon=0}$



(1) 11 limit cycles (2) The system $(8.4.11)_{\varepsilon=0}$



(3) A system of degree 7

Fig.8.4.3 Four copies of system (8.4.11)

has 49 finite singular points which are intersection points of the straight lines $x = 0, x = \pm\sqrt{3 \pm \dfrac{1}{\sqrt{2}}}, x = \pm\sqrt{3}$ and $y = 0, y = \pm\sqrt{2}, y = \pm\sqrt{3}, y = \pm 2$. The phase portrait of $(PH_3)_{\varepsilon=0}$ is shown in figure (3) of Fig. 8.4.3.

The first perturbed terms of (PH_3) have the forms:

$$P_3^{(1)}(x, y) = y(x^2 - 3)\left[(y^2 - 3)^2 - (x^2 - 3)^2 - \lambda\right],$$
$$Q_3^{(1)}(x, y) = x(y^2 - 3)\left[(y^2 - 3)^2 - (x^2 - 3)^2 - \lambda\right].$$

These are polynomials of degree 7. Hence, firstly, we have from Lemma 8.4.2 that there exist $4 \times 11 = 44$ limit cycles of (PH_3) under the first perturbations $P_3^{(1)}$ and $Q_3^{(1)}$. By Lemma 8.4.3, the above perturbations do not create limit cycle around the centers on the y-axis. Thus, secondly, we use Lemma 8.4.1 to add new perturbation terms $P_3^{(2)}$ and $Q_3^{(2)} = 0$ such that 3×3 limit cycles appear around the $3 = 2^2 - 1$ centers of $(PH_3)_{\varepsilon=0}$ on the y-axis. To sum up, two sets of perturbations give rise to $S_3 = 4 \times 11 + 3 \times 3 = 53$ limit cycles of (PH_3).

By using inductive method, similar to that in §8.4.2, from the "quadruple transformation"

$$(x, y) \to (x^2 - 3^{k-1}, y^2 - 3^{k-1})$$

and the bifurcations of small amplitude limit cycles around the centers on the y-axis for the system $(PH_{k+1}), k = 3, 4, \cdots$, we have

$$S_{k+1} = 4 \times S_k + (2^k - 1)^2 \tag{8.4.13}$$

limit cycles. Note that $S_2 = 11$. Thus we obtain from (8.4.12) and (8.4.8) that

$$S_k = 4^{k-1}\left(k - \frac{1}{6}\right) + 2^k - \frac{1}{3}. \tag{8.4.14}$$

Proposition 8.4.2. *By using the Z_2-equivariant systems (PH_k) to yield limit cycles, where (PH_2) is* (8.4.11), *we have*

$$H(n) \geqslant \frac{1}{4}(n+1)^2\left[\frac{\ln(n+1)}{\ln 2} - \frac{1}{6}\right] + n + \frac{2}{3}. \tag{8.4.15}$$

8.4.4 Lower Bound for $H(3 \times 2^{k-1} - 1)$

In this subsection, we consider the perturbed Z_2-equivariant vector field of degree 5 (see [Li Jibin etc, 2001]):

$$\begin{aligned}\frac{dx}{dt} &= -y(1 - by^2 + y^4)\\ &\quad -\varepsilon x(px^4 + qy^4 + gx^2y^2 + mx^2 + ny^2 - \lambda),\\ \frac{dy}{dt} &= x(1 - ax^2 + x^4)\\ &\quad -\varepsilon y(px^4 + qy^4 + gx^2y^2 + mx^2 + ny^2 - \lambda).\end{aligned} \tag{8.4.16}$$

or its polar coordinate form:

$$\begin{aligned}\frac{dr}{dt} &= \frac{1}{4}\sin 2\theta\big[(b-a) - (b+a)\cos 2\theta + 2r^2\cos 2\theta\big]r^3\\ &\quad -\varepsilon r^5(p\cos^4\theta + q\sin^4\theta + g\cos^2\theta\sin^2\theta)\\ &\quad -\varepsilon r[r^2(m\cos^2\theta + n\sin^2\theta) - \lambda],\\ \frac{d\theta}{dt} &= 1 - \frac{1}{8}\big[3(a+b) + 4(a-b)\cos 2\theta + (a+b)\cos 4\theta\big]r^2\\ &\quad + \frac{1}{8}(5 + 3\cos 4\theta)r^4,\end{aligned} \tag{8.4.17}$$

where $a > b > 2$. $(8.4.16)_{\varepsilon=0}$ and $(8.4.17)_{\varepsilon=0}$ have the Hamiltonian functions as follows:

$$H(x,y) = -\frac{1}{2}(x^2 + y^2) + \frac{1}{4}(ax^4 + by^4) - \frac{1}{6}(x^6 + y^6), \tag{8.4.18}$$

$$\begin{aligned}H_1(r,\theta) &= -\frac{1}{2}r^2 + \frac{1}{32}(3(a+b) + 4(a-b)\cos 2\theta\\ &\quad + (a+b)\cos 4\theta)r^4 - \frac{1}{48}(5 + 3\cos 4\theta)r^6.\end{aligned} \tag{8.4.19}$$

Denote that

$$\xi_1 = \sqrt{\frac{a - \sqrt{a^2-4}}{2}}, \quad \xi_2 = \sqrt{\frac{a + \sqrt{a^2-4}}{2}},$$
$$\eta_1 = \sqrt{\frac{b - \sqrt{b^2-4}}{2}}, \quad \eta_2 = \sqrt{\frac{b + \sqrt{b^2-4}}{2}}.$$

It is easy to see that the system (8.4.16) has 13 centers at

$$(0,0), (\xi_1,\eta_1), (\xi_1,-\eta_1), (\xi_2,0), (\xi_2,\eta_2), (\xi_2,-\eta_2), (0,\eta_2)$$

and their Z_2-equivariant symmetric points, 12 saddle points at

$$(0,\eta_1), (\xi_1,0), (\xi_1,\eta_2), (\xi_1,-\eta_2), (\xi_2,\eta_1), (\xi_2,-\eta_1)$$

and their Z_2-equivariant symmetric points. We have from (8.4.18) that

$$\begin{aligned}
h_0^c =& H(0,0) = 0,\\
h_1^c =& H(\xi_1,\eta_1) = H(\xi_1,-\eta_1)\\
=& -\frac{1}{24}\left[6(a+b) - (a^3+b^3) + (a^2-4)^{\frac{3}{2}} + (b^2-4)^{\frac{3}{2}}\right],\\
h_2^c =& H(\xi_2,0) = -\frac{1}{24}\left[6a - a^3 - (a^2-4)^{\frac{3}{2}}\right],\\
h_3^c =& H(0,\eta_2) = H(0,-\eta_2) = -\frac{1}{24}\left[6b - b^3 - (b^2-4)^{\frac{3}{2}}\right],\\
h_4^c =& H(\xi_2,\eta_2) = H(\xi_2,-\eta_2)\\
=& -\frac{1}{24}\left[6(a+b) - (a^3+b^3) - (a^2-4)^{\frac{3}{2}} - (b^2-4)^{\frac{3}{2}}\right];
\end{aligned}$$

and

$$\begin{aligned}
h_1^s &= H(\xi_1,0) = -\frac{1}{24}\left[6a - a^3 + (a^2-4)^{\frac{3}{2}}\right],\\
h_2^s &= H(0,\eta_1) = -\frac{1}{24}\left[6b - b^3 + (b^2-4)^{\frac{3}{2}}\right],\\
h_3^s &= H(\xi_1,\eta_2) = -\frac{1}{24}\left[6(a+b) - (a^3+b^3) + (a^2-4)^{\frac{3}{2}} - (b^2-4)^{\frac{3}{2}}\right],\\
h_4^s &= H(\xi_2,\eta_1) = -\frac{1}{24}\left[6(a+b) - (a^3+b^3) - (a^2-4)^{\frac{3}{2}} + (b^2-4)^{\frac{3}{2}}\right].
\end{aligned}$$

Suppose that $(a,b) = (2.5, 2.3)$. We have

$$\xi_1 = 0.7071067812, \quad \xi_2 = 1.415213562,$$
$$\eta_1 = 0.762960789, \quad \eta_2 = 1.310683347,$$

and

$$\begin{aligned}
&h_1^c = -0.2436732647, \quad h_2^s = -0.1290899314,\\
&h_3^s = -0.1215767351, \quad h_1^s = -0.1145833333,\\
&h_3^c = -0.0069934018, \quad h_4^s = 0.03757673603,\\
&h_4^c = 0.1596732652, \quad h_2^c = 0.16666666667,
\end{aligned}$$

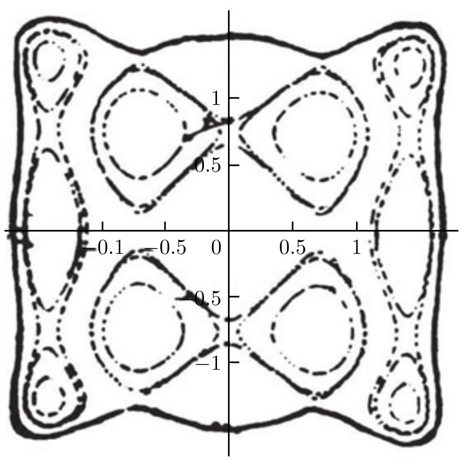
$$-\infty < h_1^c < h_2^s < h_3^s < h_1^s < h_3^c < 0 < h_4^s < h_4^c < h_2^c.$$

The unperturbed system $(8.4.16)_{\varepsilon=0}$ has the phase portrait of figure (2) in Fig.8.4.4. In [Li Jibin etc, 2001], we showed that when

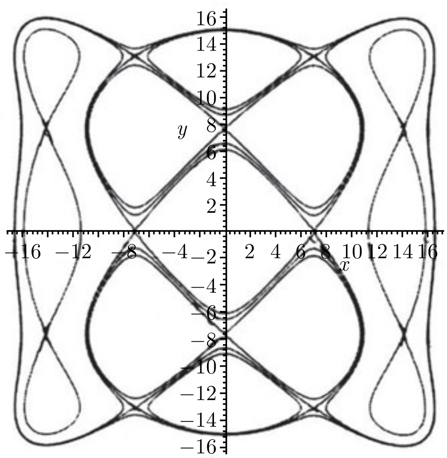
$$\begin{aligned}
&(p, q, g, m, n)\\
=&(-.144543,\ 1.157350656,\ -3.328234861,\ 3.014502,\ 6.564525872),\\
&\lambda \in \Big(\lambda_1(h_2^s),\ \min\big(\max(\lambda_1(h),\ \ \max(\lambda_6(h))\big)\Big)\\
\approx&(9.319050412,\ \ 9.319051762),
\end{aligned}$$

the system (8.4.16) has at least 23 limit cycles having the configuration shown in figure (1) of Fig.8.4.4.

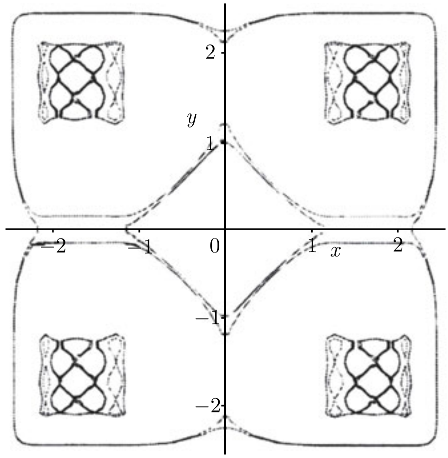
We now take (8.4.16) as (PH_2). Under the map $(x, y) \to (x^2 - 3, y^2 - 3)$, the new system $(PH_3)_{\varepsilon=0}$ has the phase portrait shown in figure (3) of Fig.8.4.4. There exist 121 finite singular points of $(PH_3)_{\varepsilon=0}$ consisting of the intersection points of the straight lines $x = 0, x = \pm\sqrt{3 \pm \xi_i}, x = \sqrt{3}$ and $y = 0, y = \pm\sqrt{3 \pm \eta_i}, y = \sqrt{3}, i = 1, 2$. There are $5 = 3 \times 2^{2-1} - 1$ centers on the y-axis.



(1) 23 limit cycles



(2) The system $(8.4.16)_{\varepsilon=0}$



(3) A system of degree 11

Fig.8.4.4 Four copies of $(8.4.16)_{\varepsilon=0}$

By Lemma 8.4.2, the perturbed terms

$$\begin{aligned}P_3^{(1)}(x,y) =&y(x^2-3)\left[p(x^2-3)^4+q(y^2-3)^4\right.\\&\left.+g(x^2-3)^2(y^2-3)^2+m(x^2-3)^2+n(y^2-3)^2-\lambda\right],\\Q_3^{(1)}(x,y) =&x(y^2-3)\left[p(x^2-3)^4+q(y^2-3)^4\right.\\&\left.+g(x^2-3)^2(y^2-3)^2+m(x^2-3)^2+n(y^2-3)^2-\lambda\right]\end{aligned}$$

quadruple the number of limit cycles of (PH_2), i.e., there exist $92 = 4\times 23$ limit cycles of (PH_3). Next, by using Lemma 8.4.1 to perform secondary perturbation for 5 centers on the y-axis of $(PH_3)_{\varepsilon=0}$, we have $P_3^{(2)} = \eta_0x^{11}-\eta_1x^9+\cdots+\eta_4x^3-\eta_5x, Q_3^{(2)}(x,y)=0$. It give rise to $5^2=(3\times 2^{2-1}-1)^2=25$ limit cycles. Thus, the system (PH_3) has at least 92+25=117 limit cycles.

Again by using inductive method, suppose that the system (PH_k) has S_k limit cycles. First, transform the system (PH_k) by the quadruple map: $(x,y)\to(x^2-3^{k-1}, y^2-3^{k-1})$. Then perform secondary perturbation to the centers on the y-axis of $(PH_{k+1})_{\varepsilon=0}$. We have

$$S_{k+1} = 4\times S_k + (3\times 2^{k-1}-1)^2. \tag{8.4.20}$$

Also let $S_k = 4^k\sigma_k$. Similar to the computation of (8.4.1), we have

$$\begin{aligned}\sigma_k =&\sigma_2+\frac{9}{16}(k-2)-\frac{3}{2}\left(\frac{1}{2^3}+\cdots+\frac{1}{2^k}\right)+\left(\frac{1}{4^3}+\cdots+\frac{1}{4^k}\right)\\=&\sigma_2+\frac{9}{16}k-\frac{71}{48}+\frac{3}{2^{k+1}}-\frac{1}{3\times 4^k}.\end{aligned}\tag{8.4.21}$$

Notice that $\sigma_2 = \dfrac{23}{16}$. Thus,

$$S_k = 4^{k-1}\left(\frac{9}{4}k - \frac{1}{6}\right) + 3 \times 2^{k-1} - \frac{1}{3}. \tag{8.4.22}$$

Let $n = 3 \times 2^{k-1} - 1$. Then, $k - 1 = \log_2\left(\dfrac{n+1}{3}\right) = \dfrac{\ln(n+1) - \ln 3}{\ln 2}$. We have from (8.4.22) that

Proposition 8.4.3. *By considering the Z_2-equivariant systems (PH_k) to yield limit cycles, where (PH_2) is (8.4.16), we have*

$$H(n) \geqslant \frac{1}{4}(n+1)^2\left[\frac{\ln(n+1) - \ln 3}{\ln 2} + \frac{25}{27}\right] + n + \frac{2}{3}.$$

Denote that $\mu = \dfrac{1}{4\ln 2} \approx 0.360673$. To sum up, the Proposition 8.4.1 $\sim$ Proposition 8.4.3 imply that

Theorem 8.4.1. *There are two sequences of $n = 2^k - 1$ and $n = 3 \times 2^{k-1} - 1, k = 2, 3, \cdots$, and a constant $\mu = (4 \times \ln 2)^{-1}$ such that the number $H(n)$ of limit cycles of the systems (PH_k) grows at least as rapidly as $\mu(n+1)^2 \ln(n+1)$.*

Bibliographical Notes

Up to now, we know that a given system (E_n) always has a finite number of limit cycles [Ilyashenko, 1991] and that $H(2) \geqslant 4$, $H(3) \geqslant 12$, $H(5) \geqslant 24$, $H(7) \geqslant 50$, $H(9) \geqslant 80$(see [Shi Songling, 1980; Chen Lansun and Wan Mingshu, 1979; Chan,H. et al, 2001; Li Jibin et al, 1987; Li Jibin et al, 2001; Yu Pei and Han Maoan,2005; Liu Yingrong and Huang Wentao,2005; Lloyd, 1988; Luo Dingjun et al, 1997; Perko, 1991; Ye Yanqian, 1995]). Also by considering a small neighborhood of a singular point, $H(n) \geqslant \dfrac{1}{2}[n^2 + 5n - 20 - 6(-1)^n]$ for $n \geqslant 6$ [Otrokov, 1954]. [Christopher and Lloyd,1995] showed that $H(2^k - 1) \geqslant 4^{k-1}\left(2k - \dfrac{35}{6}\right) + 3.2^k - \dfrac{5}{3}$ (for example $H(7) \geqslant 25$) by perturbing some families of closed orbits of a Hamiltonian system sequence in small neighborhoods of some center points and using a "quadruple transformation". The method given by them is very interesting. Unfortunately, the computation of a lower bound is not correct (see Remark 8.4.1). Therefore, we need to correct and develop their work.

Chapter 9

Center-Focus Problem and Bifurcations of Limit Cycles for a Z_2-Equivariant Cubic System

In this chapter, we study the Z_2-equivariant cubic system which is represented by $(E_3^{Z_2})$. We first solve completely the problem of center-focus for this class of systems. Then, considering the bifurcation of limit cycle created from infinity, we show that a cubic system has at least 13 limit cycles.

9.1 Standard Form of a Class of System $(E_3^{Z_2})$

In this section, we consider the following system $(E_3^{Z_2})$ having at least two finite elementary focuses

$$\begin{aligned}\frac{dx}{dt} &= X_1(x,y) + X_3(x,y) = X(x,y),\\ \frac{dy}{dt} &= Y_1(x,y) + Y_3(x,y) = Y(x,y),\end{aligned} \tag{9.1.1}$$

where $X_k(x,y),\ Y_k(x,y)$ are homogeneous real polynomials of order k with respect to x and y, $k = 1, 3$.

Suppose that system (9.1.1) has at least two finite elementary focuses. We assume that two of them are at the points $(\pm 1, 0)$ (otherwise, we make proper linear transformation).

Definition 9.1.1. *If the functions of the right hand of system* (9.1.1) *satisfy*

$$\begin{aligned}&X(1,0) = Y(1,0) = 0,\\ &\frac{\partial X(1,0)}{\partial x} = \frac{\partial Y(1,0)}{\partial y} = \delta,\\ &\frac{\partial X(1,0)}{\partial y} = -1, \quad \frac{\partial Y(1,0)}{\partial x} = 1,\end{aligned} \tag{9.1.2}$$

then we say that system (9.1.1) *has the standard form.*

Obviously, when system (9.1.1) has the standard form, the linearized systems of (9.1.1) at the point $(\pm 1, 0)$ have the forms:

$$\frac{dx}{dt} = \delta(x \mp 1) - y, \quad \frac{dy}{dt} = (x \mp 1) + \delta y. \tag{9.1.3}$$

Lemma 9.1.1. *If system* (9.1.1) *has at least two finite elementary focuses, then by using suitable linear transformation, it makes* (9.1.1) *become the standard form.*

Proof. Without lose of generality, suppose that system (9.1.1) has two elementary focuses at the points $(\pm 1, 0)$. We have

$$X(1,0) = Y(1,0) = 0, \quad \frac{X(1,0)}{\partial y}\frac{\partial Y(1,0)}{\partial x} \neq 0. \tag{9.1.4}$$

Write that

$$\begin{aligned} p &= \frac{\partial X(1,0)}{\partial x} + \frac{\partial Y(1,0)}{\partial y}, \\ q &= \frac{\partial X(1,0)}{\partial x}\frac{\partial Y(1,0)}{\partial y} - \frac{\partial X(1,0)}{\partial y}\frac{\partial Y(1,0)}{\partial x} \end{aligned} \tag{9.1.5}$$

and

$$p^2 - 4q = -4s^2, \quad p = 2s\delta, \quad s > 0. \tag{9.1.6}$$

Then, by the transformation

$$\begin{aligned} &\frac{\partial Y(1,0)}{\partial x}\,\tilde{x} = \frac{\partial Y(1,0)}{\partial x}\,x + \left(\frac{\partial Y(1,0)}{\partial y} - \delta s\right) y, \\ &\frac{\partial Y(1,0)}{\partial x}\,\tilde{y} = sy, \quad \tilde{t} = st, \end{aligned} \tag{9.1.7}$$

system (9.1.1) becomes the standard form. □

On the basis of (9.1.2) and Lemma 9.1.1, we have

Theorem 9.1.1. *If system* (9.1.1) *has at least two finite elementary focuses, by a proper linear transformation,* (9.1.1) *can be reduced to the following standard form*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\delta}{2}x - (a_1 + 1)y + \frac{\delta}{2}x^3 + a_1x^2y + a_2xy^2 + a_3y^3, \\ \frac{dy}{dt} &= -\frac{1}{2}x + (\delta - a_4)y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3. \end{aligned} \tag{9.1.8}$$

Obviously, $(\pm 1, 0)$ are elementary focuses of system (9.1.8) and the linearized systems at the points $(\pm 1, 0)$ are (9.1.3). Letting

$$x = \pm(u + 1), \quad y = \pm v, \quad t = \pm\tau, \tag{9.1.9}$$

system (9.1.8) becomes

$$\begin{aligned}\frac{du}{d\tau}&=\delta u-v+\frac{3\delta}{2}u^2+2a_1uv+a_2v^2+\frac{\delta}{2}u^3+a_1u^2v+a_2uv^2+a_3v^3,\\ \frac{dv}{d\tau}&=u+\delta v+\frac{3}{2}u^2+2a_4uv+a_5v^2+\frac{1}{2}u^3+a_4u^2v+a_5uv^2+a_6v^3.\end{aligned}\quad(9.1.10)$$

9.2 Liapunov Constants, Invariant Integrals and the Necessary and Sufficient Conditions of the Existence for the Bi-Center

When $\delta=0$, system (9.1.8) becomes

$$\begin{aligned}\frac{dx}{dt}&=-(a_1+1)y+a_1x^2y+a_2xy^2+a_3y^3,\\ \frac{dy}{dt}&=-\frac{1}{2}x-a_4y+\frac{1}{2}x^3+a_4x^2y+a_5xy^2+a_6y^3\end{aligned}\qquad(9.2.1)$$

and system(9.1.10) becomes

$$\begin{aligned}\frac{du}{d\tau}&=-v+a_1(2+u)uv+a_2(1+u)v^2+a_3v^3,\\ \frac{dv}{d\tau}&=u+\frac{3}{2}u^2+\frac{1}{2}u^3+a_4(2+u)uv+a_5(1+u)v^2+a_6v^3.\end{aligned}\qquad(9.2.2)$$

We denote that

$$\begin{aligned}A_1&=2(1+a_5)a_4+(2+a_1+a_5)a_2,\\ A_2&=2(1+a_1)(1+a_5)-a_3,\\ A_3&=3a_4+(a_1+a_5)(5a_2+4a_4),\\ A_4&=6(a_1+a_5)(1+a_1)+(5a_2-2a_4)a_4.\end{aligned}\qquad(9.2.3)$$

By using the transformation $z=u+iv,\ w=u-iv,\ T=it$, system (9.2.2) can become its associated system. Applying the formula in Theorem 2.3.6, we obtain the first Liapunov constants at $(\pm1,0)$ of (9.2.1). We find that there exist 6 terms in V_3 and when $V_3=0$, there are 25, 118, 350, 831, 1717 terms in $V_5,\ V_7,\ V_9\ V_{11}\ V_{13}$, respectively. By some tricks of the simplification, we know that

Theorem 9.2.1. *The first 6 Liapunov constants of* (9.2.1) *at the singular points* $(\pm1,0)$ *are as follows*:

$$\begin{aligned}V_3&=\frac{1}{4}(a_2+2a_1a_2-2a_4+2a_2a_5-2a_4a_5+3a_6),\\ V_5&\sim\frac{1}{36}(-3A_2A_3+2A_1A_4),\end{aligned}$$

$$\begin{aligned}
V_7 &\sim \frac{1}{864} h_0 h_3,\\
V_9 &\sim \frac{1}{45360} h_0 h_4,\\
V_{11} &\sim \frac{1}{698544} h_0 h_5,\\
V_{13} &\sim \frac{-44}{85562001} a_2(a_2-a_4)(2a_2-a_4)(4a_2-a_4)\\
&\quad \times(5a_2-a_4)(2a_2+a_4)(4a_2+a_4)h_0h_6,
\end{aligned} \tag{9.2.4}$$

where

$$\begin{aligned}
h_0 &= 3(a_1+a_5)A_2 - 2a_4A_1,\\
h_3 &= 30a_2 - 54a_1a_2 - 84a_1^2a_2 + 70a_2^3 - 105a_2a_3 + 36a_4 + 36a_1a_4\\
&\quad -70a_2^2a_4 - 84a_3a_4 - 52a_2a_4^2 + 16a_4^3 + 126a_2a_5 + 126a_1a_2a_5,\\
h_4 &= 108(1+a_1)^2(5a_2-8a_4)\\
&\quad -12(1+a_1)(525a_2^3 - 420a_2^2a_4 - 110a_2a_4^2 + 92a_4^3)\\
&\quad +5(539a_2^5 - 560a_2^3a_4^2 + 448a_2^2a_4^3 + 96a_2a_4^4 - 64a_4^5),\\
h_5 &= \big[-4(1+a_1)^2(128a_2 - 97a_4)\\
&\quad +4(1+a_1)(3136a_2^3 - 2401a_2^2a_4 + 440a_2a_4^2 + 94a_4^3)\\
&\quad -a_4(7007a_2^4 - 784a_2^3a_4 + 2352a_2^2a_4^2 - 1256a_2a_4^3 + 16a_4^4)\big]\\
&\quad \times(9+9a_1+4a_4^2),\\
h_6 &= 9038315a_2^2 + 4146497a_2a_4 + 191510a_4^2.
\end{aligned} \tag{9.2.5}$$

Lemma 9.2.1. *The resultant of h_4, h_5 with respect to a_1 is*

$$\begin{aligned}
R(h_4,h_5,a_1) &= a_2^5(a_2-a_4)^3(2a_2-a_4)(4a_2-a_4)^2\\
&\quad \times(5a_2-a_4)(2a_2+a_4)(4a_2+a_4)\Delta(a_2,a_4),
\end{aligned} \tag{9.2.6}$$

where

$$\Delta(a_2,a_4) = 20a_2^3 - 35a_2^2a_4 - 20a_2a_4^2 - a_4^3. \tag{9.2.7}$$

This lemma tell us that if the first six Liapunov constants at $(\pm1,0)$ of (9.2.1) are all zeros, then, the common factor between V_{13} and $R(h_4,h_5,a_1)$ is zero. Thus we have

Lemma 9.2.2. *If the first six Liapunov constants at the singular points* $(\pm1,0)$ *of* (9.2.1) *are all zeros, then,*

$$a_2(a_2-a_4)(2a_2-a_4)(4a_2-a_4)(5a_2-a_4)(2a_2+a_4)(4a_2+a_4)h_0 = 0. \tag{9.2.8}$$

Theorem 9.2.2. *The first six Liapunov constants of* (9.2.1) *at the singular points* $(\pm 1, 0)$ *are all zero, if and only if one of the following conditions is satisfied*:

$$(C_1):\ a_4 = 0,\ a_1 = -a_5,\ a_6 = -\frac{1}{3}a_2;$$

$$(C_2):\ a_4 = 0,\ a_1 + a_5 \neq 0,\ a_2 = a_6 = 0;$$

$$(C_3):\ h_0 = 0,\ a_1 + a_5 \neq 0,$$
$$a_6 = \frac{1}{3}(-a_2 - 2a_1a_2 + 2a_4 - 2a_2a_5 + 2a_4a_5),$$
$$2(1+a_1)(a_1+a_5)^2 - a_4^2(1+2a_1+2a_5) = 0;$$

$$(C_4):\ 2(1+a_5)a_4 + (2+a_1+a_5)a_2 = 0,$$
$$a_3 = 2(1+a_1)(1+a_5),$$
$$a_6 = \frac{1}{3}(-a_2 - 2a_1a_2 + 2a_4 - 2a_2a_5 + 2a_4a_5);$$

$$(C_5):\ a_4 \neq 0,\ a_1 = \frac{1}{2}(-2+3a_4^2),\ a_2 = a_4,$$
$$a_3 = a_4^2(1 - a_4^2 + a_5),\ a_6 = a_4(1 - a_4^2);$$

$$(C_6):\ a_4 \neq 0,\ a_1 = \frac{1}{8}(-8+5a_4^2),\ a_2 = \frac{1}{2}a_4,$$
$$a_5 = -\frac{1}{8}(8+a_4^2),\ a_3 = -\frac{5}{32}a_4^4,\ a_6 = \frac{1}{4}a_4(2-a_4^2);$$

$$(C_7):\ a_4 \neq 0,\ a_1 = -\frac{1}{32}(32+15a_4^2),$$
$$a_2 = \frac{1}{4}a_4, a_3 = \frac{1}{512}a_4^2(64+15a_4^2),$$
$$a_5 = -\frac{1}{32}(96+17a_4^2), a_6 = -\frac{3}{16}a_4(4+a_4^2);$$

$$(C_8):\ a_4 \neq 0,\ a_1 = -\frac{1}{50}(50+21a_4^2),$$
$$a_2 = \frac{1}{5}a_4, a_3 = \frac{1}{1250}a_4^2(250+63a_4^2),$$
$$a_5 = -\frac{1}{50}(200+39a_4^2), a_6 = -\frac{1}{25}a_4(35+9a_4^2);$$

$$(C_9):\ a_4 \neq 0,\ a_1 = -\frac{1}{9}(9+4a_4^2),$$
$$a_2 = 0,\ a_3 = 0,\ a_6 = \frac{2}{3}a_4(1+a_5);$$

$$(C_{10}):\ a_4 \neq 0,\ a_1 = -\frac{1}{8}(8+3a_4^2),\ a_2 = -\frac{1}{2}a_4,$$
$$a_3 = \frac{3}{16}a_4^2(4+a_4^2+4a_5),\ a_6 = \frac{1}{8}a_4(4-a_4^2+8a_5);$$

$$(C_{11}):\ a_4 \neq 0,\ a_1 = -\frac{1}{32}(32+15a_4^2),$$

$$a_2 = -\frac{1}{4}a_4, a_3 = \frac{1}{512}a_4^2(832 + 495a_4^2),$$
$$a_5 = \frac{1}{32}(160 + 111a_4^2), a_6 = \frac{1}{16}a_4(76 + 45a_4^2).$$

Proof. We first prove the sufficiency.

Substituting each condition (C_j) $(j = 1, 2, \cdots, 11)$ into the above formulas of V_3, $V_5, \cdots, V_{13}$, respectively, it follows that $V_3 = V_5 = \cdots = V_{13} = 0$. Thus, the sufficiency of this theorem holds.

We next prove that every condition (C_j) is the necessary condition such that $V_3 = V_5 = \cdots = V_{13} = 0$. By Lemma 9.2.2, we need to consider the following four cases.

(1) If $a_4 = 0$ and $h_0 \neq 0$ then Lemma 9.2.2 implies that $a_2 = 0$. Hence, the relationship

$$\begin{aligned} &4V_3 = 3a_6 = 0, \\ &h_0 = 3(a_1 + a_5)(2 + 2a_1 - a_3 + 2a_5 + 2a_1a_5) \neq 0 \end{aligned} \tag{9.2.9}$$

follows the condition (C_2).

(2) If $h_0 \neq 0$ and $a_2(a_2 - a_4)(2a_2 + a_4) = 0$, by solving $V_3 = V_5 = \cdots = V_{13} = 0$, we have

$$\begin{aligned} &18a_1 = -18 + 25a_2^2 + 10a_2a_4 - 8a_4^2, \\ &12a_3 = -a_2(-20a_2 + 8a_4 + 5a_2a_4^2 + 7a_4^3 - 20a_2a_5 + 8a_4a_5), \\ &6a_6 = 2a_2 + 4a_4 - 5a_2^2a_4 - a_2a_4^2 - 4a_2a_5 + 4a_4a_5. \end{aligned} \tag{9.2.10}$$

Thus, when $a_2 = 0$, we obtain the condition (C_9). When $a_2 - a_4 = 0$, we have the condition (C_5). When $2a_2 + a_4 = 0$, it gives rise to the condition (C_{10}).

(3) If $h_0 \neq 0$ and $(2a_2 - a_4)(4a_2 - a_4)(5a_2 - a_4)(4a_2 + a_4) = 0$. By solving $V_3 = V_5 = \cdots = V_{13} = 0$, we have

$$\begin{aligned} 6a_4a_1 =& 160a_2^3 - 6a_4 - 45a_2^2a_4 - 10a_2a_4^2, \\ 18a_4^2a_3 =& a_2(1920a_2^3 - 1344a_2^2a_4 + 132a_2a_4^2 + 647a_2^3a_4^2 \\ &+30a_4^3 - 556a_2^2a_4^3 + 103a_2a_4^4 + a_4^5), \\ 6a_4^3a_5 =& -(960a_2^3 - 672a_2^2a_4 + 36a_2a_4^2 + 320a_2^3a_4^2 \\ &+36a_4^3 - 237a_2^2a_4^3 + 28a_2a_4^4 + 6a_4^5), \\ 9a_4^2a_6 =& -(960a_2^3 - 672a_2^2a_4 + 39a_2a_4^2 + 400a_2^3a_4^2 \\ &+24a_4^3 - 269a_2^2a_4^3 + 29a_2a_4^4 + 5a_4^5). \end{aligned} \tag{9.2.11}$$

Hence, when $2a_2 - a_4 = 0$, we have the condition (C_6). When $4a_2 - a_4 = 0$, we obtain the condition (C_7). When $5a_2 - a_4 = 0$, we have the condition (C_8). When $4a_2 + a_4 = 0$, we obtain the condition (C_{11}).

(4) If $h_0 = V_3 = V_5 = 0$, we consider the following three subcases. $A_1 = A_2 = 0$ imply that (C_4) holds. If $|A_1|+|A_2| \neq 0$ and $|a_4|+|a_1+a_5| \neq 0$, (C_3) holds. Finally, we get condition (C_1) when $|A_1| + |A_2| \neq 0$ and $a_4 = a_1 + a_5 = 0$. □

We now prove that the origin of (9.2.2) is a center when each condition $(C_1) \sim (C_{11})$ of Theorem 9.2.2 holds.

(1) Suppose that the condition (C_1) holds. Under this parameter condition, system (9.2.1) becomes

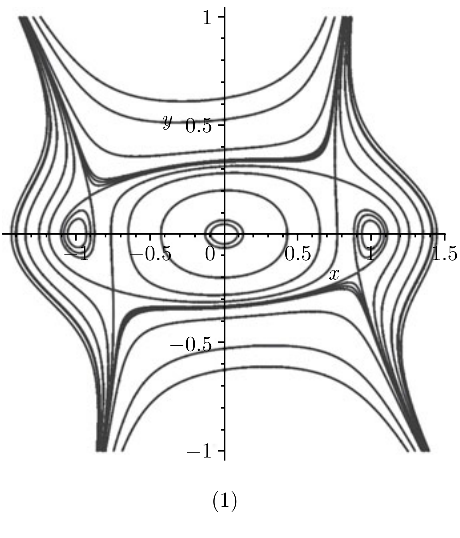
$$\begin{aligned}\frac{dx}{dt} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3,\\ \frac{dy}{dt} &= -\frac{1}{2}x + \frac{1}{2}x^3 - a_1xy^2 - \frac{1}{3}a_2y^3.\end{aligned} \tag{9.2.12}$$

It is easy to see that the following conclusion holds.

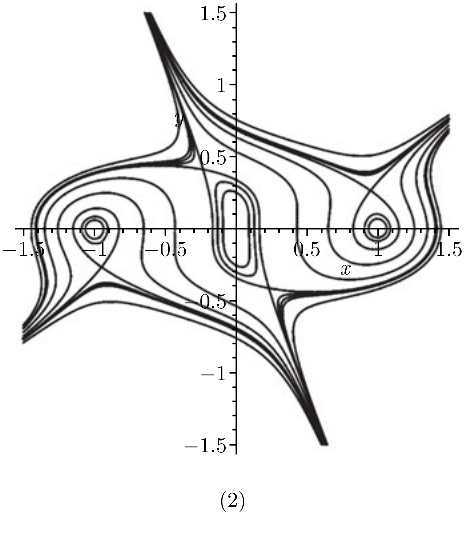
Proposition 9.2.1. *System* (9.2.12) *is a Hamiltonian system with the Hamiltonian quantity*

$$\begin{aligned}F_1(x,y) = &\frac{1}{4}[x^2 - 2(1+a_1)y^2]\\ &-\frac{1}{24}[3x^4 - 12a_1x^2y^2 - 8a_2xy^3 - 6a_3y^4].\end{aligned} \tag{9.2.13}$$

As an example, we use Fig.9.2.1 to show some phase portraits of system (9.2.13).



(1)



(2)

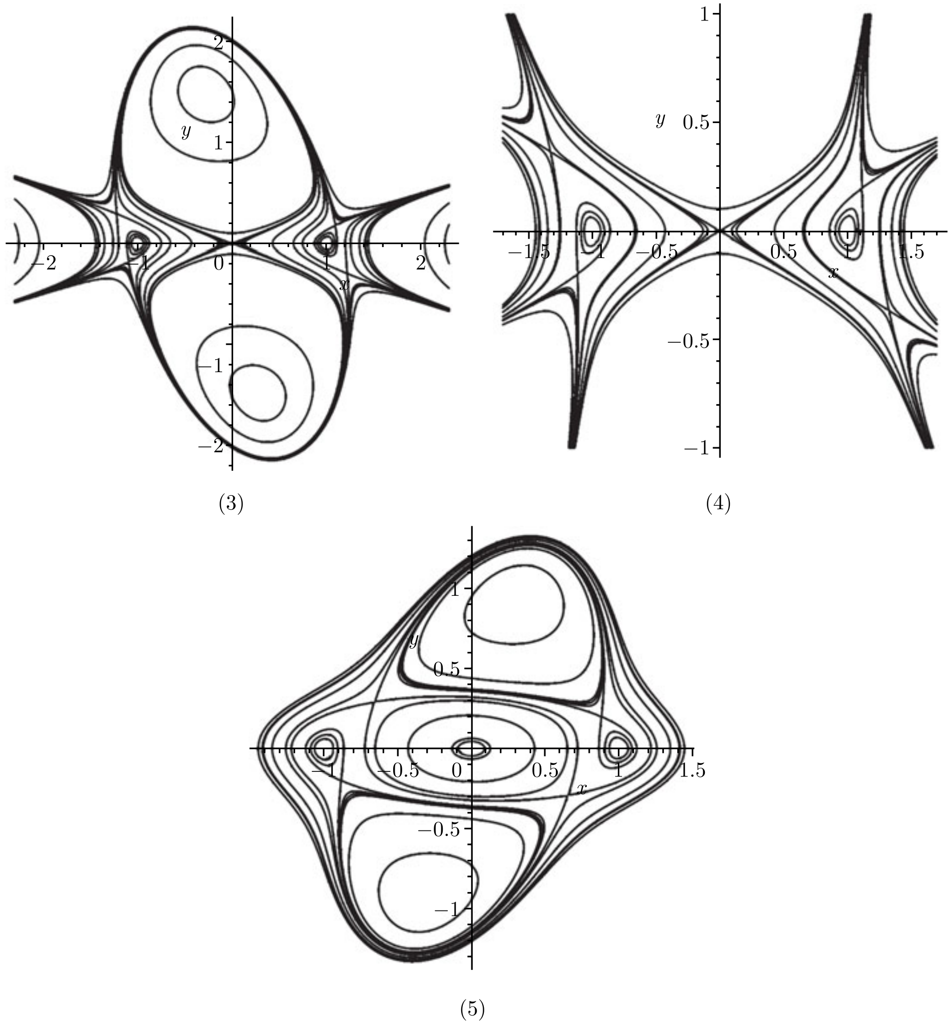


Fig.9.2.1 Some phase portraits of (9.2.12)

(2) Suppose that the condition (C_2) holds. Under this parameter condition, system (9.2.1) becomes

$$\frac{dx}{dt} = y(-1 - a_1 + a_1x^2 + a_3y^2), \quad \frac{dy}{dt} = \frac{1}{2}x(-1 + x^2 + 2a_5y^2). \tag{9.2.14}$$

Proposition 9.2.2. *Let*

$$\begin{aligned} g_1 &= (2 + a_1 + a_5) - (a_1 + a_5)x^2 - 2(a_3 - a_1a_5 + a_5^2)y^2, \\ g_2 &= -1 + x^2 + 2a_5y^2, \\ \gamma_1 &= 2a_3 + (a_1 - a_5)^2. \end{aligned} \tag{9.2.15}$$

System (9.2.14) *has an integral factor*

$$M_2(x,y) = f_1^{-1} \tag{9.2.16}$$

and a first integral

$$F_2(x,y) = f_1 f_2^{(a_1+a_5)}, \tag{9.2.17}$$

where

$$f_1 = g_1^2 - \gamma_1 g_2^2,$$
$$f_2 = \begin{cases} \left(\dfrac{g_1+\sqrt{\gamma_1}g_2}{g_1-\sqrt{\gamma_1}g_2}\right)^{\frac{1}{\sqrt{\gamma_1}}}, & if\ \gamma_1 > 0; \\ \exp\dfrac{2g_2}{g_1}, & if\ \gamma_1 = 0; \\ \exp\left(\dfrac{2}{\sqrt{-\gamma_1}}\arctan\dfrac{g_2}{g_1}\right), & if\ \gamma_1 < 0. \end{cases} \tag{9.2.18}$$

(3) Suppose that the condition (C_3) holds, we have

Proposition 9.2.3. *Under the condition* (C_3), *system* (9.2.1) *has an integral factor*

$$M_3(x,y) = f_3^{2(a_1+a_5)} \tag{9.2.19}$$

and a first integral

$$F_3(x,y) = f_3^{(1+2a_1+2a_5)} f_4, \tag{9.2.20}$$

where

$$\begin{aligned} f_3 &= (a_1+a_5)x - a_4 y, \\ f_4 &= (2+a_1+a_5)[a_4x + 2(1+a_1)(a_1+a_5)y] \\ &\quad -(1+a_1+a_5)[a_4x^3 + 2(1+a_1)(a_1+a_5)x^2y \\ &\quad +2a_4(1+a_5)xy^2 + 2a_3(a_1+a_5)y^3]. \end{aligned} \tag{9.2.21}$$

Remark 9.2.1. *Suppose that the condition* (C_3) *holds. Let*

$$a_1 + a_5 = \gamma, \quad a_4 = b_4\gamma. \tag{9.2.22}$$

By solving $2(1+a_1)(a_1+a_5)^2 - a_4^2(1+2a_1+2a_5) = 0$, *we obtain the expression of* a_1. *By solving* $h_0 = 0$, *we obtain the expression* a_3. *Thus, condition* (C_3) *can be reduced to*

$$(\tilde{C}_3): \begin{cases} \gamma \neq 0, \quad a_4 = b_4\gamma, \quad a_1 = \dfrac{1}{2}(-2 + b_4^2 + 2b_4^2\gamma), \\ a_5 = \dfrac{1}{2}(2 - b_4^2 + 2\gamma - 2b_4^2\gamma), \\ a_6 = \dfrac{1}{3}(-a_2 - 2a_2r + 4b_4\gamma - b_4^3 r + 2b_4\gamma^2 - 2b_4^3\gamma^2), \\ a_3 = -\dfrac{1}{6}b_4(8a_2 - 12b_4 + 3b_4^3 + 4a_2\gamma \\ \qquad -14b_4\gamma + 8b_4^3\gamma - 4b_4\gamma^2 + 4b_4^3\gamma^2). \end{cases} \tag{9.2.23}$$

(4) Suppose that the condition (C_4) holds. Let

$$\begin{aligned}&\gamma_2 = 1 + a_1 + a_5, \quad \gamma_3 = 2 + 2a_1 + a_4^2,\\ &g_3 = 1 - x^2 - 2y^2 - 2a_5y^2.\end{aligned} \tag{9.2.24}$$

We have

Proposition 9.2.4. *Under the condition (C_4), system (9.2.1) has an integral factor*

$$M_4(x, y) = f_5^{-\gamma_2} f_6^{-1} \tag{9.2.25}$$

and a first integral

$$F_4(x, y) = f_5 f_6 f_7^{a_4}, \tag{9.2.26}$$

where

$$\begin{aligned}
&f_5 = \begin{cases} (1 + \gamma_2 g_3)^{\frac{1}{\gamma_2}}, & if \quad \gamma_2 \neq 0;\\ e^{g_3}, & if \quad \gamma_2 = 0,\end{cases}\\
&f_6 = x^2 + 2a_4xy - 2(1 + a_1)y^2,\\
&f_7 = \begin{cases} \left(\dfrac{x + a_4y - \sqrt{\gamma_3}\ y}{x + a_4y + \sqrt{\gamma_3}\ y}\right)^{\frac{1}{\sqrt{\gamma_3}}}, & if \quad \gamma_3 > 0;\\ \exp \dfrac{-2y}{x + a_4y}, & if \quad \gamma_3 = 0;\\ \exp\left(\dfrac{-2}{\sqrt{-\gamma_3}} \arctan \dfrac{y}{x + a_4y}\right), & if \quad \gamma_3 < 0.\end{cases}
\end{aligned} \tag{9.2.27}$$

Remark 9.2.2. *Under the condition (C_4), if $a_2 = a_4 = 0$, then, $a_6 = A_1 = 0$. Thus, when $a_1 + a_5 = 0$, the condition (C_1) holds; When $a_1 + a_5 \neq 0$, the condition (C_2) holds. In addition, if $|a_2| + |a_4| \neq 0$, then, $A_1 = 0$ implies that there exits a constant γ such that*

$$2 + a_1 + a_5 = 2\gamma a_4, \quad 1 + a_5 = -\gamma a_2. \tag{9.2.28}$$

In this case, the condition (C_4) can be changed to

$$(\tilde{C}_4) : \begin{cases} a_2a_4 \neq 0,\ a_1 = -1 + (a_2 + 2a_4)\gamma,\\ a_3 = -2a_2(a_2 + 2a_4)\gamma^2,\\ a_5 = -1 - a_2\gamma, \quad a_6 = a_2(1 - 2a_4\gamma).\end{cases} \tag{9.2.29}$$

(5) Suppose that the condition (C_5) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt} &= -\frac{3}{2}a_4^2y + \frac{1}{2}(-2 + 3a_4^2)x^2y + a_4xy^2 + a_4^2(1 - a_4^2 + a_5)y^3,\\ \frac{dy}{dt} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_4(1 - a_4^2)y^3.\end{aligned} \tag{9.2.30}$$

Proposition 9.2.5. *Let*

$$\begin{aligned}\gamma_4 =& -2 + a_4^2 + 2a_5,\\ g_4 =& 2a_4^2(2 + a_4^2 + 2a_5) - \gamma_4(2 + a_4^2 + 2a_5)(x^2 - a_4^2y^2)\\ &-(a_4^2 - 2a_5)\gamma_4(-x + a_4y)^2\\ &\times(-x^2 - 2a_4xy - 2y^2 + 2a_4^2y^2 - 2a_5y^2),\\ g_5 =& a_4^2 - 2(x^2 - a_4^2y^2)\\ &+(-1 + a_4^2)(-x + a_4y)^2(-x^2 - 2a_4xy - 4y^2 + 3a_4^2y^2).\end{aligned} \tag{9.2.31}$$

System (9.2.30) *has an integral factor*

$$M_5(x, y) = f_8^2 f_9^{\frac{-4+3a_4^2+2a_5}{2}} \tag{9.2.32}$$

and a first integral

$$F_5 = f_9 f_{10}^2, \tag{9.2.33}$$

where

$$\begin{aligned} f_8 =& x - a_4y,\\ f_9 =& \begin{cases} [a_4^2 + (-1 + a_4^2)(-x^2 + a_4^2y^2)]^{\frac{1}{1-a_4^2}}, & if\ \ 1 - a_4^2 \neq 0;\\ e^{-1+x^2-y^2}, & if\ \ 1 - a_4^2 = 0,\end{cases}\\ f_{10} =& \begin{cases} g_4^{\frac{1}{\gamma_4}}, & if\ \ \gamma_4 \neq 0;\\ \exp\dfrac{g_5}{4a_4^2}, & if\ \ \gamma_4 = 0.\end{cases}\end{aligned} \tag{9.2.34}$$

(6) Suppose that the condition (C_6) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt} =& -\frac{5}{8}a_4^2y + \frac{1}{8}(-8 + 5a_4^2)x^2y + \frac{1}{2}a_4xy^2 - \frac{5}{32}a_4^4y^3,\\ \frac{dy}{dt} =& -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y\\ &-\frac{1}{8}(8 + a_4^2)xy^2 + \frac{1}{4}a_4(2 - a_4^2)y^3.\end{aligned} \tag{9.2.35}$$

Proposition 9.2.6. *System* (9.2.35) *has an integral factor*

$$M_6(x, y) = f_{11}^{\frac{-16+5a_4^2}{6}} f_{12}^{\frac{-8+a_4^2}{6}} \tag{9.2.36}$$

and a first integral

$$F_6 = f_{11}^5 f_{12} f_{13}^6, \tag{9.2.37}$$

where

$$f_{11}=2x-a_4y,\quad f_{12}=2x+5a_4y,$$
$$f_{13}=\begin{cases}\left[8+(-2+a_4^2)(4-4x^2+a_4^2y^2)\right]^{\frac{1}{-2+a_4^2}}, & if\quad -2+a_4^2\neq 0;\\ \exp\dfrac{2-2x^2+y^2}{4}, & if\quad -2+a_4^2=0.\end{cases}\qquad(9.2.38)$$

(7) Suppose that the condition (C_7) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt}=&\frac{15}{32}a_4^2y-\frac{1}{32}(32+15a_4^2)x^2y\\&+\frac{1}{4}a_4xy^2+\frac{1}{512}a_4^2(64+15a_4^2)y^3,\\ \frac{dy}{dt}=&-\frac{1}{2}x-a_4y+\frac{1}{2}x^3+a_4x^2y\\&-\frac{1}{32}(96+17a_4^2)xy^2-\frac{3}{16}a_4(4+a_4^2)y^3.\end{aligned}\qquad(9.2.39)$$

Proposition 9.2.7. *System* (9.2.39) *has an integral factor*

$$M_7(x,y)=f_{14}^{\frac{-8}{3}}\qquad(9.2.40)$$

and a first integral

$$F_7(x,y)=f_{14}^{-5}f_{15}^3,\qquad(9.2.41)$$

where

$$\begin{aligned}f_{14}=&48a_4^2(4x+3a_4y)-(8+3a_4^2)(4x+a_4y)^3,\\ f_{15}=&2560a_4^4(4x+5a_4y)\\&-80a_4^2(16+5a_4^2)(4x+a_4y)^2(4x+3a_4y)\\&+(8+3a_4^2)(16+5a_4^2)(4x+a_4y)^5.\end{aligned}\qquad(9.2.42)$$

(8) Suppose that the condition (C_8) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt}=&\frac{21}{50}a_4^2y-\frac{1}{50}(50+21a_4^2)x^2y\\&+\frac{1}{5}a_4xy^2+\frac{1}{1250}a_4^2(250+63a_4^2)y^3,\\ \frac{dy}{dt}=&-\frac{1}{2}x-a_4y+\frac{1}{2}x^3+a_4x^2y\\&-\frac{1}{50}(200+39a_4^2)xy^2-\frac{1}{25}a_4(35+9a_4^2)y^3.\end{aligned}\qquad(9.2.43)$$

Proposition 9.2.8. *System* (9.2.43) *has an integral factor*

$$M_8(x,y) = f_{16}^{\frac{-10}{3}} \tag{9.2.44}$$

and a first integral

$$F_8(x,y) = f_{16}^{-7} f_{17}^{3}, \tag{9.2.45}$$

where

$$\begin{aligned} f_{16} =& 225a_4^2(5x+3a_4y) - (25+9a_4^2)(5x+a_4y)^3,\\ f_{17} =& 1968750a_4^6(5x+7a_4y)\\ &+525a_4^2(25+7a_4^2)(25+9a_4^2)(5x+a_4y)^4(5x+3a_4y)\\ &-78750a_4^4(25+7a_4^2)(5x+a_4y)(5x+3a_4y)^2\\ &-(25+7a_4^2)(25+9a_4^2)^2(5x+a_4y)^7. \end{aligned} \tag{9.2.46}$$

(9) Suppose that the condition (C_9) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned} \frac{dx}{dt} =& \frac{4}{9}a_4^2y - \frac{1}{9}(9+4a_4^2)x^2y,\\ \frac{dy}{dt} =& -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + \frac{2}{3}a_4(1+a_5)y^3. \end{aligned} \tag{9.2.47}$$

Proposition 9.2.9. *System* (9.2.47) *has an integral factor*

$$M_9(x,y) = f_{18}^{-3} f_{19}^{\frac{9-8a_4^2+18a_5}{2}} \tag{9.2.48}$$

and a first integral

$$\begin{aligned} F_9 =& (3x+4a_4y)f_{18}^{-2} f_{19}^{\frac{9(3+2a_5)}{2}}\\ &-6(1+a_5)\int \frac{f_{19}^{\frac{9(3+2a_5)}{2}}\mathrm{d}x}{9+(9+4a_4^2)(-1+x^2)}, \end{aligned} \tag{9.2.49}$$

where

$$\begin{aligned} f_{18} =& 3x+2a_4y,\\ f_{19} =& \begin{cases} [9+(9+4a_4^2)(-1+x^2)]^{\frac{1}{9+4a_4^2}}, & if \ \ 9+4a_4^2 \neq 0;\\ e^{\frac{-1+x^2}{9}}, & if \ \ 9+4a_4^2 = 0. \end{cases} \end{aligned} \tag{9.2.50}$$

Remark 9.2.3. *When* $9+4a_4^2=0$, a_4 *is a complex number, the above results of the integrability of* (9.2.1) *are also true when the parameter group* $(a_1,a_2,a_3,a_4,a_5,a_6)$ *is complex.*

(10) Suppose that the condition (C_{10}) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt} =& \frac{3}{8}a_4^2 y - \frac{1}{8}(8+3a_4^2)x^2 y \\ & -\frac{1}{2}a_4 x y^2 + \frac{3}{16}a_4^2(4+a_4^2+4a_5)y^3, \\ \frac{dy}{dt} =& -\frac{1}{2}x - a_4 y + \frac{1}{2}x^3 + a_4 x^2 y \\ & + a_5 x y^2 + \frac{1}{8}a_4(4-a_4^2+8a_5)y^3. \end{aligned} \tag{9.2.51}$$

Proposition 9.2.10. *The singular points* $(\pm 1, 0)$ *of system* (9.2.51)*are centers.*

Proof. By using the transformation

$$\begin{aligned}\xi &= \frac{1}{2} - \frac{2(2x+3a_4 y)}{(2x+a_4 y)^3}, \\ \eta &= \frac{2y}{2x+a_4 y}, \quad d\tau = \frac{1+a_4\eta}{1-2\xi}dt, \end{aligned} \tag{9.2.52}$$

system (9.2.51) becomes

$$\begin{aligned}\frac{d\xi}{d\tau} &= \frac{1}{16}\eta(1-2\xi)\left[-16+3a_4^2(8+a_4^2+8a_5)\eta^2\right], \\ \frac{d\eta}{d\tau} &= \frac{1}{8}\left[8\xi + (8-3a_4^2+8a_5)\eta^2 - a_4^2(8+a_4^2+8a_5)\eta^4\right]. \end{aligned} \tag{9.2.53}$$

The above transformation makes $(\pm 1, 0)$ of (9.2.51) become the origin $(0,0)$ of (9.2.53). In addition,

$$\begin{aligned}\xi &= (x-1) + h.o.t., \quad \eta = y + h.o.t., \quad \text{near } (1,0), \\ \xi &= -(x+1) + h.o.t., \quad \eta = -y + h.o.t., \quad \text{near } (-1,0). \end{aligned} \tag{9.2.54}$$

Clearly, the vector field defined by (9.2.53) is symmetric with respect to the ξ-axis. It implies that the conclusion of this proposition. □

(11) Suppose that the condition (C_{11}) holds. Under this parameter condition, system (9.2.1) becomes

$$\begin{aligned}\frac{dx}{dt} =& \frac{15}{32}a_4^2 y - \frac{1}{32}(32+15a_4^2)x^2 y \\ & -\frac{1}{4}a_4 x y^2 + \frac{1}{512}a_4^2(832+495a_4^2)y^3, \\ \frac{dy}{dt} =& -\frac{1}{2}x - a_4 y + \frac{1}{2}x^3 + a_4 x^2 y \\ & + \frac{1}{32}(160+111a_4^2)x y^2 + \frac{1}{16}a_4(76+45a_4^2)y^3. \end{aligned} \tag{9.2.55}$$

Proposition 9.2.11. *The singular points* $(\pm 1, 0)$ *of system* (9.2.55) *are centers.*

Proof. Let

$$g_6 = (256x^3 + 576a_4x^2y - 528a_4^2xy^2 - 540a_4^4xy^2 - 932a_4^3y^3 - 585a_4^5y^3). \quad (9.2.56)$$

By using the transformation

$$\begin{aligned} \xi &= \frac{4y}{4x+3a_4y}\sqrt{\frac{12x+13a_4y}{3(4x+3a_4y)}}, \\ \eta &= \frac{1}{2} - \frac{8(4x+5a_4y)^3}{(4x+3a_4y)^5} - \frac{4y^2(12x+13a_4y)}{3(4x+3a_4y)^6}\, g_6, \\ d\tau &= \frac{(4x+3a_4y)^3}{16(4x+5a_4y)}\sqrt{\frac{3(4x+3a_4y)}{12x+13a_4y}}dt, \end{aligned} \quad (9.2.57)$$

system (9.2.55) becomes the following Lienard system

$$\begin{aligned} \frac{d\xi}{d\tau} &= \eta + \frac{1}{64}\xi^2[448 + 192a_4^2 - a_4^2(528 + 297a_4^2\xi^2], \\ \frac{d\eta}{d\tau} &= -\frac{1}{512}\xi(4 - 3a_4^2\xi^2)[4 + (48 + 27a_4^2)\xi^2] \\ &\quad \times[32 - a_4^2(240a_4^2 + 135a_4^2)\xi^2]. \end{aligned} \quad (9.2.58)$$

The transformation (9.2.57) makes the singular points $(\pm 1, 0)$ of (9.2.55) become the origin of (9.2.58) and we have

$$\begin{aligned} &\xi = y + h.o.t., \qquad \eta = (x-1) + h.o.t., \qquad \text{near } (1,0), \\ &\xi = -y + h.o.t., \quad \eta = -(x+1) + h.o.t., \quad \text{near } (-1,0). \end{aligned} \quad (9.2.59)$$

Obviously, the vector field defined by (9.2.58) is symmetric with respect to the η-axis. It implies that the conclusion of this proposition. □

Theorem 9.2.1,Theorem 9.2.2 and Propositions 9.2.1∼ Propositions 9.2.11 imply that

Theorem 9.2.3. *The singular points* $(\pm 1, 0)$ *of system* (9.2.1) *are centers if and only if the first* 6 *Liapunov constants are all zeros, namely, one of* 11 *conditions in Theorem* 9.2.2 *holds.*

9.3 The Conditions of Six-Order Weak Focus and Bifurcations of Limit Cycles

We know from Theorem 9.2.1 and Lemma 9.2.1 that if $(\pm 1, 0)$ of system (9.2.1) are weak focus of order 6, then we have

$$\Delta(a_2, a_4) = 20a_2^3 - 35a_2^2a_4 - 20a_2a_4^2 - a_4^3 = 0, \quad a_4h_0 \neq 0. \tag{9.3.1}$$

Let $a_2 = \lambda a_4$. It is easy to see that the function $\Delta(\lambda, 1)$ has three zeros at

$$\begin{aligned}
\lambda_1 &= \frac{7}{12} + \frac{\sqrt{97}}{6}\cos\theta_0 = 2.21224585\cdots, \\
\lambda_2 &= \frac{7}{12} + \frac{\sqrt{97}}{6}\cos\left(\theta_0 - \frac{2\pi}{3}\right) = -0.05557708\cdots, \\
\lambda_3 &= \frac{7}{12} + \frac{\sqrt{97}}{6}\cos\left(\theta_0 + \frac{2\pi}{3}\right) = -0.40666876\cdots,
\end{aligned} \tag{9.3.2}$$

where

$$\theta_0 = \frac{1}{3}\arctan\left(\frac{36\sqrt{2319}}{4451}\right). \tag{9.3.3}$$

By Theorem 9.2.1, we have

Theorem 9.3.1. *For system* (9.2.1), $(\pm 1, 0)$ *are weak focuses of order 6 if and only if*

$$\Delta(\lambda, 1) = 0, \quad a_4h_0 \neq 0, \quad (a_1, a_2, a_3, a_5, a_6) = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6), \tag{9.3.4}$$

where

$$\begin{aligned}
\tilde{a}_1 &= \frac{1}{18}[-18 + (-8 + 154\lambda + 385\lambda^2)a_4^2], \\
\tilde{a}_2 &= \lambda a_4, \\
\tilde{a}_3 &= -\frac{1}{64800}a_4^2[2880(8 + 74\lambda + 143\lambda^2) \\
&\quad + (3578819 + 73223024\lambda + 158462585\lambda^2)a_4^2], \\
\tilde{a}_5 &= -\frac{1}{90}[126 + (44 + 320\lambda + 797\lambda^2)a_4^2], \\
\tilde{a}_6 &= -\frac{1}{675}a_4[45(4 - 19\lambda) + (502 + 6820\lambda + 16105\lambda^2)a_4^2].
\end{aligned} \tag{9.3.5}$$

Lemma 9.3.1. *Suppose that* $\Delta(\lambda, 1) = 0$ *and* $(a_1, a_2, a_3, a_5, a_6) = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6)$. *Then* $h_0 = 0$ *if and only if*

$$\lambda = \lambda_1, \quad a_4^2 = \omega_1, \quad \text{or} \quad a_4^2 = \omega_2, \tag{9.3.6}$$

where

$$\omega_1 = 0.03274565\cdots, \quad \omega_2 = 0.03453237\cdots. \tag{9.3.7}$$

Proof. When $(a_1, a_2, a_3, a_5, a_6) = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6)$, we have

$$h_0 = \frac{a_4^2(5468369 + 111981560\lambda + 242266355\lambda^2)}{593806218257103750}\,\tilde{h}_0, \tag{9.3.8}$$

where

$$\begin{aligned}\tilde{h}_0 = {} & 158348324868561a_4^4 \\ & +260090(1927442096 + 735778623\lambda - 734798180\lambda^2)a_4^2 \\ & +275(1591628188157 + 1932176311266\lambda - 1198485512456\lambda^2)\end{aligned} \tag{9.3.9}$$

is a quadratic polynomial in a_4^2. $\Delta(\lambda, 1) = \tilde{h}_0 = 0$ imply that

$$\begin{aligned}a_4^2 = {} & \frac{-5(1927442096+735778623\lambda-734798180\lambda^2)}{6088212729} \\ & \pm\frac{25\sqrt{6(-2600381406258223-14988190268053629\lambda+7306440411744220\lambda^2)}}{6088212729}.\end{aligned} \tag{9.3.10}$$

When $\lambda = \lambda_2$ and $\lambda = \lambda_3$, (9.3.10) follows that a_4^2 is not a positive real number. But when $\lambda = \lambda_1$, $a_4^2 = \omega_1,\ \omega_2$, where $\omega_1 = 0.03274565\cdots$, $\omega_2 = 0.03453237\cdots$. Namely Lemma 9.3.1 holds □

According to Theorem 9.3.1 and Lemma 9.3.1, we have

Theorem 9.3.2. $(\pm 1, 0)$ *are weak focuses of order* 6 *of system* (9.2.1) *if and only if*

$$\Delta(\lambda, 1) = 0, \quad a_4 \neq 0, \quad (a_1, a_2, a_3, a_5, a_6) = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6) \tag{9.3.11}$$

and when $\lambda = \lambda_1$, $a_4^2 \neq \omega_1,\ \omega_2$.

Theorem 9.3.3. *Suppose that* $(\pm 1, 0)$ *are weak focuses of order* 6 *of system* (9.2.1). *Then, when*

$$|\delta| + \sum_{k=1}^{6} |a_k - \tilde{a}_k| \ll 1, \tag{9.3.12}$$

by making a small perturbation of the coefficient group of $(\delta, a_1, a_2, a_3, a_5, a_6)$ *of system* (9.1.8), *there exist* 6 *small amplitude limit cycles in a small neighborhood of* $(\pm 1, 0)$, *respectively.*

Proof. Because $(\pm 1, 0)$ are weak focuses of order 6 of system (9.2.1) and (9.3.4) holds, Theorem 9.2.1 follows that when $\Delta(\lambda, 1) = 0$, $a_4 h_0 \neq 0$, the Jacbin of the function group $(V_3, V_5, V_7, V_9, V_{11})$ with respect to $(a_1, a_2, a_3, a_5, a_6)$ at $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6)$ is given by

$$J=\frac{\partial(V_3,\ V_5,\ V_7,\ V_9,\ V_{11})}{\partial(a_1,\ a_2,\ a_3,\ a_5,\ a_6)}=\begin{vmatrix} \frac{\partial V_3}{\partial a_1} & \frac{\partial V_3}{\partial a_2} & \frac{\partial V_3}{\partial a_3} & \frac{\partial V_3}{\partial a_5} & \frac{3}{4} \\ \frac{\partial V_5}{\partial a_1} & \frac{\partial V_5}{\partial a_2} & \frac{\partial V_5}{\partial a_3} & \frac{\partial V_5}{\partial a_5} & 0 \\ \frac{\partial V_7}{\partial a_1} & \frac{\partial V_7}{\partial a_2} & \frac{\partial V_7}{\partial a_3} & \frac{\partial V_7}{\partial a_5} & 0 \\ \frac{\partial V_9}{\partial a_1} & \frac{\partial V_9}{\partial a_2} & 0 & 0 & 0 \\ \frac{\partial V_{11}}{\partial a_1} & \frac{\partial V_{11}}{\partial a_2} & 0 & 0 & 0 \end{vmatrix}$$

$$=\frac{539(8602456533509+175937693579696\lambda+380614976209391\lambda^2)}{12538266255360}a_4^{13}h_0^3\neq 0. \quad (9.3.13)$$

Thus Theorem 3.1.4 follows that Theorem 9.3.3 holds. □

Under conditions of Theorem 9.2.1 and Theorem 9.3.1, $V_{13}=a_4^{11}\kappa\neq 0$, where

$$\begin{aligned}\kappa=&\frac{-11}{354294000000}[(56794007957132160\\&-3379058706136258840a_4^2+5022530151757757587a_4^4)\\&+(1161552717525657600-6910860774908941888 0a_4^2\\&+10272093395220754789 92a_4^4)\lambda+(2512846172532009600\\&-14950617206152492756 0a_4^2+22222142980486728092 25a_4^4)\lambda^2)]. \quad (9.3.14)\end{aligned}$$

Theorem 9.2.1 and Theorem 9.3.2 imply that

Lemma 9.3.2. *If*

$$\begin{aligned}&a_4\neq 0,\ \Delta(\lambda,1)=0,\ \delta=259200\kappa a_4^{11}\varepsilon^{12},\\&a_1=\tilde{a}_1+c_0\varepsilon^4,\\&a_2=\tilde{a}_2+\frac{104}{45}(13+220\lambda+460\lambda^2)a_4^3\varepsilon^2,\\&a_3=\tilde{a}_3+c_1\varepsilon^2+c_2\varepsilon^4+c_3\varepsilon^6+c_4\varepsilon^8,\\&a_5=\tilde{a}_5+c_5\varepsilon^2+c_6\varepsilon^4+c_7\varepsilon^6+c_8\varepsilon^8,\\&a_6=\tilde{a}_6+c_9\varepsilon^2+c_{10}\varepsilon^4+c_{11}\varepsilon^6+c_{12}\varepsilon^8+c_{13}\varepsilon^{10}, \quad (9.3.15)\end{aligned}$$

and $a_4^2\neq\omega_1,\ \omega_2$, *when* $\lambda=\lambda_1$, *where* $c_0\ \sim c_{13}$ *are given by §9.7, then the first six focal values at* $(\pm 1,0)$ *of system* (9.1.8) *are as follow*

$$\begin{aligned}&\nu_1(2\pi)-1=518400a_4^{11}\kappa\pi\varepsilon^{12}+o(\varepsilon^{12}),\\&\nu_3(2\pi)=-773136a_4^{11}\kappa\pi\varepsilon^{10}+o(\varepsilon^{10}),\\&\nu_5(2\pi)=296296a_4^{11}\kappa\pi\varepsilon^8+o(\varepsilon^8),\end{aligned}$$

$$\nu_7(2\pi) = -44473a_4^{11}\kappa\pi\varepsilon^6 + o(\varepsilon^6),$$
$$\upsilon_9(2\pi) = 3003a_4^{11}\kappa\pi\varepsilon^4 + o(\varepsilon^4),$$
$$\nu_{11}(2\pi) = -91a_4^{11}\kappa\pi\varepsilon^2 + o(\varepsilon^2),$$
$$\nu_{13}(2\pi) = a_4^{11}\kappa\pi + o(1). \tag{9.3.16}$$

Theorem 9.3.4. *Under the condition in Lemma* 9.3.2*, there exists a positive number* $\varepsilon_0 > 0$*, such that system* (9.1.8) *has exactly* 12 *limit circles, which are close to the circles* $(x \mp 1)^2 + y^2 = k^2\varepsilon^2$ *when* $0 < |\varepsilon| < \varepsilon_0$*,* $k = 1, 2, 3, 4, 5, 6$.

Proof. By Lemma 9.3.2, the quasi succession function at $(\pm 1, 0)$ of system (9.1.8) is that

$$\begin{aligned} L(h,\varepsilon) &= a_4^{11}\kappa\pi\eta(518400\varepsilon^{12} - 773136h^2\varepsilon^{10} + 296296h^4\varepsilon^8 \\ &\quad -44473h^6\varepsilon^6 + 3003h^8\varepsilon^4 - 91h^{10}\varepsilon^2 + h^{12}) \\ &= a_4^{11}\kappa\pi\prod_{k=1}^{6}(h^2 - k^2\varepsilon^2), \end{aligned} \tag{9.3.17}$$

(9.3.17) and Theorem 3.3.3 imply this theorem. □

9.4 A Class of $(E_3^{Z_2})$ System With 13 Limit Cycles

In this section, we consider the following system having two weak focuses of order 6:

$$\begin{aligned} \frac{dx}{dt} &= -(\tilde{a}_1 + 1)y + \tilde{a}_1 x^2 y + \tilde{a}_2 x y^2 + \tilde{a}_3 y^3, \\ \frac{dy}{dt} &= -\frac{1}{2}x + \frac{1}{2}x^3 - \tilde{a}_1 x y^2 - \frac{1}{3}\tilde{a}_2 y^3, \end{aligned} \tag{9.4.1}$$

where $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6)$ are given by (9.3.5), $\Delta(\lambda, 1) = 0$, $a_4 \neq 0$ and $a_4^2 \neq \omega_1$, ω_2 when $\lambda = \lambda_1$.

Write the functions of the right hand of system (9.4.1) as follows:

$$\begin{aligned} &X_1(x,y) = -(\tilde{a}_1 + 1)y, \quad Y_1(x,y) = -\frac{1}{2}x - a_4 y, \\ &X_3(x,y) = \tilde{a}_1 x^2 y + \tilde{a}_2 x y^2 + \tilde{a}_3 y^3, \\ &Y_3(x,y) = \frac{1}{2}x^3 + a_4 x^2 y + \tilde{a}_5 x y^2 + \tilde{a}_6 y^3. \end{aligned} \tag{9.4.2}$$

By (9.3.5), every component of the group $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_5, \tilde{a}_6)$ is a polynomial in λ, a_4 with the rational coefficients. $\lambda \in \{\lambda_1, \lambda_2, \lambda_3\}$ and a_4 is a free parameter. We are going to consider the bifurcation condition of limit cycles of (9.4.1) created from the infinity (i.e., the equator Γ_∞ of the Poincar'e sphere). Let

$$\begin{aligned}
P_2(x,y) &= 2[xX_1(x,y) + yY_1(x,y)] = -(3+2\tilde{a}_1)xy - 2a_4y^2,\\
Q_2(x,y) &= 2[xY_1(x,y) - yX_1(x,y)] = -x^2 - 2a_4xy + 2(1+\tilde{a}_1)y^2,\\
P_4(x,y) &= 2[xX_3(x,y) + yY_3(x,y)]\\
&= (1+2\tilde{a}_1)x^3y + 2(\tilde{a}_2 + a_4)x^2y^2 + 2(\tilde{a}_3+\tilde{a}_5)xy^3 + 2\tilde{a}_6y^4,\\
Q_4(x,y) &= 2[xY_3(x,y) - yX_3(x,y)]\\
&= x^4 + 2a_4x^3y + 2(\tilde{a}_5 - \tilde{a}_1)x^2y^2 + 2(\tilde{a}_6 - \tilde{a}_2)xy^3 - 2\tilde{a}_3y^4. \quad (9.4.3)
\end{aligned}$$

It is known that system (9.4.1) has no real singular point on the equator Γ_∞ if and only if $Q_4(x,y)$ is positive definite.

Suppose that $Q_4(x,y)$ is positive definite. In order to investigate the stability and bifurcations of limit circles on the equator Γ_∞ of (9.4.1), by making the transformation

$$x = \frac{\cos\theta}{\rho}, \quad y = \frac{\sin\theta}{\rho}, \quad (9.4.4)$$

system (9.4.1) becomes

$$\frac{d\rho}{d\theta} = -\rho\frac{P_4(\cos\theta,\sin\theta) + P_2(\cos\theta,\sin\theta)\rho^2}{Q_4(\cos\theta,\sin\theta) + Q_2(\cos\theta,\sin\theta)\rho^2}. \quad (9.4.5)$$

According to (9.4.3),

$$Q_4(\cos\theta,\sin\theta)|_{\theta=0} = 1. \quad (9.4.6)$$

Clearly, the right hand side of (9.4.5) is an odd function with respect to ρ, Hence, the solution of (9.4.5) satisfying the initial condition $\rho|_{\theta=0} = h$ has the form

$$\rho = \sum_{k=0}^{\infty} \tilde{\nu}_{2k+1}(\theta)h^{2k+1}. \quad (9.4.7)$$

Substituting (9.4.7) into (9.4.5), we obtain

$$\tilde{\nu}_1(\theta) = \exp\int_0^\theta \frac{-P_4(\cos\varphi,\sin\varphi)}{Q_4(\cos\varphi,\sin\varphi)}d\varphi, \quad (9.4.8)$$

$$\tilde{\nu}_3(\theta) = \tilde{\nu}_1(\theta)\int_0^\theta \begin{vmatrix} P_4(\cos\varphi,\sin\varphi), & Q_4(\cos\varphi,\sin\varphi)\\ P_2(\cos\varphi,\sin\varphi), & Q_2(\cos\varphi,\sin\varphi)\end{vmatrix} \frac{\tilde{\nu}_1^2(\varphi)d\varphi}{Q_4^2(\cos\varphi,\sin\varphi)}. \quad (9.4.9)$$

Because

$$4P_4(\cos\theta,\sin\theta) \equiv 2R_2(\cos\theta,\sin\theta) - \frac{d}{d\theta}Q_4(\cos\theta,\sin\theta), \quad (9.4.10)$$

where

$$R_2 = a_4\cos^2\theta + 2(\tilde{a}_1+\tilde{a}_5)\cos\theta\sin\theta + (\tilde{a}_2 + 3\tilde{a}_6)\sin^2\theta. \quad (9.4.11)$$

(9.4.8) and (9.4.10) follows that

$$\begin{aligned}&\tilde{\nu}_1(\theta) = \sqrt[4]{Q_4(\cos\theta, \sin\theta)} \cdot e^{G(\theta)},\\&\tilde{\nu}_1(2\pi) - 1 = e^{G(2\pi)} - 1,\end{aligned} \tag{9.4.12}$$

where

$$G(\theta) = -\frac{1}{2}\int_0^\theta \frac{R_2(\cos\varphi, \sin\varphi)}{Q_4(\cos\varphi, \sin\varphi)} d\varphi. \tag{9.4.13}$$

By (9.4.12), we know that the infinity is a weak focus of (9.4.1) only if $Q_4(x, y)$ is positive definite and $G(2\pi) = 0$.

We see from (9.4.9), (9.4.12) and (9.4.13), when $G(2\pi) = 0$, $\tilde{\nu}_1(\theta)$ is a periodic function of period π and

$$\tilde{\nu}_3(2\pi) = 2\int_0^\pi \begin{vmatrix} P_4(\cos\varphi, \sin\varphi), Q_4(\cos\varphi, \sin\varphi) \\ P_2(\cos\varphi, \sin\varphi), Q_2(\cos\varphi, \sin\varphi) \end{vmatrix} \frac{e^{2G(\varphi)} d\varphi}{Q_4^{\frac{3}{2}}(\cos\varphi, \sin\varphi)}. \tag{9.4.14}$$

Lemma 9.4.1. *Suppose that system* (9.4.1) *has no real singular point on the equator* Γ_∞ *and* $G(2\pi) = 0$*. Then,* a_4 *is a real zero of the following polynomial* H_1 *of degree* 15 *in* a_4^2*:*

$$\begin{aligned}H_1 = {}& 244888012800000000000 + 770943744000000000000 a_4^2\\&-469421964289260000000000 a_4^4\\&+360750952055123055000000000 a_4^6\\&-11431107404380004968125000000 a_4^8\\&+110138721573434196447703125000 a_4^{10}\\&+672943076906684356581168750000 a_4^{12}\\&+82160450429898196694971093750 a_4^{14}\\&-4557126546224962577450661875000 a_4^{16}\\&-8171720224658402005924077250000 a_4^{18}\\&-5754997658961605207559458750000 a_4^{20}\\&+12864759493450383065060070737500 a_4^{22}\\&+16372281536221812443601993215000 a_4^{24}\\&+9348427295888702301153433555000 a_4^{26}\\&+2631700867457517734619874849000 a_4^{28}\\&+296158609528957974567827931710 a_4^{30}.\end{aligned} \tag{9.4.15}$$

Theorem 9.4.1. *System* (9.4.1) *has no real singular point on the equator* Γ_∞ *and* $G(2\pi) = 0$ *if and only if* $a_4 = \pm a_4^*$, $\lambda = \lambda_2$, *where* $a_4^*(= 0.81233628\cdots)$ *is the largest real zero of the function* H_1.

Lemma 9.4.1 and Theorem 9.4.1 will be proved in Section 9.5.

Lemma 9.4.2.

$$\left.\frac{dG(2\pi)}{da_4}\right|_{a_4=\pm a_4^*,\ \lambda=\lambda_2} = 17.00901058\cdots > 0. \tag{9.4.16}$$

Lemma 9.4.3.

$$\left.\tilde{\nu}_3(2\pi)\right|_{a_4=\pm a_4^*,\ \lambda=\lambda_2} \approx \pm 5.36546 \times 10^{11}. \tag{9.4.17}$$

The proofs of Lemma 9.4.2 and Lemma 9.4.3 will be given in Section 9.6.

We see from Theorem 9.4.1, Lemma 9.4.2 and Lemma 9.4.3 that the equator Γ_∞ of system (9.4.1) is a unstable (stable) inner limit cycle when $\lambda = \lambda_2,\ a_4 = a_4^*$ (or $a_4 = -a_4^*$). When $\lambda = \lambda_2,\ a_4 = a_4^*(1-\sigma)$ (or $a_4 = -a_4^*(1-\sigma)$) and $0 < \sigma \ll 1$, the equator Γ_∞ of system (9.4.1) is a stable (unstable) inner limit cycle.

By using well known Hopf bifurcation theorem, we have

Theorem 9.4.2. *When $\lambda = \lambda_2,\ a_4 = a_4^*(1-\sigma)$ (or $a_4 = -a_4^*(1-\sigma)$) for $0 < \sigma \ll 1$, system (9.4.1) has exactly a unstable (or a stable) limit cycle near the equator.*

Theorem 9.3.3 and Theorem 9.4.2 imply the following main result.

Theorem 9.4.3. *For system $(E_3^{Z_2})$, 6 limit cycles can be created respectively in two small neighborhoods of two weak focuses of 6 order. In addition, in a inner neighborhood of the equator, there exists a larger limit cycle. Therefore, there exist 13 limit cycles with the scheme $1 \supset (6 \bigcup 6)$.*

When $\lambda = \lambda_2,\ a_4 = \pm a_4^*$, by solving (9.3.14), we have $\kappa = 0.00037809\cdots > 0$. Thus, Theorem 9.3.4 and Theorem 9.4.3 follow that

Theorem 9.4.4. *Suppose that the coefficients of system (9.1.8) are given by (9.3.15), where $\lambda = \lambda_2,\ a_4 = a_4^*(1-\sigma)$ (or $a_4 = -a_4^*(1-\sigma)$). Then, we have*

(1) When $\varepsilon = \sigma = 0$, $(\pm 1, 0)$ are unstable (or stable) weak focus of 6 order, the equator Γ_∞ of system (9.1.8) is a unstable (or a stable) inner limit cycle.

(2) When $\varepsilon = 0,\ 0 < \sigma \ll 1$, $(\pm 1, 0)$ are unstable (or stable) weak focus of 6 order, and in the neighborhood of the equator, there exists a unique unstable (or stable) limit cycle.

(3) When $0 < |\varepsilon| \ll \sigma \ll 1$, there exist 6 limit cycles in a neighborhoods of $(\pm 1, 0)$, respectively. In a neighborhood of the equator, there exists a unique unstable (or stable) limit cycle. Namely, there exist 13 limit cycles. Furthermore, the equator Γ_∞ is a inner stable (or unstable) limit cycle.

9.5 Proofs of Lemma 9.4.1 and Theorem 9.4.1

In order to know the exact value of $G(2\pi)$, we need to have factorization of $Q_4(x,y)$. Because the coefficient of the term x^4 in $Q_4(x,y)$ is 1. It implies that $Q_4(x,y)$ is positive definite if and only if there exist two positive numbers α,β and two constants γ_1,γ_2, such that

$$Q_4(x,y)=[(x+\gamma_1 y)^2+\alpha^2y^2][(x+\gamma_2 y)^2+\beta^2y^2]. \tag{9.5.1}$$

Expanding the right hand of (9.5.1) and comparing the coefficients of the same powers with (9.4.3), it follows from (9.3.5) that $\alpha,\beta,\gamma_1,\gamma_2$ are solutions of the equations $f_1=f_2=f_3=f_4=0$, where

$$\begin{aligned}
f_1=&a_4-\gamma_1-\gamma_2,\\
f_2=&36+4a_4^2+45\alpha^2+45\beta^2+1090a_4^2\lambda+2722a_4^2\lambda^2\\
&+45\gamma_1^2+180\gamma_1\gamma_2+45\gamma_2^2,\\
f_3=&180a_4+502a_4^3-180a_4\lambda+6820a_4^3\lambda+16105a_4^3\lambda^2\\
&+675\beta^2\gamma_1+675\alpha^2\gamma_2+675\gamma_1^2\gamma_2+675\gamma_1\gamma_2^2,\\
f_4=&23040a_4^2+3578819a_4^4-32400\alpha^2\beta^2+213120a_4^2\lambda\\
&+73223024a_4^4\lambda+411840a_4^2\lambda^2+158462585a_4^4\lambda^2\\
&-32400\beta^2\gamma_1^2-32400\alpha^2\gamma_2^2-32400\gamma_1^2\gamma_2^2.
\end{aligned} \tag{9.5.2}$$

Remark 9.5.1. *It is easy to show that if* $(\alpha,\beta,\gamma_1,\gamma_2)$ *is a complex solution group of the equations* $f_1=f_2=f_3=f_4=0$, *then* $Q_4(x,y)$ *always has the factorization* (9.5.1) *whether* $Q_4(x,y)$ *is positive definite or not.*

Lemma 9.5.1. *System* (9.4.1) *has no real singular point on the equator if and only if there exist three positive numbers* α,β,γ, *such that*

$$Q_4(x,y)=[(x+\frac{1}{2}a_4y-a_4\gamma y)^2+\alpha^2y^2][(x+\frac{1}{2}a_4y+a_4\gamma y)^2+\beta^2y^2]. \tag{9.5.3}$$

Proof. Because $a_4\neq 0$, we see from $f_1=0$ that there exists a γ such that

$$\gamma_1=\frac{1}{2}a_4-\gamma a_4,\quad \gamma_2=\frac{1}{2}a_4+\gamma a_4. \tag{9.5.4}$$

We can assume that $\gamma\geqslant 0$ and prove that $\gamma\neq 0$. In fact, we see from $f_1=f_2=0$ that

$$\alpha^2+\beta^2=-\frac{4}{5}+2\gamma^2a_4^2-\frac{1}{90}(143+2180\lambda+5444\lambda^2)a_4^2. \tag{9.5.5}$$

Since we have $\Delta(\lambda,\ 1)=0$, (9.5.5) implies that $\alpha^2+\beta^2<-\frac{4}{5}+2\gamma^2a_4^2$. It follows that $\gamma\neq 0$. □

Substituting (9.5.4) into (9.5.2), we know that $f_2 = f_3 = 0$ if and only if

$$\alpha^2 = f_5, \quad \beta^2 = f_6, \tag{9.5.6}$$

where

$$\begin{aligned} f_5 =& \frac{1}{2700\gamma}(180 - 269a_4^2 - 1080\gamma - 2145a_4^2\gamma + 2700a_4^2\gamma^3 \\ &+360\lambda + 2710a_4^2\lambda - 32700a_4^2\gamma\lambda + 8620a_4^2\lambda^2 - 81660a_4^2\gamma\lambda^2), \\ f_6 =& \frac{1}{2700\gamma}(-180 + 269a_4^2 - 1080\gamma - 2145a_4^2\gamma + 2700a_4^2\gamma^3 \\ &-360\lambda - 2710a_4^2\lambda - 32700a_4^2\gamma\lambda - 8620a_4^2\lambda^2 - 81660a_4^2\gamma\lambda^2). \end{aligned} \tag{9.5.7}$$

Lemma 9.5.2. *System* (9.4.1) *has no real singular point on the equator if and only if* $F_1 = 0,\ f_5 > 0,\ f_6 > 0$ *and* $\gamma > 0$, *where*

$$\begin{aligned} F_1 =& 32400 + 213480a_4^2 + 8910016a_4^4 + 129600\lambda \\ &+6988320a_4^2\lambda + 179010340a_4^4\lambda + 129600\lambda^2 \\ &+15915600a_4^2\lambda^2 + 386328865a_4^4\lambda^2 \\ &-180(6480 - 5760a_4^2 + 255463a_4^4 + 136800a_4^2\lambda \\ &+5439976a_4^4\lambda + 465120a_4^2\lambda^2 + 11877880a_4^4\lambda^2)\gamma^2 \\ &+162000a_4^2(72 + 143a_4^2 + 2180a_4^2\lambda + 5444a_4^2\lambda^2)\gamma^4 \\ &-29160000a_4^4\gamma^6. \end{aligned} \tag{9.5.8}$$

Proof. Submitting (9.5.4), (9.6.6) into f_4, it is easy to show that $f_4 = \Delta(\lambda, 1) = 0$ if and only if $F_1 = 0$. Thus Lemma 9.5.2 holds. □

Denote that

$$\begin{aligned} n_1 =& (\beta^2 - \alpha^2) \\ &\times(-1440 + 943a_4^2 + 900\alpha^2 - 4320\lambda + 31780a_4^2\lambda + 75700a_4^2\lambda^2) \\ &-60(3\alpha^2 + \beta^2)(-72 - 43a_4^2 + 150a_4^2\lambda + 376a_4^2\lambda^2)\gamma \\ &+4a_4^2(-1440 + 943a_4^2 - 675\alpha^2 - 225\beta^2 \\ &-4320\lambda + 31780a_4^2\lambda + 75700a_4^2\lambda^2)\gamma^2 \\ &-240a_4^2(-72 - 43a_4^2 + 150a_4^2\lambda + 376a_4^2\lambda^2)\gamma^3 - 3600a_4^4\gamma^4, \\ n_2 =& -(\beta^2 - \alpha^2) \\ &\times(-1440 + 943a_4^2 + 900\beta^2 - 4320\lambda + 31780a_4^2\lambda + 75700a_4^2\lambda^2) \\ &+60(\alpha^2 + 3\beta^2)(-72 - 43a_4^2 + 150a_4^2\lambda + 376a_4^2\lambda^2)\gamma \\ &+4a_4^2(-1440 + 943a_4^2 - 225\alpha^2 - 675\beta^2 \\ &-4320\lambda + 31780a_4^2\lambda + 75700a_4^2\lambda^2)\gamma^2 \end{aligned}$$

$$+240a_4^2(-72-43a_4^2+150a_4^2\lambda+376a_4^2\lambda^2)\gamma^3-3600a_4^4\gamma^4,$$

$$n_3=-15(\alpha^2-\beta^2)(-72-43a_4^2+150a_4^2\lambda+376a_4^2\lambda^2)$$
$$+a_4^2(-1440+943a_4^2+450\alpha^2+450\beta^2-4320\lambda$$
$$+31780a_4^2\lambda+75700a_4^2\lambda^2)\gamma+900a_4^4\gamma^3. \quad (9.5.9)$$

Substituting (9.5.3) into (9.4.13) and using the method of partial integration, we have

Lemma 9.5.3. *When* $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$,

$$G(\theta)=\frac{G_1(\theta)-G_1(0)}{1800\alpha\beta(\alpha^2-2\alpha\beta+\beta^2+4a_4^2\gamma^2)(\alpha^2+2\alpha\beta+\beta^2+4a_4^2\gamma^2)}, \quad (9.5.10)$$

where

$$G_1(\theta)=a_4\beta n_1\arctan\frac{2a_4(1-2\gamma)+[a_4^2(1-2\gamma)^2+4\alpha^2)]\tan\theta}{4\alpha}$$
$$+a_4\alpha n_2\arctan\frac{2a_4(1+2\gamma)+[a_4^2(1+2\gamma)^2+4\beta^2]\tan\theta}{4\beta}$$
$$+2\alpha\beta n_3\log\frac{4\alpha^2\tan^2\theta+(2+a_4-2a_4\gamma\tan\theta)^2}{4\beta^2\tan^2\theta+(2+a_4+2a_4\gamma\tan\theta)^2}. \quad (9.5.11)$$

Lemma 9.5.4.

$$G(2\pi)=\frac{-a_4\pi(\alpha f_7+\beta f_8)}{900\alpha\beta(\alpha^2+2\alpha\beta+\beta^2+4a_4^2\gamma^2)}, \quad (9.5.12)$$

where

$$f_7=1440-943a_4^2+900\beta^2+4320\gamma$$
$$+2580a_4^2\gamma+900a_4^2\gamma^2+4320\lambda-31780a_4^2\lambda$$
$$-9000a_4^2\gamma\lambda-75700a_4^2\lambda^2-22560a_4^2\gamma\lambda^2,$$
$$f_8=1440-943a_4^2+900\alpha^2-4320\gamma$$
$$-2580a_4^2\gamma+900a_4^2\gamma^2+4320\lambda-31780a_4^2\lambda$$
$$+9000a_4^2\gamma\lambda-75700a_4^2\lambda^2+22560a_4^2\gamma\lambda^2. \quad (9.5.13)$$

Proof. Because the integrand of the right hand of (9.4.13) is a periodic function of period π, Lemma 9.5.3 implies that

$$G(2\pi)=2\left[G\left(\frac{\pi}{2}\right)-G\left(-\frac{\pi}{2}\right)\right]$$
$$=\frac{2\left[G_1\left(\frac{\pi}{2}\right)-G_1\left(-\frac{\pi}{2}\right)\right]}{1800\alpha\beta(\alpha^2-2\alpha\beta+\beta^2+4a_4^2\gamma^2)(\alpha^2+2\alpha\beta+\beta^2+4a_4^2\gamma^2)}$$
$$=\frac{2\pi a_4(\beta n_1+\alpha n_2)}{1800\alpha\beta(\alpha^2-2\alpha\beta+\beta^2+4a_4^2\gamma^2)(\alpha^2+2\alpha\beta+\beta^2+4a_4^2\gamma^2)}. \quad (9.5.14)$$

(9.5.9) and (9.5.14) give the conclusion of this Lemma. □

Remark 9.5.2. *When $\alpha f_7+\beta f_8 \neq 0$, the function $G(\theta)$ has jump discontinuous points at $\theta=\dfrac{\pi}{2}\pm k\pi,\ k=0,1,2,\cdots$. When $\alpha f_7+\beta f_8=0$, for $\theta\in(-\infty,\infty)$, $G(\theta)$ is a continuous periodic function of period π.*

Lemma 9.5.5. *Suppose that system (9.4.1) has no real singular point on the equator. Then $G(2\pi)=0$ if and only if $f_7f_8\leqslant 0$ and $F_2=0$, where*

$$\begin{aligned}F_2=&777600a_4^2(48600-99630a_4^2-4001235a_4^4+142751540a_4^6\\&+194400\lambda-1590300a_4^2\lambda-78937845a_4^4\lambda+2920922183a_4^6\lambda\\&-3877200a_4^2\lambda^2-170418660a_4^4\lambda^2+6319248215a_4^6\lambda^2)\gamma^4\\&-216(104976000+1138989600a_4^2-83650309200a_4^4\\&-949863385800a_4^6+68019121524943a_4^8+419904000\lambda\\&+24686856000a_4^2\lambda-1705595216400a_4^4\lambda-19423812453480a_4^6\lambda\\&+1391129313186520a_4^8\lambda+51858144000a_4^2\lambda^2\\&-3691578067200a_4^4\lambda^2-42021926420400a_4^6\lambda^2\\&+30095004770480 05a_4^8\lambda^2)\gamma^2+1133740800+6858432000a_4^2\\&-3027453103440a_4^4-30040565308416a_4^6\\&+2789782868404321a_4^8+15116544000\lambda+170222083200a_4^2\lambda\\&-61921065003840a_4^4\lambda-614438279223480a_4^6\lambda\\&+57056708189236408a_4^8\lambda-133959025108800a_4^4\lambda^2\\&+30233088000\lambda^2+376625894400a_4^2\lambda^2\\&-1329260609063880a_4^6\lambda^2+123433679684533387a_4^8\lambda^2.\end{aligned}\tag{9.5.15}$$

Proof. Lemma 9.5.2 follows that if system (9.4.1) has no real singular point on the equator, then, $F_1=0$. We see from (9.5.12) that $G(2\pi)=0$ if and only if $f_7f_8\leqslant 0$ and $\alpha^2f_7^2-\beta^2f_8^2=0$. From (9.6.6) and (9.5.13), by using $F_1=0$ and $\Delta(\lambda,1)=0$, we have

$$\alpha^2f_7^2-\beta^2f_8^2=\frac{1}{30375a_4^2\gamma^3}F_2.\tag{9.5.16}$$

It follows the Lemma 9.5.5. □

Proof of Lemma 9.4.1. Lemma 9.5.2 and Lemma 9.5.5 follow that if system (9.4.1) has no real singular point on the equator Γ_∞ and $G(2\pi)=0$, then $F_1=F_2=\Delta(\lambda,1)=0,\ a_4\gamma\neq 0$. Thus, using Mathematica program, we obtain

$$\mathrm{Res}(\mathrm{Res}(F_1,F_2,\gamma),\Delta(\lambda,1),\lambda)=a_4^{36}H_0^6H_1^4=0,\tag{9.5.17}$$

where H_1 is given by (9.4.15),

$$H_0=-28800-1389200a_4^2+1293760a_4^4+5630539a_4^6.\tag{9.5.18}$$

Next we prove that $H_0 \neq 0$ by finding contradiction. Suppose that $H_0 = 0$, then

$$M_1 = \text{Res}(H_0, F_1, a_4) = 0, \quad M_2 = \text{Res}(H_0, F_2, a_4) = 0.$$

By using Mathematica, we know that $M_1,\ M_2$ are two polynomials with respect to $\gamma,\ \lambda$. The highest common factor of $\text{Res}(M_1, \Delta(\lambda,1), \lambda)$ and $\text{Res}(M_2, \Delta(\lambda,1), \lambda)$ is γ^{12}. Thus, when $H_0 = F_1 = F_2 = \Delta(\lambda,1) = 0$, we have $\gamma = 0$. This contradicts the conditions of Lemma 9.5.2. Thus Lemma 9.4.1 holds.

Lemma 9.5.6. *When $F_1 = F_2 = H_1 = \Delta(\lambda,1) = 0,\ a_4 \neq 0,\ \lambda$ is a polynomial of degree 14 of a_4^2 with rational coefficients, i.e., $\lambda = H_2$, where*

$$H_2 = \frac{1}{m}\sum_{k=0}^{14} b_k a_4^{2k} \tag{9.5.19}$$

and $m,\ b_0,\ b_1,\ \cdots,\ b_{14}$ are given by §9.7.

Proof. By using Mathematica, we see that $\sqrt{\text{Res}(F_1, F_2, \gamma)}$ is a polynomial in $a_4,\ \lambda$. Making this polynomial with $\Delta(\lambda,1)$ to do mutual division with respect to λ, we have $a_4^8(F_3 - F_4\lambda) = 0$. Hence, when $F_1 = F_2 = H_1 = \Delta(\lambda,1) = 0,\ a_4 \neq 0$, we have

$$\lambda = \frac{F_3}{F_4}, \tag{9.5.20}$$

where F_3, F_4 are two polynomials of a_4^2 with rational coefficients. The highest common factor of F_4 and H_1 is 1. By the polynomials theory, there exist two polynomials F_5, F_6 in a_4^2 with rational coefficients, such that

$$F_4F_5 + H_1F_6 \equiv 1. \tag{9.5.21}$$

(9.5.20) and (9.5.21) imply that

$$\lambda = \frac{F_3F_5}{F_4F_5} = F_3F_5, \tag{9.5.22}$$

when $F_1 = F_2 = H_1 = \Delta(\lambda,1) = 0,\ a_4 \neq 0$. Using $H_1 = 0$ to eliminate the terms of a_4 with power exponents larger than 28 in the expansion of F_3F_5. We obtain the conclusion of the Lemma 9.5.6. □

Remark 9.5.3. *Using Mathematica, we obtain*

$$\Delta(H_2, 1) = H_1F_7, \tag{9.5.23}$$

where F_7 is a polynomial of a_4^2. Thus, $H_1 = 0$ follows that $\Delta(H_2,1) = 0$.

Lemma 9.5.7. *Suppose that $\gamma \neq 0,\ H_1 = 0,\ \lambda = H_2$. Then, $F_1 = F_2 = 0$ if and only if $F_2 = 0$.*

Proof. By using Mathematica, it is easy to verify from $\Delta(H_2, 1) = 0$ that

$$48q_0^2F_1 + (1800a_4^2q_0\gamma^2 - q_1)F_2 = H_3(a_4, \lambda)\gamma^2 + H_4(a_4, \lambda), \tag{9.5.24}$$

where

$$\begin{aligned} q_0 =& 48600 - 99630a_4^2 - 4001235a_4^4 + 142751540a_4^6 + 194400\lambda \\ & -1590300a_4^2\lambda - 78937845a_4^4\lambda + 2920922183a_4^6\lambda \\ & -3877200a_4^2\lambda^2 - 170418660a_4^4\lambda^2 + 6319248215a_4^6\lambda^2, \\ q_1 =& -17496000 - 42573600a_4^2 + 11777807700a_4^4 \\ & -640423404750a_4^6 + 11135043548357a_4^8 - 69984000\lambda \\ & -1567836000a_4^2\lambda + 240482822400a_4^4\lambda \\ & -13097306243370a_4^6\lambda + 227734631514230a_4^8\lambda \\ & -3316464000a_4^2\lambda^2 + 520536499200a_4^4\lambda^2 \\ & -28333646277600a_4^6\lambda^2 + 492670092977045a_4^8\lambda^2. \end{aligned} \tag{9.5.25}$$

In (9.5.24), $H_3(a_4, \lambda)$ and $H_4(a_4, \lambda)$ are two polynomials in a_4, λ with rational coefficients, for which the highest power exponent of λ is 2. And we have

$$H_3(a_4, H_2) = H_1F_8, \quad H_4(a_4, H_2) = H_1F_9,$$

where $F_8,\ F_9$ are two polynomials in a_4^2. Thus when $H_1 = 0, \quad \lambda = H_2$, we have $H_3 = H_4 = 0$. Hence, when $H_1 = 0, \quad \lambda = H_2$, (9.5.24) follows that

$$48q_0^2F_1 + (1800a_4^2q_0\gamma^2 - q_1)F_2 = 0. \tag{9.5.26}$$

Again using Mathematica, we know that $\text{Res}(q_0, H_1, a_4)$ is a polynomial in λ, for which with $\Delta(\lambda, 1)$ are relatively prime. Thus, when $H_1 = \Delta(\lambda, 1) = 0,\ q_0 \neq 0$. Therefore, from (9.5.26), we obtain the conclusion of this Lemma 9.5.7. □

Remark 9.5.4. *Notice that all operations in the above lemmas are rational operations, by using Mathematica to polynomials of a_4, λ, γ with rational coefficients. So that, they have no any rounding error.*

The Proof of Theorem 9.4.1. Necessary: It follows from Lemma 9.4.1, Lemma 9.5.5 and Lemma 9.5.6 that if system (9.4.1) has no real singular point on the equator and $G(2\pi) = 0$, then $H_1 = F_2 = 0,\ \lambda = H_2$. We can find that $H_1|_{a_4^2=\zeta}$ has exact four positive zeros $a_4^2 = \zeta_k,\ k = 1, 2, 3, 4$, where

$$\begin{aligned} &\zeta_1 = 0.65989022\cdots, \quad \zeta_2 = 0.37330788\cdots, \\ &\zeta_3 = 0.03359415\cdots, \quad \zeta_4 = 0.01780119\cdots \end{aligned} \tag{9.5.27}$$

and $\zeta_1 = (a_4^*)^2$. Thus, in order to make $H_1 = 0$ and $\lambda = H_2$, it has to satisfy the following 4 conditions:

$$\begin{aligned} &C_1 : a_4^2 = \zeta_1,\ \lambda = \lambda_2,\\ &C_2 : a_4^2 = \zeta_2,\ \lambda = \lambda_2,\\ &C_3 : a_4^2 = \zeta_3,\ \lambda = \lambda_1,\\ &C_4 : a_4^2 = \zeta_4,\ \lambda = \lambda_1. \end{aligned} \tag{9.5.28}$$

From (9.5.26), when $H_1 = 0, \lambda = H_2$ and

$$\gamma^2 = \frac{q_1}{1800 a_4^2 q_0}, \tag{9.5.29}$$

we have $F_1 = 0$. For γ^2 given by (9.5.29), we obtain the following computational results:

$$\gamma^2 = \begin{cases} -0.08092370\cdots, & \text{if } a_4^2 = \zeta_1,\ \lambda = \lambda_2,\\ 0.01951104\cdots, & \text{if } a_4^2 = \zeta_2,\ \lambda = \lambda_2,\\ 0.22561073\cdots, & \text{if } a_4^2 = \zeta_3,\ \lambda = \lambda_1,\\ 190.45459957\cdots, & \text{if } a_4^2 = \zeta_4,\ \lambda = \lambda_1. \end{cases} \tag{9.5.30}$$

By (9.5.30), if one of the conditions $C_2,\ C_3,\ C_4$ is satisfied and γ^2 is defined by (9.5.29), then $F_1 = 0, \gamma^2 > 0$. Furthermore, (9.5.5) and (9.5.30) imply that

$$\alpha^2 + \beta^2 = \begin{cases} -0.94577861\cdots, & \text{if } a_4^2 = \zeta_2,\ \lambda = \lambda_2,\\ -12.58341028\cdots, & \text{if } a_4^2 = \zeta_3,\ \lambda = \lambda_1,\\ -0.27130003\cdots, & \text{if } a_4^2 = \zeta_4,\ \lambda = \lambda_1. \end{cases} \tag{9.5.31}$$

Clearly, $\alpha^2 + \beta^2$ is negative. By Lemma 9.5.2 and Remark 9.5.1, we obtain the necessary of this theorem.

Sufficiency: When the condition C_1 holds, we see from $F_2 = 0$, (9.6.6) and (9.5.13) that

$$\begin{aligned} &\gamma = 0.95518279\cdots,\\ &\alpha^2 = 0.02182871\cdots, \quad \beta^2 = 0.09886412\cdots,\\ &f_7 = 8243.65696363\cdots, \quad f_8 = -3873.59848007\cdots. \end{aligned} \tag{9.5.32}$$

Lemma 9.5.2, Lemma 9.5.5 and Lemma 9.5.7 imply the sufficiency of this theorem.

9.6 The Proofs of Lemma 9.4.2 and Lemma 9.4.3

When $\lambda = \lambda_2$ and a_4 is varied in a small neighborhood of $\pm a_4^*$, Theorem 9.4.1 implies that $Q_4(x, y)$ is positive definite. Thus, we see from Lemma 9.5.1 and Lemma 9.5.2 that $Q_4(x, y)$ has the factorization as (9.5.3), where α, β, γ satisfy

$$\alpha^2 = f_5, \quad \beta^2 = f_6, \quad F_1 = 0. \tag{9.6.1}$$

It implies that $\alpha,\ \beta, \gamma$ are continuous functions of a_4^2. When $\dfrac{\partial F_1}{\partial \gamma} \neq 0$, we can calculate $\dfrac{d\alpha}{da_4}$, $\dfrac{d\beta}{da_4}$, $\dfrac{d\gamma}{da_4}$ and by using (9.5.14), we can find $\dfrac{dG(2\pi)}{da_4}$. By (9.5.14) and (9.5.16), we have

$$G(2\pi) = H_5 F_2, \tag{9.6.2}$$

where

$$H_5 = \frac{-\pi}{27337500 a_4 \alpha\beta\gamma^3(\alpha^2 + 2\alpha\beta + \beta^2 + 4a_4^2\gamma^2)(\alpha f_7 - \beta f_8)}. \tag{9.6.3}$$

By (9.5.32), when $\lambda = \lambda_2$ and a_4 is varied in a small neighborhood of $\pm a_4^*$, we obtain $\alpha f_7 - \beta f_8 > 0$.

Lemma 9.6.1. *When $F_1 = H_1 = \Delta(\lambda, 1) = 0$, we have* $\dfrac{\partial F_1}{\partial \gamma} \neq 0$.

Proof. By using Mathematica, we know that $\mathrm{Res}\left(F_1, \dfrac{\partial F_1}{\partial \gamma}, \gamma\right)$ is a polynomial in a_4, λ. For the resultant of this polynomial and $\Delta(\lambda, 1)$ with respect to λ, it is a polynomial of a_4, which is prime with the polynomial H_1. Thus, this lemma holds. □

Proof of Lemma 9.4.2

Proof. When $\lambda = \lambda_2$ and a_4 is varied in a small neighborhood of $\pm a_4^*$, Lemma 9.5.2 implies that $F_1 = 0$. We see from Lemma 9.6.1 that

$$\frac{d\gamma}{da_4} = -\frac{\partial F_1}{\partial a_4} \Bigg/ \frac{\partial F_1}{\partial \gamma}. \tag{9.6.4}$$

(9.6.2) and (9.6.4) follow that when $\lambda = \lambda_2$, at $a_4 = \pm a_4^*$, we have

$$\frac{dG(2\pi)}{da_4} = H_5 \frac{dF_2}{da_4} = H_5 \left(\frac{\partial F_1}{\partial \gamma}\frac{\partial F_2}{\partial a_4} - \frac{\partial F_2}{\partial \gamma}\frac{\partial F_1}{\partial a_4} \right) \Bigg/ \frac{\partial F_1}{\partial \gamma}. \tag{9.6.5}$$

Using (9.5.32), (9.6.3) and (9.6.5), we obtain

$$\left.\frac{dG(2\pi)}{da_4}\right|_{a_4 = \pm a_4^*,\ \lambda = \lambda_2} = 17.00901058\cdots. \tag{9.6.6}$$

Then, Lemma 9.4.2 holds. □

Proof of Lemma 9.4.3

Proof. By using (9.4.14), (9.5.10) and (9.5.32) to compute, we have that Lemma 9.4.3 holds. □

9.7 Appendix

$$
\begin{aligned}
c_0 &= \frac{4004}{939195}(594561331+12159853840\lambda+26305992745\lambda^2)a_4^6,\\
c_1 &= -\frac{104}{455625}(-1074510+452297684a_4^2-21857085\lambda\\
&\quad +9250222925a_4^2\lambda-47246400\lambda^2+20011444745a_4^2\lambda^2)a_4^4,\\
c_2 &= -\frac{13}{31697831250}(2558716387449552+1013924199420034321a_4^2\\
&\quad +52332524917779600\lambda+20736808593910992256a_4^2\lambda\\
&\quad +113214060155936160\lambda^2+44860994915155953859a_4^2\lambda^2)a_4^6,\\
c_3 &= \frac{-26}{24962042109375}(-2505649578712461828360+14354474223509378504546 57a_4^2\\
&\quad -51245617654399769951760\lambda+293578127807206225981924 16a_4^2\lambda\\
&\quad -110862255246939470854440\lambda^2+635112521001243005054556 35a_4^2\lambda^2)a_4^8,\\
c_4 &= \frac{-13}{113295472094941275609059081852580522954503906 25}\\
&\quad \times(-318344602907139195790832810191361997798537630432 00000\\
&\quad +502802609324616398556238728655205908122384142668 00000a_4^2\\
&\quad +636502132469179681540092970784216750248951361751 6480000a_4^4\\
&\quad +572595610835705032560917201962397075270298231765 30770400a_4^6\\
&\quad -574628361037648308775123789593894035720634362368 8256986400a_4^8\\
&\quad +649846631231070052107661520445798338133894874027 033648917807a_4^{10}\\
&\quad -651171308515396915028594335554609014646473804062 080000\lambda\\
&\quad +102832246613996050439308046371190775640572904891 3680000a_4^2\lambda\\
&\quad +130177523057425237007563600201174967602039292098 802208000a_4^4\lambda\\
&\quad +117107424423942414653278170557609654613289928529 3528528000a_4^6\lambda\\
&\quad -117523160886860981291242154839583128481670374217 405899553280a_4^8\lambda\\
&\quad +132906823590317024038964373693252104925608688212 955976352100 00a_4^{10}\lambda\\
&\quad -140873976624935009804456898850163195818301073009 4080000\lambda^2\\
&\quad +222461873262328412108332974845634277652848155753 5040000a_4^2\lambda^2\\
&\quad +281619644127217978175922220374204092025950234649 115680000a_4^4\lambda^2\\
&\quad +253344458190021601738013570205253403334482840567 8997364000a_4^6\lambda^2\\
&\quad -254243841462741792188480943393426556725577682816 903219855200a_4^8\lambda^2\\
&\quad +287524102757874152741155924232342276866520511624 0496844307844 5a_4^{10}\lambda^2)a_4^2,\\
c_5 &= \frac{-104}{3375}(180+18317a_4^2-2025\lambda+373439a_4^2\lambda-6660\lambda^2+807620a_4^2\lambda^2)a_4^2,\\
c_6 &= \frac{52}{65221875}(-14640825890+277794403047a_4^2-297642955430\lambda
\end{aligned}
$$

$$+5682054029130a_4^2\lambda-643354561400\lambda^2+12292426157715a_4^2\lambda^2)a_4^4,$$

$$c_7 = \frac{-416}{184904015625}(-20183385331212507+29713827094181744 42a_4^2-412796851484198355\lambda$$

$$+60770804241575838161a_4^2\lambda-893026165172960340\lambda^2+13146857641817590 7831a_4^2\lambda^2)a_4^6,$$

$$c_8 = \frac{1664}{5116450048601050002949373428963765593 75}$$

$$\times(108860103505395151810421890871440964979720$$

$$+1741036818794145207611361594589037525718 15a_4^2$$

$$-10557357616993051187063587884723483133192010a_4^4$$

$$+15686143312926105403786154188555484685737 8251a_4^6$$

$$+231194208646927270158043264887992690997 5400\lambda$$

$$+362110884136236001661928956215431793967 8125a_4^2\lambda$$

$$-215846749820364988164980114966120038295542310a_4^4\lambda$$

$$+320815810217294758657087072819027049560738 5813a_4^6\lambda$$

$$+502739435764743188838616096657754842311 1200\lambda^2$$

$$+785173973247023647368396744401907895530 5700a_4^2\lambda^2$$

$$-46693039136553350301932891108919115922041 2440a_4^4\lambda^2$$

$$+694037897309232007465350684619532534642721 0480a_4^6\lambda^2),$$

$$c_9 = \frac{-104}{10125}(-2679+67652a_4^2-53790\lambda+1389194a_4^2\lambda-117060\lambda^2+3007532a_4^2\lambda^2)a_4^3,$$

$$c_{10} = \frac{-104}{1760990625}(152217275580+25959189233645a_4^2+3106111900860\lambda+530914901675948a_4^2\lambda$$

$$+6717064481280\lambda^2+114855466752411 5a_4^2\lambda^2)a_4^5,$$

$$c_{11} = \frac{-208}{924520078125}(-55324412125423200+2791171411083646 3963a_4^2$$

$$-1131514912456026120\lambda+57085117967070391 4920a_4^2\lambda$$

$$-2447869640821698600\lambda^2+12349514366650004 10505a_4^2\lambda^2)a_4^7,$$

$$c_{12} = \frac{1664}{83922571922178722673377097668578165151484375}$$

$$\times(-15583426332122373999281701317970676482828104000$$

$$-23891149373767040074636852963343188844405944500a_4^2$$

$$+13984948593728642804527774861856397824230587222 00a_4^4$$

$$-20793724322697563749127889076387518827736461552550a_4^6$$

$$-97070022249405258023382735078287506074716486086518 12a_4^8$$

$$+14290357053686912791257940479993811033933719235030495 63a_4^{10}$$

$$-30883855817164666692767602530702698198193211 2000\lambda$$

$$-48165772555616026842525276541874274240260329 6500a_4^2\lambda$$

$$
\begin{aligned}
&+2861044325838236724662844419125233995374909500900 0a_4^4\lambda\\
&-42527114994771807783087641910903817044100213908330 0a_4^6\lambda\\
&-198527896338605010832955718278602835685067281710568300a_4^8\lambda\\
&+2922668015264806810391100739864253410991098016405208 1012a_4^{10}\lambda\\
&-66512004691197497766409033642646119765621728000 0\lambda^2\\
&-1039909056612189336615325693624189379525811690000a_4^2\lambda^2\\
&+6189697081472272787221328789224751613318162683400 0a_4^4\lambda^2\\
&-92000991900334571871404544435120019866852923814255 0a_4^6\lambda^2\\
&-4294855133108026423605461611719603665250026180725810 80a_4^8\lambda^2\\
&+63227566203473001110005154354345428715380731301618997941a_4^{10}\lambda^2)a_4,
\end{aligned}
$$

$$
c_{13}=\frac{13}{33569028768871489069350839067431266060593750}\times
$$

$$
\begin{aligned}
&(-4040205092075217603862475032779911919096132362403840\\
&-6313720500235766190675753060783427329906110269870080a_4^2\\
&+3756514631927643332614253654874378653413612028427878 40a_4^4\\
&-5583567470484805513994084581188403174298092037520826368a_4^6\\
&+4693766296112163253393898683177878958618737696194359680a_4^8\\
&-2792638244410956748913064223853468387927415032496168833 20a_4^{10}\\
&+41508926021424039839269358246035302497876757577188942687 51a_4^{12}\\
&-82632819838886726884258829868258612352273189240012800\lambda\\
&-1291301092120391159576024420926522870809256440339456 00a_4^2\lambda\\
&+7682832680274963868683889895540070630201612896542924800a_4^4\lambda\\
&-1141952863134362875298317219011376937213623277821486325 76a_4^6\lambda\\
&+959970460403958407026807121829334615029672130721653248 00a_4^8\lambda\\
&-5711511926905246144144229474804367855377698640154882394240a_4^{10}\lambda\\
&+84894177226224001992666441721005962217714575442125870700816a_4^{12}\lambda\\
&-17876450372938340420340360198026951200213420703416320 0\lambda^2\\
&-279354249526317031945953522931124592655758960850944000a_4^2\lambda^2\\
&+16620662751872805587639269962363364226394854431066030080a_4^4\lambda^2\\
&-247044481054784797022208066454576065774106626431621836800a_4^6\lambda^2\\
&+207675300550152899924828413787343853013720775912715420800a_4^8\lambda^2\\
&-12356004739316987035148856963617616057492861928103728203880a_4^{10}\lambda^2
\end{aligned}
$$

$$+1836558987489021987584264257768209268751351385593823710309\,25a_4^{12}\lambda^2)a_4^3,$$

$$\begin{aligned}
m =&4176886109973281150700559361164637423680536853272279630426655719719555270000000 0\\
&4026974966083378410273075486824117322587531354095802103064493892783593792005066 6\\
&601400899676699463166114331559690403514600153445424055110355701173751520627 10,\\
b_0 =&1330447322416882501298976454000210660907874191353796463329152111818510042485 41\\
&3307444325520614416008284829887138178742285656954421717638695442199027329201580 1\\
&9411003918727592808217608879303912683124924141568534669810047705844257082500000 0,\\
b_1 =&193423433375474894684156726235133905616106288168037646397551765284572330 2\\
&3759345561204468294471204805699239539695787283866484205383766762921229991500727 6\\
&5450055602803114392713055900512548736710402698508094834731289270886348800000000 0,\\
b_2 =&-13040863486523610015045283297008755320568186012203932475171433985752306553 19\\
&6601517281280017980497856070130402051196784936706021645902683719235294674638559 9\\
&7285182703103163256601936132281147754381772100910013677939141612107054080000000 0,\\
b_3 =&10357297966501306381768117764140359046244019390290288467651198841287839018639 3\\
&2790802492163877860905680420052017632318874664289592249361428234983273190992099 2\\
&1206256524556242620167246023030964800207593082424239696335784088199447884000000 0,\\
b_4 =&-3460089866550368464115993377149611861085983900612920426835200721074487360806731\\
&3564022725319195470948867446053731721554298270374482288781537649319291518388748 6\\
&8169309942092422666333081342420720478294396406443164444824136615376686467900000 0,\\
b_5 =&4026848040553526466602633576220978109863265422985632175443883154088545529911080 7\\
&8453612120849065290622418429441790371071226149863160427549976569606511248897451 1\\
&135767476450346824987055328131196613915316856204172555350204783974586683544725000,\\
b_6 =&7803945403996117091438477204088099694013714135511292028821694599197938257356830 4\\
&9639671625696203505319816739775989822174327326660062653998349360694156194611103 9\\
&474243929675638380440146857795066515075830476750511754022709068188237611266218 75,\\
b_7 =&-30018245714023117860425186337115186414988172844838337563226238757426051319821589 9\\
&2001639240832450736414579424199305910065711705976794003727323397764109348175099 5\\
&4720292756354220205949030665213386290055325967782717513770536378224228427865250 0,\\
b_8 =&-86956585101975727542431225947371462620193556601890287484858665867838483629961326 1\\
&7858456426683447454963141973513281562672074630525206776058220289962388887921963 2\\
&1038804307040278123936001492407437780087816554406769721560419708213727859527500 0,
\end{aligned}$$

$$
\begin{aligned}
b_9 =&-351215895944591709669958029992291472177924645159423648585456748610439038468724173\\
&318768684164340778775208327283875535492960932960686006371749300218146207525283369\\
&19760268964065707016059698892649273598719535138119499434467174672913633103496500,\\
b_{10} =&11367344502181841718687033234026248991937789350678227823438105540371440927846939 46\\
&19694936062270258789013022233978282296183597310914589599475309120050094795980126\\
&39783601995465443638579964701322287663009660627694773248681384474422193923011110,\\
b_{11} =&175853783662598970767592041988059870583516571313116267086209183847016675137279632 6\\
&49545635386158117842156071811159020046528481582544856554269477832520957759143 48\\
&36976364035424660761708522472839528481644522771064844337498003568656527343117700,\\
b_{12} =&10902075245595493878748617113460724331028374339835164459659498956461532529389412 81\\
&87679233530160151431933195777501859766129895098057209186046743779570574334805468\\
&12913648850418781253711176096485628875358010200191617158012607460091024125124080,\\
b_{13} =&32265189884081291685483768001133685096674918018839981059000963320010040808745823 6\\
&41291757383448775615551522083856805273297884172070469517508008409636013814501143\\
&41821263609286230076704935433258418799768104803836971289380534968847770021460476,\\
b_{14} =&37581541338721502665448857904407239421659394133860592481017760576209065297804531\\
&14149594176635059879272781147765426412700898207260332189646547073028908102374075\\
&37509832177899639544708691594205316929398289424810073677303066024011374998517151.
\end{aligned}
$$

Bibliographical Notes

The center-focus problem at the origin of system $(E_3^{Z_2})$ was solved by [Sibirskii, 1965], furthermore, shortened expressions of the first five Liapunov constants of the origin had been given by [Liu Y.R., 1987] and [Liu Y.R. etc, 1989]. If system $(E_3^{Z_2})$ have two elementary focuses, the first result of there exist at least 12 small-amplitude limit cycles was proved by [Yu P. etc 2004; Yu P. etc 2005a; Yu P. etc 2005b]. In [Liu Y.R. etc, 2005], for a class of system $(E_3^{Z_2})$ with five free parameters and two elementary focuses, shortened expressions of the first six Liapunov constants were obtained, center-focus problem was solved, a new proof of there existing 12 limit circles for $(E_3^{Z_2})$ system was given.

In [Li J.B. etc, 2010] and [Liu Y.R. etc, 2011b], the authors considered the most general case that system $(E_3^{Z_2})$ is a six-parameter system which has two elementary focuses (or centers). For this system, shortened expressions of the first six Liapunov constants of two elementary focuses are obtained, center-focus problem was solved completely. The conclusion that there exist at most 12 small-amplitude limit cycles with the scheme $6\bigcup 6$ was proved. Because the system having two elementary weak focuses of order 6 keep to have a free parameter, in [Li J.B. etc, 2010] and [Liu Y.R. etc, 2011c], the authors obtained a larger limit cycle by the bifurcation from the equator. Therefore, the existence of 13 limit cycles with the scheme $1 \supset (6 \bigcup 6)$ had been proved.

By considering Poincar'e bifurcations from some period annuluses, i.e., investigating the numbers of some Abelian integrals, [Li C.Z. etc, 2009a] also obtained the existence of existence of 13 limit cycles for a symmetric cubic system with different disposition.

Chapter 10

Center-Focus Problem and Bifurcations of Limit Cycles for Three-Multiple Nilpotent Singular Points

Suppose that the origin of a real planar analytic system is an isolate singular point and at this point, the eigenvalues of the coefficient matrix of the linearized system are all zeros, but the coefficients of the linear terms are not all zero. In this case, the origin is called a nilpotent singular point, for which the study on the center-focus problem and bifurcations of limit cycles is more difficult. Recent years, the authors of this book made some new contributions on this study direction. In this chapter, we introduce their new results.

10.1 Criteria of Center-Focus for a Nilpotent Singular Point

Let the origin be a isolate singular point and it is nilpotent. Then, by making a proper linear transformation, a planar autonomous analytic system can be reduced to the following form:

$$\begin{aligned}\frac{dx}{dt} &= \Phi(x,y) = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j,\\ \frac{dy}{dt} &= \Psi(x,y) = \sum_{k+j=2}^{\infty} b_{kj}x^k y^j,\end{aligned} \tag{10.1.1}$$

where $\Phi(x,y), \Psi(x,y)$ are analytic in a neighborhood of the origin. Clearly, a nilpotent singular point is a multiple point. By the discussion in Section 1.2, we have

Proposition 10.1.1. *Suppose that $y = f(x)$ is the unique solution of $f(x) + \Phi(x, f(x)) = 0$ in a neighborhood of the origin of system* (10.1.1)*, where $f(0) = 0$. If we have*

$$\Psi(x, f(x)) = \alpha x^m + o(x^m), \quad \alpha \neq 0, \tag{10.1.2}$$

then the multiplicity of the origin is m.

The result in Section 1.2 tell us that a m-multiple singular point can be decomposed exactly into m complex elementary singular point.

On the basis of the discussion in [Amelikin etc, 1982] we have

Proposition 10.1.2. *The origin of system* (10.1.1) *is a focus* (*or an enter*) *if and only if*

$$\begin{aligned}&\Psi(x, f(x) = \alpha x^{2n-1} + o(x^{2n-1}), \quad \alpha \neq 0,\\&\left[\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right]_{y=f(x)} = \beta x^{n-1} + o(x^{n-1}),\\&\beta^2 + 4n\alpha < 0,\end{aligned} \tag{10.1.3}$$

where n is a positive integer.

In order to solve the center and focus problem, in [Amelikin etc, 1982], the authors made the transformation

$$x = (-\alpha)^{\frac{-1}{2n-2}} x_1, \quad y = (-\alpha)^{\frac{-1}{2n-2}} y_1 + f(x) \tag{10.1.4}$$

and introduces the Lyapunov polar coordinates

$$x_1 = rCs\vartheta, \quad y_1 = -r^n Sn\vartheta, \tag{10.1.5}$$

such that the concepts of focal values and the successor function of the origin for system (10.1.1) were defined.

The normal forms of system (10.1.1) are discussed in some paper. Form Theorem 19.10 in [Amelikin etc, 1982], we have

Theorem 10.1.1. *If condition* (10.1.3) *is satisfied, then there exist the following formal series*

$$\begin{aligned}u &= x + \sum_{k+j=2}^{\infty} a'_{kj} x^k y^j,\\v &= y + \sum_{k+j=2}^{\infty} b'_{kj} x^k y^j,\\\frac{dt}{d\tau} &= 1 + \sum_{k+j=1}^{\infty} c'_{kj} x^k y^j,\end{aligned} \tag{10.1.6}$$

such that by the above transformation, system (10.1.1) *is reduced to the following Liénard equations*

$$\frac{du}{d\tau} = v + F(u), \quad \frac{dv}{d\tau} = \alpha u^{2n-1}, \tag{10.1.7}$$

where

$$F(u) = \frac{1}{n}\beta u^n + o(u^n). \tag{10.1.8}$$

According to [Álvarez etc, 2006], we have

Theorem 10.1.2. *If condition* (10.1.3) *holds, then there exist the power series having the form* (10.1.6) *with non-zero convergence radius, such that by the above transformation, system* (10.1.1) *is reduced to the following Liénard equations*

$$\frac{du}{d\tau} = v, \quad \frac{dv}{d\tau} = \alpha u^{2n-1} + v \sum_{k=n-1}^{\infty} B_k u^k, \quad B_{n-1} = \beta. \tag{10.1.9}$$

In addition, for all k, B_{2k} play the role of Lyapunov constants of the origin of (10.1.1).

To find the coefficients $a'_{kj},\ b'_{kj},\ c'_{kj}$ of transformation (10.1.6) and B_k in (10.1.9), it is very tedious and hard work.

Especially, if system (10.1.1) is symmetric with the origin, which can be written as

$$\frac{dx}{dt} = y + \sum_{k=1}^{\infty} X_{2k+1}(x,y), \quad \frac{dy}{dt} = \sum_{k=1}^{\infty} Y_{2k+1}(x,y), \tag{10.1.10}$$

where $X_{2k+1}(x,y),\ Y_{2k+1}(x,y)$ are homogenous polynomial of degree $2k+1$ in x, y. Amelikin et al in Chapter 18 of [Amelikin etc, 1982] gave the following conclusion.

Theorem 10.1.3. *For system* (10.1.10), *if the conditions in* (10.1.3) *are satisfied, then there exists a positive definite formal power series $F(x,y)$ in a neighborhood of the origin, such that*

$$\left.\frac{dF}{dt}\right|_{(10.1.10)} = \sum_{k=[\frac{3n+1}{2}]}^{\infty} V_k x^{2k}. \tag{10.1.11}$$

Remark 10.1.1. *Generally, for* (10.1.1), *when the conditions in* (10.1.3) *hold, there always exists a positive definite Lyapunov functions in a neighborhood of the origin. However, the Lyapunov functions may be not a formal power series of x, y.*

Example 10.1.1. *For system*

$$\frac{dx}{dt} = y + \mu x^2 + \lambda x^3, \quad \frac{dy}{dt} = -2x^3 + 2\mu xy + 2\lambda\mu x^4, \tag{10.1.12}$$

we have the Lyapunov function

$$F = (x^4 + y^2)\exp\left(2\mu \arctan \frac{y}{x^2}\right), \tag{10.1.13}$$

such that

$$\frac{dF}{dt} = 4\lambda(1+\mu^2)x^6 \exp\left(2\mu \arctan \frac{y}{x^2}\right). \tag{10.1.14}$$

Clearly, F is not a formal power series of x, y.

10.2 Successor Functions and Focus Value of Three-Multiple Nilpotent Singular Point

In this section, we assume that $m = 3$ in (10.1.2). Then, Proposition 10.1.2 follows that

Proposition 10.2.1. *The origin of system* (10.1.1) *is a three-multiple focus*(*or center*), *if and only if*

$$b_{20} = 0,\ (2a_{20} - b_{11})^2 + 8b_{30} < 0. \tag{10.2.1}$$

Without loss of generality, under condition (10.2.1), we can assume that

$$a_{20} = \mu, \quad b_{20} = 0, \quad b_{11} = 2\mu, \quad b_{30} = -2. \tag{10.2.2}$$

Otherwise, by letting

$$2a_{20} + b_{11} = 4\mu, \quad (2a_{20} - b_{11})^2 + 8b_{30} = -16\lambda^2 \tag{10.2.3}$$

and making the transformation

$$\xi = \lambda x, \quad \eta = \lambda y + \frac{1}{4}(2a_{20} - b_{11})\lambda^2 x^2, \tag{10.2.4}$$

it gives rise to the mentioned case. When (10.2.2) holds, system (10.1.1) becomes the following:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{k+2j=3}^{\infty} a_{kj} x^k y^j = X(x, y), \\ \frac{dy}{dt} &= 2\mu xy - 2x^3 + \sum_{k+2j=4}^{\infty} b_{kj} x^k y^j = Y(x, y). \end{aligned} \tag{10.2.5}$$

By using the generalized polar coordinate transformation

$$x = r\cos\theta, \quad y = r^2 \sin\theta, \tag{10.2.6}$$

system (10.2.5) is changed as follows:

$$\begin{aligned} \frac{dr}{dt} &= \frac{r\cos\theta X(r\cos\theta, r^2\sin\theta) + \sin\theta Y(r\cos\theta, r^2\sin\theta)}{r(1+\sin^2\theta)} \\ &= \frac{r}{1+\sin^2\theta} \sum_{k=1}^{\infty} R_k(\theta) r^k, \\ \frac{d\theta}{dt} &= \frac{\cos\theta Y(r\cos\theta, r^2\sin\theta) - 2r\sin\theta X(r\cos\theta, r^2\sin\theta)}{r^2(1+\sin^2\theta)} \\ &= \frac{r}{1+\sin^2\theta} \sum_{k=0}^{\infty} Q_k(\theta) r^k, \end{aligned} \tag{10.2.7}$$

where

$$\begin{aligned}
R_1(\theta) &= \cos\theta[\sin\theta(1-2\cos^2\theta)+\mu(\cos^2\theta+2\sin^2\theta)],\\
Q_0(\theta) &= -2(\cos^4\theta+\sin^2\theta) \leqslant -\frac{3}{2}.
\end{aligned} \tag{10.2.8}$$

Thus, we have

$$\frac{dr}{d\theta} = \frac{r\sum_{k=1}^{\infty} R_k(\theta)r^{k-1}}{\sum_{k=0}^{\infty} Q_k(\theta)r^k} = \frac{R_1(\theta)}{Q_0(\theta)}\,r + o(r). \tag{10.2.9}$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta)h^k, \tag{10.2.10}$$

be a solution of (10.2.9) satisfying the initial condition $r|_{\theta=0} = h$ where h is small and

$$\nu_1(\theta) = \exp\int_0^{\theta} \frac{R_1(\theta)}{Q_0(\theta)}d\theta, \quad \nu_k(0) = 0, \quad k = 2, 3, \cdots. \tag{10.2.11}$$

Submitting (10.2.10), (10.2.11) into (10.2.9), $\nu_k(\theta)$ can be solved successively. Especially, we have

$$\nu_1(\theta) = \frac{1}{(\cos^4\theta+\sin^2\theta)^{\frac{1}{4}}\exp\left(\frac{1}{2}\mu\arctan\frac{\sin\theta}{\cos^2\theta}\right)}. \tag{10.2.12}$$

It follows that

$$\nu_1(k\pi) = 1, \quad k = 0,\ \pm 1,\ \pm 2, \cdots. \tag{10.2.13}$$

Proposition 10.2.2. *Under the transformation $r \to -r,\ \theta \to \pi - \theta$, equation* (10.2.9) *is invariant.*

Proposition 10.2.3. *When $|\theta| < 4\pi,\ |h| \ll 1$, we have*

$$-\tilde{r}(\pi-\theta, h) \equiv \tilde{r}(\theta, -\tilde{r}(\pi, h)) \tag{10.2.14}$$

Proof. By Proposition 10.2.2, $r = -\tilde{r}(\pi-\theta, h)$ is a solution of (10.2.9) satisfying $r|_{\theta=0} = -\tilde{r}(\pi, h)$. On the other hand, $r = \tilde{r}(\theta, -\tilde{r}(\pi, h))$ is a solution of (10.2.9) satisfying the same initial condition. Thus, by the uniqueness of solution, (10.2.14) holds. □

Moreover, (10.2.9) is invariant under the transformation $r \to -r, \theta \to \theta - 2\pi$. It implies that

Proposition 10.2.4. *When* $|\theta| < 4\pi,\ |h| \ll 1$,

$$\tilde{r}(\theta - 2\pi, h) \equiv \tilde{r}(\theta, \tilde{r}(-2\pi, h)), \tag{10.2.15}$$

$$\tilde{r}(\theta + 2\pi, h) \equiv \tilde{r}(\theta, \tilde{r}(2\pi, h)). \tag{10.2.16}$$

Because for all sufficiently small r, we have $\dfrac{d\theta}{dt} < 0$. Hence, we can define the successor functions of system (10.2.9) in a small neighborhood of the origin as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} \nu_k(-2\pi)h^k. \tag{10.2.17}$$

Proposition 10.2.3 and Proposition 10.2.4 follow the following result.

Theorem 10.2.1. *For any positive integer* m, $\nu_{2m+1}(-2\pi)$ *has the form*

$$\nu_{2m+1}(-2\pi) = \sum_{k=1}^{m} \xi_m^{(k)} \nu_{2k}(-2\pi), \tag{10.2.18}$$

where $\xi_m^{(k)}$ *is a polynomial of* $\nu_j(\pi),\ \nu_j(2\pi),\ \nu_j(-2\pi)\ (j = 2, 3, \cdots, 2m)$ *with rational coefficients.*

Definition 10.2.1. *Let* $f_k,\ g_k$ *be continuous and boundary functions with respect to* μ *and* $a_{ij}, b_{ij},\ k = 1, 2, \cdots$. *Suppose that for an integer* m, *there exists a group* $\left(\xi_1^{(m)}, \xi_2^{(m)}, \cdots, \xi_{m-1}^{(m)}\right)$ *of continuous and boundary functions in* μ, a_{ij}, b_{ij} *such that*

$$f_m = g_m + \left(\xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \cdots + \xi_{m-1}^{(m)} f_{m-1}\right). \tag{10.2.19}$$

Then, we say that f_m *and* g_m *are equivalent, written by* $f_m \sim g_m$. *If for any integer* m, *we have* $f_m \sim g_m$, *we say that the sequences of functions* $\{f_m\}$ *and* $\{g_m\}$ *are equivalent, written by* $\{f_m\} \sim \{g_m\}$.

Remark 10.2.1. *It is easy to see from Definition* 10.2.1 *that the following conclusions hold*:

(1) *The equivalent relationship of the sequences of functions is self-reciprocal, symmetric and transmissible.*

(2) *If for some integer* m, $f_m \sim g_m$, *then, when* $f_1 = f = \cdots = f_{m-1} = 0$, *we have* $f_m = g_m$.

(3) *The relationship* $f_1 \sim g_1$ *implies that* $f_1 = g_1$.

By Theorem 10.2.1, we have

Proposition 10.2.5. *For any positive integer* m, $\nu_{2m+1}(-2\pi)\ \sim\ 0$.

Remark 10.2.2. *We know from Theorem* 10.2.1 *that when* $k > 1$ *for the first non-zero* $\nu_k(-2\pi)$, k *is an even integer. This fact is different from the center-focus problem for the elementary singular points.*

Definition 10.2.2. *For system* (10.2.5):

(1) *For any positive integer* m, $\nu_{2m}(-2\pi)$ *is called the* m*-order focal value of the origin*;

(2) *If* $\nu_2(-2\pi) \neq 0$, *the origin is called* 1*-order weak focus; if there is an integer* $m > 1$, *such that* $\nu_2(-2\pi) = \nu_4(-2\pi) = \cdots = \nu_{2m-2}(-2\pi) = 0$, *but* $\nu_{2m}(-2\pi) \neq 0$, *then, the origin is called* m*-order weak focus*;

(3) *If for all positive integer we have* $\nu_{2m}(-2\pi) = 0$, *then, the origin is called a center.*

Theorem 10.2.1 follows that

Theorem 10.2.2. *If the origin of system* (10.2.5) *is a* m*-order weak focus, it is stable when* $\nu_{2m}(-2\pi) < 0$ *and unstable when* $\nu_{2m}(-2\pi) > 0$. *If the origin of system* (10.2.5) *is a nilpotent center, then in a neighborhood of the origin of* (10.2.5), *all solutions are periodic solutions.*

10.3 Bifurcation of Limit Cycles Created from Three-Multiple Nilpotent Singular Point

In this section, we consider the perturbed system of system (10.2.5)

$$\frac{dx}{dt} = \delta x + X(x,y) = \delta x + y + \mu x^2 + \sum_{k+2j=3}^{\infty} a_{kj} x^k y^j,$$
$$\frac{dy}{dt} = 2\delta y + Y(x,y) = 2\delta y + 2\mu xy - 2x^3 + \sum_{k+2j=4}^{\infty} b_{kj} x^k y^j. \tag{10.3.1}$$

Clearly, when $0 < |\delta| \ll 1$, in a neighborhood of the origin, there exist one elementary node at the origin and two complex singular points of system (10.2.5) at (x_1, y_1) and (x_2, y_2) where

$$x_{1,2} = \frac{-\delta}{(\mu \pm i)} + o(\delta), \quad y_{1,2} = \frac{\pm i\delta^2}{(\mu \pm i)^2} + o(\delta^2). \tag{10.3.2}$$

When $\delta \to 0$, those three singular points coincide to become a three-multiple singular point $O(0,0)$.

By generalized polar coordinate transformation (10.2.6), the linearized system

$$\frac{dx}{dt} = \delta x, \quad \frac{dy}{dt} = 2\delta y \tag{10.3.3}$$

becomes

$$\frac{dr}{dt} = \delta r = \frac{rR_0(\theta)}{1+\sin^2\theta}, \quad \frac{d\theta}{dt} = 0, \tag{10.3.4}$$

where

$$R_0(\theta) = (1+\sin^2\theta)\delta. \tag{10.3.5}$$

Nonlinear system (10.3.1) is transformed into

$$\frac{dr}{d\theta} = \frac{\sum_{k=0}^{\infty} R_k(\theta) r^k}{\sum_{k=0}^{\infty} Q_k(\theta) r^k} = \frac{R_0(\theta)}{Q_0(\theta)} + O(r), \tag{10.3.6}$$

by the same transformation (10.2.6).

Let

$$r = \tilde{r}(\theta, h, \delta) = \nu_0(\theta,\delta) + \sum_{k=1}^{\infty} \nu_k(\theta,\delta) h^k, \tag{10.3.7}$$

be a solution of system (10.3.6) satisfying the initial condition $r|_{\theta=0} = h$, where h is sufficiently small and

$$\nu_0(0,\delta) = 0, \quad \nu_1(0,\delta) = 1, \quad \nu_k(0,\delta) = 0, \quad k = 2, 3, \cdots. \tag{10.3.8}$$

Denote that

$$\begin{aligned} &\nu_0(\theta,\delta) = A_0(\theta)\delta + o(\delta), \\ &\nu_k(\theta,\delta) = \nu_k(\theta,0) + A_k(\theta)\delta + o(\delta), \quad k = 1, 2, \cdots. \end{aligned} \tag{10.3.9}$$

We have from (10.2.13), (10.3.7) and (10.3.9) that the successive function in a neighborhood of the origin of system (10.3.1) is as follows:

$$\begin{aligned} \Delta(h,\delta) &= \tilde{r}(-2\pi, h, \delta) - h \\ &= [A_0(-2\pi)\delta + o(\delta)] + [A_1(-2\pi)\delta + o(\delta)]\, h \\ &\quad + \sum_{k=2}^{\infty} [\nu_k(-2\pi, 0) + A_k(-2\pi)\delta + o(\delta)]\, h^k. \end{aligned} \tag{10.3.10}$$

Proposition 10.3.1. *In* (10.3.9),

$$A_0(\theta) = \frac{-\nu_1(\theta,0)}{2} \int_0^{\theta} \frac{(1+\sin^2\theta)d\theta}{\nu_1(\theta,0)(\cos^4\theta + \sin^2\theta)}, \tag{10.3.11}$$

where

$$\nu_1(\theta,0) = \frac{1}{(\cos^4\theta+\sin^2\theta)^{\frac{1}{4}} \exp\left(\dfrac{1}{2}\mu \arctan\dfrac{\sin\theta}{\cos^2\theta}\right)}. \tag{10.3.12}$$

From (10.3.11), *we have*

$$A_0(-2\pi) = \frac{1}{2}\int_0^{2\pi} \frac{(1+\sin^2\theta)d\theta}{\nu_1(\theta,0)(\cos^4\theta+\sin^2\theta)} > 0. \tag{10.3.13}$$

Proof. Submitting (10.3.7), (10.3.9) into (10.3.6), we have

$$\frac{dA_0(\theta)}{d\theta} = \frac{1}{Q_1(\theta)}\left[(1+\sin^2\theta)+\cos\theta R_1(\theta)A_0(\theta)\right]. \tag{10.3.14}$$

(10.3.8) and (10.3.14) follow the results of (10.3.11) and (10.3.12). □

We see from (10.3.7) and (10.3.9) that $\tilde{r}(\theta,h,\delta)=\nu_1(\theta,0)h+o(h)$ if $\delta=o(h)$, when $0<h\ll 1$, $|\theta|<4\pi$. Thus we have

Theorem 10.3.1. *If the origin of system* $(10.3.1)_{\delta=0}$ *is a center, then there is no periodic solution in a neighborhood of the origin of* (10.3.1) *when* $0<|\delta|\ll 1$.

Theorem 10.3.2. *Suppose that the origin of system* $(10.3.1)_{\delta=0}$ *is a m-order weak focus. Then, we have*

(1) *When* $\delta\nu_{2m}(-2\pi,0)>0$, *there is no periodic solution in a neighborhood of the origin.*

(2) *When* $\delta\nu_{2m}(-2\pi,0)<0$, *there exists a unique limit cycle which encloses the elementary node* $O(0,0)$ *with initial value*

$$r|_{\theta=0} = \left[\frac{-A_0(-2\pi)\delta}{\nu_{2m}(-2\pi,0)}\right]^{\frac{1}{2m}} + o(\delta^{\frac{1}{2m}}). \tag{10.3.15}$$

Proof. Under the condition of this theorem, we have from (10.3.10) that

$$\Delta(h,0)=\nu_{2m}(-2\pi,0)h^{2m}+o(h^{2m}),\quad \Delta(0,\delta)=A_0(-2\pi)\delta+o(\delta). \tag{10.3.16}$$

By using the implicit function theorem to solve

$$\Delta(h,\delta)=0,\quad \delta|_{h=0}=0, \tag{10.3.17}$$

we have uniquely the following result:

$$\delta = -\frac{\nu_{2m}(-2\pi,0)}{A_0(-2\pi)}h^{2m}+o(h^{2m}). \tag{10.3.18}$$

Thus, we obtain

$$h = \pm\left[\frac{-A_0(-2\pi)\delta}{\nu_{2m}(-2\pi,0)}\right]^{\frac{1}{2m}} + o(\delta^{\frac{1}{2m}}). \tag{10.3.19}$$

This means that Theorem 10.3.2 holds. □

Remark 10.3.1. *Under the condition of Theorem* 10.3.2*, when* δ *is varied from zero to nonzero, the origin* (*three-multiple nilpotent singular point*) *of system* $(10.3.1)_{\delta=0}$ *is split to become one elementary node at the origin and two complex singular points. A limit cycle enclosing the elementary node* $O(0,0)$ *is created. This an interesting bifurcation behavior which is different from usual Hopf bifurcation.*

Example 10.3.1. *Considering the system*

$$\frac{dx}{dt} = \delta x + y + +ax(x^4 + y^2)^k,$$
$$\frac{dy}{dt} = 2\delta y - 2x^3 + 2ay(x^4 + y^2)^k, \tag{10.3.20}$$

where k *is a positive integer. When* $\delta = a = 0$ *system* (10.3.20) *is a Hamiltonian system with the Hamiltonian*

$$H(x, y) = x^4 + y^2. \tag{10.3.21}$$

We see from

$$\left.\frac{dH}{dt}\right|_{(10.3.20)} = 4H(\delta + aH^k) \tag{10.3.22}$$

that there exists a unique limit cycle $aH^k = -\delta$ *enclosing elementary node* $O(0,0)$ *of system* (10.3.20) *when* $\delta a < 0$. □

In order to consider multiple bifurcations of the origin, we next discuss the perturbed system as follows:

$$\frac{dx}{dt} = \delta x + y + \mu(\varepsilon,\delta)x^2 + \sum_{k+2j=3}^{\infty} a_{kj}(\varepsilon,\delta)x^k y^j = X(x, y, \varepsilon, \delta),$$
$$\frac{dy}{dt} = 2\delta y + 2\mu(\varepsilon,\delta)xy - 2x^3 + \sum_{k+2j=4}^{\infty} b_{kj}(\varepsilon,\delta)x^k y^j = Y(x, y, \varepsilon, \delta), \tag{10.3.23}$$

where $X(x, y, \varepsilon, \delta)$, $Y(x, y, \varepsilon, \delta)$ are power series of $x, y, \varepsilon, \delta$ with real coefficients and non-zero convergence radius.

Under generalized polar coordinates (10.2.5), write the solution of (10.3.23) satisfying the initial condition $r|_{\theta=0} = h$ as follows:

$$r = \tilde{r}(\theta, h, \varepsilon, \delta) = \nu_0(\theta, \varepsilon, \delta) + \sum_{k=1}^{\infty} \nu_k(\theta, \varepsilon, \delta)h^k, \tag{10.3.24}$$

where $|h|$ is sufficiently small and

$$\nu_0(0, \varepsilon, \delta) = 0, \quad \nu_1(0, \varepsilon, \delta) = 1, \quad \nu_k(0, \varepsilon, \delta) = 0, \quad k = 2, 3, \cdots. \tag{10.3.25}$$

Denote that

$$\nu_k(\theta,\varepsilon,\delta)=\nu_k(\theta,\varepsilon,0)+\delta g_k(\theta,\varepsilon,\delta), \tag{10.3.26}$$

where $g_k(\theta,\varepsilon,\delta)$ are analytic for sufficient small ε,δ when $|\theta|<4\pi$. It is easy to prove that

Proposition 10.3.2. $\nu_0(\theta,\varepsilon,0)\equiv 0,\ g_0(\theta,0,0)=A_0(\theta),\nu_1(-2\pi,\varepsilon,0)\equiv 1.$ (10.3.27)

We now suppose that $\delta=\delta(\varepsilon)$ be a power series of ε with real coefficients and non-zero convergence radius, $\delta(0)=0$. Then successor function of $(10.3.23)_{\delta=\delta(\varepsilon)}$ in a small neighborhood of the origin is given by

$$\begin{aligned}\Delta(h,\varepsilon)=&\tilde{r}(-2\pi,h,\varepsilon,\delta(\varepsilon))-h\\=&\nu_0(-2\pi,\varepsilon,\delta(\varepsilon))+[\nu_1(-2\pi,\varepsilon,\delta(\varepsilon))-1]h+\sum_{k=2}^{\infty}\nu_k(-2\pi,\varepsilon,\delta(\varepsilon))h^k.\end{aligned} \tag{10.3.28}$$

We see from (10.3.26), 10.3.27 and (10.3.28) that

$$\begin{aligned}\Delta(h,\varepsilon)=&\delta(\varepsilon)\sum_{k=0}^{\infty}g_k(-2\pi,\varepsilon,\delta(\varepsilon))h^k\\&+\sum_{k=2}^{\infty}\nu_k(-2\pi,\varepsilon,0)h^k.\end{aligned} \tag{10.3.29}$$

Definition 10.3.1. *Suppose that for* $|\varepsilon)|\ll 1$, $h=h(\varepsilon)$ *is a continuous function of real variable* ε, *which takes its value in the complex field. If* $h(0)=0$ *and* $\Delta(h(\varepsilon),\varepsilon)\equiv 0$, *then* $h=h(\varepsilon)$ *is called a zero of* $\Delta(h,\varepsilon)$.

Similar to Proposition 10.2.3, we obtain

Proposition 10.3.3. *For* $\tilde{r}(\theta,h,\varepsilon,\delta)$ *defined by* (10.3.24), *when* $|\theta|<4\pi,\ |h|\ll 1$, *we have*

$$-\tilde{r}(\pi-\theta,h,\varepsilon,\delta)\equiv\tilde{r}(\theta,-\tilde{r}(\pi,h,\varepsilon,\delta),\varepsilon,\delta). \tag{10.3.30}$$

It follows that

Proposition 10.3.4. *For a sufficiently small* ε, *if* $h=h(\varepsilon)$ *is a real zero of the successor function* $\Delta(h,\varepsilon)$, *then so is* $h=-\tilde{r}(\pi,h(\varepsilon),\varepsilon,\delta(\varepsilon))$.

Proposition 10.3.4 tell us that there exists a pair of the real zeros of $\Delta(h,\varepsilon)$.

Definition 10.3.2. *Suppose that* k *is a positive integer and* $k\geqslant 2$. *If there exist* $k-1$ *power series* $\xi_2(\varepsilon),\ \xi_3(\varepsilon),\cdots,\ \xi_{k-1}(\varepsilon)$ *and* $\tilde{\nu}_k(\varepsilon)$ *in* ε *with non-zero convergence radius, such that*

$$\nu_k(-2\pi,\varepsilon,0)=\tilde{\nu}_k(\varepsilon)+\sum_{s=2}^{k-1}\xi_s(\varepsilon)\nu_s(-2\pi,\varepsilon,0), \tag{10.3.31}$$

then, we say that $\nu_k(-2\pi,\varepsilon,0)$ *and* $\tilde{\nu}_k(\varepsilon)$ *are analytic equivalence, denoted by*

$$\nu_k(-2\pi,\varepsilon,0) \simeq \tilde{\nu}_k(\varepsilon). \tag{10.3.32}$$

Furthermore, if for any positive integer $k>2$, $\nu_2(-2\pi,\varepsilon,0)=\tilde{\nu}_2(\varepsilon)$ *and* $\nu_k(-2\pi,\varepsilon,0) \simeq \tilde{\nu}_k(\varepsilon)$, *then we say that the two sequences of functions* $\{\nu_k(-2\pi,\varepsilon,0)\}$ *and* $\{\tilde{\nu}_k(\varepsilon)\}$ *are analytic equivalent, written by*

$$\{\nu_k(-2\pi,\varepsilon,0)\} \simeq \{\tilde{\nu}_k(\varepsilon)\}. \tag{10.3.33}$$

By Theorem 10.2.1, we have

Proposition 10.3.5. *For any positive integer* k,

$$\nu_{2k+1}(\varepsilon,0) \simeq 0. \tag{10.3.34}$$

Because for any k, $\nu_k(-2\pi,h,\varepsilon,\delta(\varepsilon))$ is power series of ε. By Theorem 10.2.1, when $0<|\varepsilon|\ll 1$, if $\Delta(h,\varepsilon)$ is not always equal to zero, the following conditions hold.

Condition 10.3.1. *There exist a natural number* N *and a positive integer* m, *such that*

$$\begin{aligned}
&A_0(-2\pi)\delta(\varepsilon)=\lambda_0\varepsilon^{l_0+N}+o(\varepsilon^{l_0+N}),\\
&\nu_2(-2\pi,\varepsilon,0)=\lambda_1\varepsilon^{l_1+N}+o(\varepsilon^{l_k+N}),\\
&\nu_{2k}(-2\pi,\varepsilon,0)\simeq\lambda_k\varepsilon^{l_k+N}+o(\varepsilon^{l_k+N}),\quad k=2,3,\cdots,m,
\end{aligned} \tag{10.3.35}$$

where $l_0,l_1,\cdots,l_{m-1}$ *are positive integers,* $\lambda_0,\lambda_1,\cdots,\lambda_m$ *are independent of* ε. *In addition,*

$$\begin{aligned}
&l_m=0,\quad \lambda_m\neq 0,\\
&\nu_{2m+2k}(-2\pi,\varepsilon,0)=O(\varepsilon^N),\quad k=1,2,\cdots.
\end{aligned} \tag{10.3.36}$$

Remark 10.3.2. (1) *In* (10.3.35) *and* (10.3.36), *if there is a* $s\in\{2,3,\cdots,m-1\}$, *such that* $\nu_{2s}(-2\pi,\varepsilon,0)\equiv 0$, *then we take* $\lambda_s=0$, $l_s=+\infty$. *If* $N=0$, *we take* $\varepsilon^N\equiv 1$.

(2) *If Condition* 10.3.1 *holds, the origin of system* $(10.3.23)_{\delta=\varepsilon=0}$ *is a* m*-order weak focus when* $N=0$ *and a center when* $N>0$.

Proposition 10.3.4 and the proof of Weierstrass preparation theorem imply the following conclusion.

Theorem 10.3.3. *Suppose that Condition* 10.3.1 *holds. Then, we have*

(1) *There exist positive number* h_0,ε_0, *such that function* $\Delta(h,\varepsilon)$ *has exactly* $2m$ *complex zeros (in the multiplicity) at the region* $\{|h|<h_0,\ |\varepsilon|<\varepsilon_0\}$.

(2) *When* $0 < |\varepsilon| \ll 1$, *in a small neighborhood of the origin, system* $(10.3.23)_{\delta=\delta(\varepsilon)}$ *has at most* m *limit cycles enclosing the elementary node* $O(0,0)$.

(3) *When* $0 < |\varepsilon| \ll 1$, *in a small neighborhood of the origin, system* $(10.3.23)_{\delta=0}$ *has at most* $m-1$ *limit cycles.*

Definition 10.3.3. *Suppose that Condition* 10.3.1 *holds. The function*

$$L(h,\varepsilon) = \lambda_0\varepsilon^{l_0} + \lambda_1\varepsilon^{l_1}h^2 + \cdots + \lambda_k\varepsilon^{l_k}h^{2k} + \cdots + \lambda_m\varepsilon^{l_m}h^{2m} \tag{10.3.37}$$

is called a quasi successor function of $(10.3.23)_{\delta=\delta(\varepsilon)}$ *in a neighborhood of the origin.*

Similar to Theorem 3.3.2, we have

Theorem 10.3.4. *Suppose that*

(1) *Condition* 10.3.1 *holds and* $0 < \varepsilon \ll 1$.

(2) *There exist exact* s *zeros having positive first term in all* $2m$ *zeros of* $L(h,\varepsilon)$.

(3) *These* s *positive first terms are different each other.*

Then,

(1) $\Delta(h,\varepsilon)$ *has exact* s *positive zeros.*

(2) *System* $(10.3.23)_{\delta=\delta(\varepsilon)}$ *has exactly* s *limit cycles in a neighborhood of the origin.*

For the case $0 < -\varepsilon \ll 1$, replacing ε by $-\varepsilon$ in the quasi successor function, we obtain the corresponding result.

Similar to the proof of Theorem 3.3.3, we have

Theorem 10.3.5. *Suppose that*

(1) *Condition* 10.3.1 *holds and* $0 < \varepsilon \ll 1$.

(2) *There exists a positive integer* d, *such that*

$$l_k = (m-k)d, \quad k = 0,\ 1,\ \cdots,\ m. \tag{10.3.38}$$

(3) $G(\eta) = \sum_{k=0}^{\infty} \lambda_k\eta^{2k}$ *has exact* m *simple positive zeros* $\eta_1,\ \eta_2,\ \cdots,\ \eta_m$.

Then

(1) $\Delta(h,\varepsilon)$ *has exact* m *positive zeros* $h = \eta_k\varepsilon^{\frac{d}{2}} + o(\varepsilon^{\frac{d}{2}})$, $k = 1,\ 2,\ \cdots,\ m$.

(2) *System* $(10.3.23)_{\delta=\delta(\varepsilon)}$ *has exactly* m *limit cycles in a sufficiently small neighborhood of the origin.*

The same as Theorem 3.3.4, we have

Theorem 10.3.6. *Suppose that Condition* 10.3.1 *holds. In addition,*

$$\begin{aligned} &\lambda_{k-1}\lambda_k < 0, \quad k = 1,\ 2,\ \cdots,\ m, \\ &l_{k-1} - l_k > l_k - l_{k+1}, \quad k = 1,\ 2,\ \cdots,\ m-1. \end{aligned} \tag{10.3.39}$$

Then, when $0 < \varepsilon \ll 1$, $\Delta(h, \varepsilon)$ *has exact* m *positive zeros* $h = h_k(\varepsilon)$ *given by*

$$h_k(\varepsilon) = \sqrt{\frac{-\lambda_{k-1}}{\lambda_k}}\ \varepsilon^{\frac{l_{k-1}-l_k}{2}} + o(\varepsilon^{\frac{l_{k-1}-l_k}{2}}), \quad k = 1,\ 2,\ \cdots, m. \tag{10.3.40}$$

Correspondingly, system $(10.3.23)_{\delta=\delta(\varepsilon)}$ *has exactly* m *limit cycles in a sufficiently small neighborhood of the origin.*

Theorem 10.3.7. *If* $\delta(\varepsilon) = \lambda_0\varepsilon + o(\varepsilon)$, $\lambda_0 \neq 0$ *and the origin is a center of system* $(10.3.23)_{\delta=\varepsilon=0}$, *then there is no limit cycle of system* $(10.3.23)_{\delta=\delta(\varepsilon)}$ *in a sufficiently small neighborhood of the origin when* $0 < |\varepsilon| \ll 1$.

Proof. Under Condition of Theorem 10.3.8, the quasi successor function of system $(10.3.23)_{\delta=\delta(\varepsilon)}$ is $L(h, \varepsilon) = \lambda_0\varepsilon$. Theorem 10.3.4 follows the conclusion of this theorem. □

Finally, we consider system

$$\begin{aligned}\frac{dx}{dt} &= \delta x + y + \mu(\boldsymbol{\gamma}, \delta)x^2 + \sum_{k+2j=3}^{\infty} a_{kj}(\boldsymbol{\gamma}, \delta)x^k y^j = X(x, y, \boldsymbol{\gamma}, \delta),\\ \frac{dy}{dt} &= 2\delta y + 2\mu(\boldsymbol{\gamma}, \delta)xy - 2x^3 + \sum_{k+2j=4}^{\infty} b_{kj}(\boldsymbol{\gamma}, \delta)x^k y^j = Y(x, y, \boldsymbol{\gamma}, \delta),\end{aligned} \tag{10.3.41}$$

where $\boldsymbol{\gamma} = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is a vector in the n-dimension parameters space. $X(x, y, \boldsymbol{\gamma}, \delta)$, $Y(x, y, \boldsymbol{\gamma}, \delta)$ are analytic with respect to $x, y, \delta, \boldsymbol{\gamma}$ in the region $\{|x| \ll 1,\ |y| \ll 1,\ |\delta| \ll 1,\ |\boldsymbol{\gamma} - \boldsymbol{\gamma}_0| \ll 1\}$.

Similar to Theorem 3.1.4, we have

Theorem 10.3.8. *Suppose that*

(1) *The origin of system* $(10.3.41)_{\boldsymbol{\gamma}=\boldsymbol{\gamma}_0,\ \delta=0}$ *is a* m*-order weak focus,* $n \geqslant m-1$.

(2) *There exist* $j_1, j_2, \cdots, j_{m-1} \in \{1, 2, \cdots, n\}$, *such that*

$$\left.\frac{D(\nu_2, \nu_4, \cdots, \nu_{2m-2})}{D(\gamma_{j_1}, \gamma_{j_2}, \cdots, \gamma_{j_{m-1}})}\right|_{\boldsymbol{\gamma}=\boldsymbol{\gamma}_0} \neq 0, \tag{10.3.42}$$

where ν_{2k} *is the* k*-th focus value of system* $(10.3.41)_{\delta=0}$.

Then, by choosing proper parameters in the region $\{0 < ||\boldsymbol{\gamma} - \boldsymbol{\gamma}_0|| \ll 1,\ 0 < |\delta| \ll 1\}$, *system* (10.3.41) *has exactly* m *limit cycles in a sufficiently small neighborhood of the origin.*

10.4 The Classification of Three-Multiple Nilpotent Singular Points and Inverse Integral Factor

On the basis of Theorem 10.1.1 and Theorem 10.1.2, we know that

Theorem 10.4.1. *For system* (10.2.5), *one can derive successively the terms of the following series with non-zero convergence radius:*

$$\begin{aligned} u(x,y) &= x + \sum_{\alpha+\beta=2}^{\infty} a'_{\alpha\beta} x^{\alpha} y^{\beta}, \\ v(x,y) &= y + \sum_{\alpha+\beta=2}^{\infty} b'_{\alpha\beta} x^{\alpha} y^{\beta}, \quad b'_{20} = -\mu, \\ \zeta(x,y) &= 1 + \sum_{\alpha+\beta=1}^{\infty} c'_{\alpha\beta} x^{\alpha} y^{\beta} \end{aligned} \tag{10.4.1}$$

such that by the transformation

$$u = u(x,y), \quad v = v(x,y), \quad dt = \zeta(x,y) d\tau, \tag{10.4.2}$$

system (10.2.5) *is reduced to the following Liénard equations*

$$\begin{aligned} \frac{du}{d\tau} &= v + 2\mu u^2 + \sum_{k=1}^{\infty} A_k u^{4k} + \sum_{k=1}^{\infty} B_k u^{4k+2} + \sum_{k=1}^{\infty} C_k u^{2k+1} = U(u,v), \\ \frac{dv}{d\tau} &= -2(1+\mu^2) u^3 = V(u,v), \end{aligned} \tag{10.4.3}$$

In addition, the origin of system (10.2.5) *is a center if and only if for all* $k, C_k = 0$.

Definition 10.4.1. *Write that* $B_0 = 2\mu$.

(1) *If* $\mu \neq 0$, *the origin of system* (10.2.5) *is called a three-multiple nilpotent singular point of* 0*-class.*

(2) *If* $\mu = 0$ *and there exists a positive integer* s, *such that* $B_0 = B_1 = \cdots = B_{s-1} = 0$, *but* $B_s \neq 0$, *the origin of system* (10.2.5) *is called a three-multiple nilpotent singular point of* s*-class.*

(3) *If* $\mu = 0$ *and for all positive integers* s, $B_s = 0$, *the origin of system* (10.2.5) *is called a three-multiple nilpotent singular point of* ∞*-class.*

If the origin of system (10.2.5) is a center, system (10.4.3) has the form

$$\begin{aligned} \frac{du}{d\tau} &= v + 2\mu u^2 + \sum_{k=1}^{\infty} A_k u^{4k} + \sum_{k=1}^{\infty} B_k u^{4k+2}, \\ \frac{dv}{d\tau} &= -2(1+\mu^2) u^3. \end{aligned} \tag{10.4.4}$$

The vector field defined by system (10.4.4) is symmetry with respect to v axis. By the transformation

$$w = u^2, \quad v = v, \quad d\tau = -\frac{d\tau'}{2u} \tag{10.4.5}$$

system (10.4.4) can reduce to

$$\begin{aligned} \frac{dw}{d\tau'} &= -v - 2\mu w - \sum_{k=1}^{\infty} A_k w^{2k} - \sum_{k=1}^{\infty} B_k w^{2k+1}, \\ \frac{dv}{d\tau'} &= (1+\mu^2)w. \end{aligned} \tag{10.4.6}$$

Obviously, in the wOv phase plane, the origin of system (10.4.6) is an elementary singular point (focus or center). For any positive integer k, let $\nu_{2k+1}^*(2\pi)$ be the k-th focal value of the origin of system $(10.4.6)_{\mu=0}$, it is easy to see that

$$\{\ \nu_{2k+1}^*(2\pi)\ \} \sim \left\{-2\ \frac{(2k+1)!!}{(2k+2)!!}\ B_k\pi\right\}. \tag{10.4.7}$$

If $\mu \neq 0$, the origin of system (10.4.6) is a rough focus. In this case, Theorem 2.1 in [Qin Y.X., 1985] follows that by using an analytic transformation

$$\xi = w + h.o.t., \quad \eta = v + \mu w + h.o.t., \tag{10.4.8}$$

system $(10.4.6)_{\mu\neq 0}$ can be reduced to a linear system

$$\frac{d\xi}{d\tau'} = -\mu\xi - \eta, \quad \frac{d\eta}{d\tau'} = \xi - \mu\eta. \tag{10.4.9}$$

When $\mu = 0$, Theorem 4.1 in [Amelikin etc, 1982] tell us that for system $(10.4.6)_{\mu=0}$, one can construct successively the terms of a formal series of w, v

$$\xi = w + h.o.t., \quad \eta = v + h.o.t., \tag{10.4.10}$$

such that system $(10.4.6)_{\mu=0}$ becomes a normal form as follows:

$$\begin{aligned} \frac{d\xi}{d\tau'} &= -\eta + \frac{1}{2}\sum_{k=1}^{\infty}(\gamma_k\xi - \tau_k\eta)(\xi^2+\eta^2)^k, \\ \frac{d\eta}{d\tau'} &= \xi + \frac{1}{2}\sum_{k=1}^{\infty}(\tau_k\xi + \gamma_k\eta)(\xi^2+\eta^2)^k, \end{aligned} \tag{10.4.11}$$

where

$$\{\ \nu_{2k+1}^*(2\pi)\ \} \sim \{\pi\gamma_k\}, \tag{10.4.12}$$

and when all $\gamma_k = 0$, ξ, η are power series of w, v with non-zero convergent radius, τ_k is the k-th period constant of the origin of system (10.4.11).

When $\mu = 0$, (10.4.7) and (10.4.12) imply that

$$\{\gamma_k\} \sim \left\{ -2\,\frac{(2k+1)!!}{(2k+2)!!}\,B_k \right\} \sim \left\{ \frac{1}{\pi}\,\nu^*_{2k+1}(2\pi) \right\}. \tag{10.4.13}$$

We see from (10.4.1), (10.4.5), (10.4.8) and (10.4.10) that when $x = r\cos\theta, y = r^2 \sin\theta$,

$$\begin{aligned} &u = x + o(r), \quad v = y - \mu x^2 + o(r^2), \quad w = x^2 + o(r^2),\\ &\xi = x^2 + o(r^2), \quad \eta = y + o(r^2). \end{aligned} \tag{10.4.14}$$

It is easy to prove that

Theorem 10.4.2. *If the origin of system* (10.2.5) *is* 0*-class, then, system* (10.4.9) *has an analytic inverse integral factor*

$$\xi^2 + \eta^2 = x^4 + y^2 + o(r^4) \tag{10.4.15}$$

and a first integral

$$(\xi^2 + \eta^2)\exp\left(-\mu \arctan\frac{\eta}{\xi}\right). \tag{10.4.16}$$

If the origin of system (10.2.5) *is s-class, then, the origin of system* (10.4.11) *is a s order weak focus and* (10.4.11) *has a formal inverse integral factor*

$$\mathcal{D}_s = H^{s+1}\left(1 + \sum_{k=1}^{\infty}\frac{\gamma_{s+k}}{\gamma_s}H^k\right) = \mathcal{H}^{s+1} \tag{10.4.17}$$

and a first integral

$$\frac{f(H)}{1 + s\gamma_s f(H)\arctan\frac{\eta}{\xi}}. \tag{10.4.18}$$

In addition, if the origin of system (10.2.5) *is ∞-class, the origin of system* (10.4.11) *is a center, and* (10.4.11) *has a formal inverse integral factor*

$$\mathcal{D}_\infty = 1 + \frac{1}{2}\sum_{k=1}^{\infty}\tau_k H^k \tag{10.4.19}$$

and a first integral H, where

$$\begin{aligned} &H = \xi^2 + \eta^2 = x^4 + y^2 + o(r^4),\\ &\mathcal{H} = H\left(1 + \sum_{k=1}^{\infty}\frac{\gamma_{s+k}}{\gamma_s}H^k\right)^{\frac{1}{s+1}} = x^4 + y^2 + o(r^4),\\ &f(H) = \left(-s\int\frac{\mathcal{D}_\infty}{\mathcal{H}^{s+1}}\,dH\right)^{-1} = H^s + o(H^s). \end{aligned} \tag{10.4.20}$$

Remark 10.4.1. *By Theorem* 10.4.2, *if the origin of system* (10.2.5) *is* ∞*-class, then, when the origin is a center, for any natural number* s, *system* (10.4.11) *has an analytic inverse integral factor* $(\xi^2+\eta^2)^{s+1}\mathcal{D}_\infty$.

In order to obtain the inverse integral factor of system (10.4.4) when the origin is a center, we need to use the following result which can be proved by the chain rule for the differentiation. Write that

$$g = -\frac{\partial(\xi,\eta)}{\partial(w,v)}\,\frac{\partial(w,v)}{\partial(u,v)}\,\frac{d\tau}{d\tau'}, \tag{10.4.21}$$

$$\mathcal{M}(u,v) = \begin{cases} (\xi^2+\eta^2)g^{-1}, & \text{if } s=0,\\ \mathcal{H}g^{\frac{-1}{s+1}}, & \text{if } 0<s<\infty,\\ (\xi^2+\eta^2)(\mathcal{D}_\infty g^{-1})^{\frac{1}{s+1}}, & \text{if } s=\infty, \end{cases} \tag{10.4.22}$$

$g,\mathcal{M}$ are formal series of u,v, and $g=1+h.o.t.$. Moreover, when $x=r\cos\theta,\ y=r^2\sin\theta$,

$$\mathcal{M}(u,v) = u^4+(v+\mu u^2)^2+o(r^4) = x^4+y^2+o(r^4). \tag{10.4.23}$$

By Theorem 10.4.2 and Proposition 1.1.4, we have

Theorem 10.4.3. *If the origin of system* (10.4.4) *is* 0*-class, then, in a neighborhood of the origin, system* (10.4.4) *has an analytic inverse integral factor* $\mathcal{M}$ *and a first integral* (10.4.16).

If the origin of system (10.4.4) *is* s*-class, in a neighborhood of the origin, system* (10.4.4) *has a formal inverse integral factor* $\mathcal{M}^{s+1}$ *and a first integral* (10.4.18).

If the origin of system (10.4.4) *is* ∞*-class, in a neighborhood of the origin, system* (10.4.4) *has an analytic inverse integral factor* $D_\infty g^{-1}$*and a first integral* $\xi^2+\eta^2$. *In addition, for any natural number* s, $\mathcal{M}^{s+1}$ *is also an inverse integral factor of system* (10.4.4).

We see from Proposition 1.1.4, if the origin of (10.2.5) is a center, then for systems (10.2.5) and (10.4.4), we have

$$\frac{\partial(u,v)}{\partial(x,y)}\frac{dt}{d\tau} = 1+h.o.t., \tag{10.4.24}$$

Theorem 10.4.3 implies the following result.

Theorem 10.4.4. *Denote that*

$$M = x^4+y^2+o(r^4) = x^4+y^2+\sum_{k+2j=5}^{\infty} c_{kj}x^ky^j. \tag{10.4.25}$$

If the origin of system (10.2.5) *is* 0*-class, then, when the origin of system* (10.2.5) *is a center, in a neighborhood of the origin, system* (10.2.5) *has an analytic inverse integral factor* M *and a first integral* (10.4.16).

If the origin of system (10.2.5) *is* s*-class, then, when the origin of system* (10.2.5) *is a center, in a neighborhood of the origin, system* (10.2.5) *has an analytic inverse integral factor* M^{s+1} *and a first integral* (10.4.18).

If the origin of system (10.2.5) *is* ∞*-class, then, when the origin of system* (10.2.5) *is a center, in a neighborhood of the origin, system* (10.2.5) *has an analytic inverse integral factor*

$$M_\infty = 1 + h.o.t. \tag{10.4.26}$$

and an analytic first integral $\xi^2 + \eta^2$.

Theorem 10.4.5. *If the origin of system* (10.2.5) *is* ∞*-class and it is a center, then, for any natural number* s*, system* (10.2.5) *has the inverse integral factor* M^{s+1}*, where* $M = x^4 + y^2 + o(r^4)$ *is a power series of* x, y *with non-zero convergent radius.*

Proof. Under the conditions of Theorem 10.4.5, Theorem 10.4.4 follows that system (10.2.5) has the inverse integral factor $M_\infty = 1 + h.o.t.$ and a first integral $\xi^2 + \eta^2 = x^4 + y^2 + o(r^4)$. Hence, for any natural number s, $M_\infty(\xi^2 + \eta^2)^{s+1}$ is the inverse integral factor of system (10.2.5). Hence Theorem 10.4.5 holds. □

Theorem 10.4.6. *If system* (10.2.5) *has a inverse integral factor* M_1^{s+1}*, where* $M_1(x, y) = x^4 + y^2 + o(r^4)$ *is a power series of* x, y*,* $s \in N$*, then* (10.2.5) *must be* s*-class or* ∞*-class.*

Proof. Suppose that system (10.2.5) is k-class which is different from natural number s, by Theorem 10.4.4, system (10.2.5) has inverse integral factor M_2^{k+1}, where $M_2(x, y) = x^4 + y^2 + o(r^4)$ is a power series of x, y. Then system (10.2.5) has a first integral $F(x, y) = M_1^{s+1} M_2^{-(k+1)}$. Therefore, we have that the origin of system (10.2.5) is a center and (10.2.5) is ∞-class. it is contradict with condition. □

10.5 Quasi-Lyapunov Constants For the Three-Multiple Nilpotent Singular Point

By Definition 10.4.1, the origin of system (10.2.5) is 0-class when $\mu \neq 0$.

Proposition 10.5.1. *If the origin of system* (10.2.5) *is* s*-class, then,*

$$s\mu = 0. \tag{10.5.1}$$

Theorem 10.5.1. *If the origin of system* (10.2.5) *is* s*-class or* ∞*-class, one can construct successively the terms of the formal power series* $M(x, y) = x^4 + y^2 + o(r^4)$, *such that*

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right)+\frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)$$
$$=\frac{1}{M^{s+2}}\sum_{m=1}^{\infty}(2m-4s-1)\lambda_m\left[x^{2m+4}+o(r^{2m+4})\right], \tag{10.5.2}$$

i.e.,

$$\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)M-(s+1)\left(\frac{\partial M}{\partial x}X+\frac{\partial M}{\partial y}Y\right)$$
$$=\sum_{m=1}^{\infty}\lambda_m\left[(2m-4s-1)x^{2m+4}+o(r^{2m+4})\right]. \tag{10.5.3}$$

Proof. If the origin of system (10.2.5) is s-class or ∞-class, Theorem 10.4.3 follows that system (10.4.4) has inverse integral factor $\mathcal{M}^{s+1}$ in a neighborhood of the origin, where $\mathcal{M}$ given by (10.4.23) is a power series of u, v. Thus, for system (10.4.3), we have

$$\frac{\partial}{\partial u}\left(\frac{U}{\mathcal{M}^{s+1}}\right)+\frac{\partial}{\partial v}\left(\frac{V}{\mathcal{M}^{s+1}}\right)=\frac{\partial}{\partial u}\left(\frac{1}{\mathcal{M}^{s+1}}\sum_{m=1}^{\infty}C_m u^{2m+1}\right)$$
$$=\frac{1}{\mathcal{M}^{s+2}}\sum_{m=1}^{\infty}C_m[(2m-4s-3)(1+\mu^2)u^{2m+4}$$
$$+2\mu(2m-2s-1)u^{2m+2}v+(2m+1)u^{2m}v^2+o(r^{2m+4})], \tag{10.5.4}$$

i.e.,

$$\left(\frac{\partial U}{\partial u}+\frac{\partial V}{\partial v}\right)\mathcal{M}-(s+1)\left(\frac{\partial \mathcal{M}}{\partial u}U+\frac{\partial \mathcal{M}}{\partial v}V\right)$$
$$=\sum_{m=1}^{\infty}C_m[(2m-4s-3)(1+\mu^2)u^{2m+4}$$
$$+2\mu(2m-2s-1)u^{2m+2}v+(2m+1)u^{2m}v^2+o(r^{2m+4})]. \tag{10.5.5}$$

Write that

$$\mathcal{M}^*=\frac{1}{s+1}\sum_{m=1}^{\infty}C_m u^{2m+1}v, \tag{10.5.6}$$

then,

$$\left(\frac{\partial U}{\partial u}+\frac{\partial V}{\partial v}\right)\mathcal{M}^*-(s+1)\left(\frac{\partial \mathcal{M}^*}{\partial u}U+\frac{\partial \mathcal{M}^*}{\partial v}V\right)$$
$$=\sum_{m=1}^{\infty}C_m\Big[2(1+\mu^2)u^{2m+4}-\frac{2\mu(2ms+2m+s-1)}{s+1}u^{2m+2}v$$
$$-(2m+1)u^{2m}v^2+o(r^{2m+4})\Big]. \tag{10.5.7}$$

Let

$$\tilde{\mathcal{M}} = \mathcal{M} + \mathcal{M}^* = u^4 + (v + \mu u^2)^2 + o(r^4). \tag{10.5.8}$$

By (10.5.5), (10.5.7) and (10.5.8), we have

$$\begin{aligned}
&\left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right)\tilde{\mathcal{M}} - (s+1)\left(\frac{\partial \tilde{\mathcal{M}}}{\partial u}U + \frac{\partial \tilde{\mathcal{M}}}{\partial v}V\right)\\
&= \sum_{m=1}^{\infty} C_m\Big[(2m-4s-1)(1+\mu^2)u^{2m+4}\\
&\quad -\frac{4s\mu(s+2)}{s+1}u^{2m+2}v + o(r^{2m+4})\Big].
\end{aligned} \tag{10.5.9}$$

By (10.5.1) and (10.5.9), we have

$$\begin{aligned}
&\left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right)\tilde{\mathcal{M}} - (s+1)\left(\frac{\partial \tilde{\mathcal{M}}}{\partial u}U + \frac{\partial \tilde{\mathcal{M}}}{\partial v}V\right)\\
&= \sum_{m=1}^{\infty} C_m\Big[(2m-4s-1)(1+\mu^2)u^{2m+4} + o(r^{2m+4})\Big].
\end{aligned} \tag{10.5.10}$$

It implies that

$$\begin{aligned}
&\frac{\partial}{\partial u}\left(\frac{U}{\tilde{\mathcal{M}}^{s+1}}\right) + \frac{\partial}{\partial v}\left(\frac{V}{\tilde{\mathcal{M}}^{s+1}}\right)\\
&= \frac{1}{\tilde{\mathcal{M}}^{s+2}}\sum_{m=1}^{\infty}(2m-4s-1)(1+\mu^2)C_m\left[u^{2m+4} + o(r^{2m+4})\right].
\end{aligned} \tag{10.5.11}$$

Denote that

$$M(x,y) = \tilde{\mathcal{M}}\left[\zeta\,\frac{\partial(u,v)}{\partial(x,y)}\right]^{\frac{-1}{s+1}} = x^4 + y^2 + o(r^4), \tag{10.5.12}$$

where $u = u(x,y), v = v(x,y), \zeta = w(x,y)$ are given by (10.4.1). (10.5.11) and Proposition 1.1.3 follow that

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)\\
&= \frac{\partial(u,v)}{\partial(x,y)}\left[\frac{\partial}{\partial u}\left(\frac{U}{\tilde{\mathcal{M}}^{s+1}}\right) + \frac{\partial}{\partial v}\left(\frac{V}{\tilde{\mathcal{M}}^{s+1}}\right)\right]\\
&= \frac{\partial(u,v)}{\partial(x,y)}\frac{1}{\tilde{\mathcal{M}}^{s+2}}\sum_{m=1}^{\infty}(2m-4s-1)(1+\mu^2)C_m\left[u^{2m+4} + o(r^{2m+4})\right].
\end{aligned} \tag{10.5.13}$$

From (10.4.1), (10.5.12) and (10.5.13), we obtain (10.5.2). Since $\lambda_m = (1+\mu^2)C_m$, it follows the conclusion of Theorem 10.5.1. □

Theorem 10.5.2. *For system* (10.2.5), *if there exists a natural number* s *and a formal series* $M(x,y) = y^2 + x^4 + o(r^4)$, *such that* (10.5.2) *holds, then*

$$\{\nu_{2m}(-2\pi)\} \sim \{\sigma_m \lambda_m\}. \tag{10.5.14}$$

where

$$\sigma_m = \frac{1}{2}\int_0^{2\pi} \frac{(1+\sin^2\theta)\cos^{2m+4}\theta d\theta}{(\cos^4\theta + \sin^2\theta)^{\frac{2m+7}{4}} \exp\left(\frac{2m-1}{2}\mu \arctan\frac{\sin\theta}{\cos^2\theta}\right)} > 0. \tag{10.5.15}$$

The proof of Theorem 10.5.2 is given in next section Section 10.6.

Definition 10.5.1. *If there exist a natural number* s *and a formal series* $M(x,y) = y^2+x^4+o(r^4)$, *such that* (10.5.2) *holds, then,* λ_m *is called the* m*-th quasi-Lyapunov constant of the origin of system* (10.2.5), $m = 1, 2, \cdots$.

We see from Theorem 10.5.1 and Theorem 10.5.2 that

Theorem 10.5.3. *If the origin of system* (10.2.5) *is* s*-class, then, the origin of system* (10.2.5) *is a center if and only if there is an inverse integral factor* M^{s+1}, *where* $M = x^4 + y^2 + o(r^4)$ *is a power series of* x, y.

Theorem 10.5.4. *If the origin of system* (10.2.5) *is* ∞*-class, then, the origin of system* (10.2.5) *is a center if and only if there exist a natural number* s *and an inverse integral factor* M^{s+1}, *where* $M = x^4 + y^2 + o(r^4)$ *is a power series of* x, y.

10.6 Proof of Theorem 10.5.2

Let $\Omega(h, h')$ be a region (see Fig.10.6.1) enclosed by the following four oriented curves $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ given by

$$\begin{aligned}
&\Gamma_1: \quad r = \tilde{r}(\theta, h), \quad \theta: \ 0 \to -2\pi,\\
&\Gamma_2: \quad y = 0, \quad x: \quad \tilde{h} \to \tilde{h}',\\
&\Gamma_3: \quad r = \tilde{r}(\theta, h'), \quad \theta: \ -2\pi \to 0,\\
&\Gamma_4: \quad y = 0, \quad x: \ h' \to h,
\end{aligned} \tag{10.6.1}$$

where $0 < h' - h \ll h \ll 1$ and

$$\tilde{h} = \tilde{r}(-2\pi, h), \quad \tilde{h}' = \tilde{r}(-2\pi, h'). \tag{10.6.2}$$

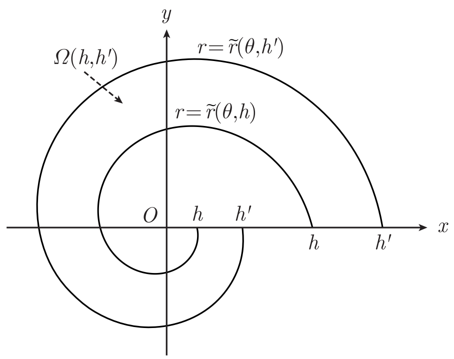


Fig.10.6.1 The region $\Omega(h,h')$

Let $M(x,y)=x^4+y^2+o(r^4)$ be a formal series of x,y, s be a natural number. Define that.

$$I=\lim_{h'\to h}\frac{1}{h'-h}\iint\limits_{\Omega(h,h')}\left[\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right)+\frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)\right]dxdy. \tag{10.6.3}$$

We have the following conclusion.

Lemma 10.6.1.

$$I=\frac{2}{h^{4s+2}}\sum_{m=1}^{\infty}(2m-4s-1)\ \nu_{2m}(-2\pi)g_m(h)h^{2m}, \tag{10.6.4}$$

where for all m, $g_m(h)$ is a formal series of h, and $g_m(0)=1$.

Proof. Notice that the integrand function of (10.6.1) has no singularity in the region $\Omega(h,h')$. By using known Green formulas, we obtain

$$I=\lim_{h'\to h}\frac{1}{h'-h}\sum_{k=1}^{4}\int_{\Gamma_k}\frac{Xdy-Ydx}{M^{s+1}}. \tag{10.6.5}$$

In the orbits Γ_1 and Γ_3, we have

$$\int_{\Gamma_1}\frac{Xdy-Ydx}{M^{s+1}}=0,\quad \int_{\Gamma_3}\frac{Xdy-Ydx}{M^{s+1}}=0. \tag{10.6.6}$$

Let

$$\frac{Y(x,0)}{M^{s+1}(x,0)}=\frac{-2g(x)}{x^{4s+1}}, \tag{10.6.7}$$

where $g(x)$ is a formal series of x, and $g(0)=1$. We see from (10.6.5) and (10.6.6) that

$$\begin{aligned}I &= 2\lim_{h'\to h}\frac{1}{h'-h}\left[\int_{\Gamma_2}\frac{g(x)}{x^{4s+1}}\,dx+\int_{\Gamma_4}\frac{g(x)}{x^{4s+1}}\,dx\right]\\ &= 2\lim_{h'\to h}\frac{1}{h'-h}\left[\int_{\tilde{h}}^{\tilde{h}'}\frac{g(x)}{x^{4s+1}}\,dx+\int_{h'}^{h}\frac{g(x)}{x^{4s+1}}\,dx\right]\\ &= 2\lim_{h'\to h}\frac{1}{h'-h}\left[\int_{h'}^{\tilde{h}'}\frac{g(x)}{x^{4s+1}}\,dx-\int_{h}^{\tilde{h}'}\frac{g(x)}{x^{4s+1}}\,dx\right].\end{aligned}\tag{10.6.8}$$

(10.6.2) and (10.6.8) follow that

$$\begin{aligned}I &= 2\frac{d}{dh}\int_{h}^{\tilde{h}}\frac{g(x)}{x^{4s+1}}\,dx\\ &= \frac{2g(\tilde{h})}{\tilde{h}^{4s+1}}\frac{d\tilde{h}}{dh}-\frac{2g(h)}{h^{4s+1}}\\ &= \frac{2g(\tilde{h})}{\tilde{h}^{4s+1}}\left[1+\sum_{k=2}^{\infty}k\nu_k(-2\pi)h^{k-1}\right]-\frac{2g(h)}{h^{4s+1}}\\ &= I_1^*+I_2^*,\end{aligned}\tag{10.6.9}$$

where

$$\begin{aligned}\tilde{h} &= \tilde{r}(-2\pi,h)=h+\sum_{k=2}^{\infty}\nu_k(-2\pi)h^k,\\ I_1^* &= \frac{2g(\tilde{h})}{\tilde{h}^{4s+1}}-\frac{2g(h)}{h^{4s+1}},\\ I_2^* &= \frac{2g(\tilde{h})}{\tilde{h}^{4s+1}}\sum_{k=2}^{\infty}k\nu_k(-2\pi)h^{k-1}.\end{aligned}\tag{10.6.10}$$

(10.6.10) implies that

$$\begin{aligned}I_1^* &= \frac{d}{dh}\left(\frac{2g(h)}{h^{4s+1}}\right)(\tilde{h}-h)\,[1+o(1)]\\ &= -2(4s+1)\frac{G_1(h)}{h^{4s+2}}\sum_{k=2}^{\infty}\nu_k(-2\pi)h^k,\\ I_2^* &= 2\frac{G_2(h)}{h^{4s+2}}\sum_{k=2}^{\infty}k\nu_k(-2\pi)h^k,\end{aligned}\tag{10.6.11}$$

where $G_k(h)$ is a formal series of h, and $G_k(0)=1$, $k=1,2$.

By (10.6.9) and (10.6.11), we know that

$$I=\frac{2}{h^{4s+2}}\sum_{k=2}^{\infty}\big[kG_2(h)-(4s+1)G_1(h)\big]\nu_k(-2\pi)h^k.\tag{10.6.12}$$

Thus, (10.6.12) and Theorem 10.2.1 imply that

$$I = \frac{2}{h^{4s+2}} \sum_{m=1}^{\infty} (2m - 4s - 1)\nu_{2m}(-2\pi)h^{2m} g_m(h), \tag{10.6.13}$$

where $g_m(h) = 1 + o(1)$ is a formal series of h. This gives rise to the proof of Lemma 10.6.1. □

Theorem 10.6.1. *If there exists a formal series of x, y given by $M(x, y) = x^4 + y^2 + o(r^4)$, such that*

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}} \sum_{m=1}^{\infty} d_m F_m(x, y), \tag{10.6.14}$$

where $\forall m,\ \kappa_m \neq 0$, then

$$\{\nu_{2m}(-2\pi)\} \sim \left\{\frac{d_m \kappa_m}{2m - 4s - 1}\right\}, \tag{10.6.15}$$

where

$$F_m(x, y) = f_m(x^2, y) + o(r^{4m+4}) \tag{10.6.16}$$

is a formal series of x, y, $f_m(x^2, y)$ is a homogeneous polynomial of degree $m+2$ of x^2, y,

$$\kappa_m = \frac{1}{2}\int_0^{2\pi} \frac{(1 + \sin^2\theta) f_m(\cos^2\theta, \sin\theta)d\theta}{(\cos^4\theta + \sin^2\theta)^{\frac{2m+7}{4}} \exp\left(\dfrac{2m-1}{2}\mu \arctan\dfrac{\sin\theta}{\cos^2\theta}\right)}. \tag{10.6.17}$$

Proof. The Jacobin of the transformation (10.2.6) is as follows:

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = r^2(1 + \sin^2\theta). \tag{10.6.18}$$

For the double integral defined by (10.6.3), making the transformation (10.2.6), then, we see from (10.6.14) that

$$\begin{aligned} I &= \lim_{h' \to h} \frac{1}{h' - h} \int_{-2\pi}^{0} d\theta \int_{\tilde{r}(\theta,h)}^{\tilde{r}(\theta,h')} \left[\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)\right] J\, dr \\ &= \int_{-2\pi}^{0} \left[\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right)\right] J\, \frac{\partial \tilde{r}(\theta, h)}{\partial h}\, d\theta. \end{aligned} \tag{10.6.19}$$

By (10.6.14), (10.6.18) and (10.6.19), we have

$$I = \sum_{m=1}^{\infty} d_m A_m(h), \tag{10.6.20}$$

where

$$A_m(h) = \int_{-2\pi}^{0} \frac{r^2(1+\sin^2\theta)F_m(x,y)}{M^{s+2}} \frac{\partial \tilde{r}(\theta,h)}{\partial h}\, d\theta. \tag{10.6.21}$$

In (10.6.21), we have set

$$x = r\cos\theta, \quad y = r^2\sin\theta, \quad r = \tilde{r}(\theta,h) = \nu_1(\theta)h + \sum_{k=2}^{\infty} \nu_k(\theta)h^k \tag{10.6.22}$$

and we have

$$\begin{aligned}
&\tilde{r} = \nu_1(\theta)h + o(h),\\
&F(x,y) = f_m(\cos^2\theta, \sin\theta)\nu_1^{2m+4}(\theta)h^{2m+4} + o(h^{2m+4}),\\
&\left.\frac{\partial \tilde{r}}{\partial h}\right|_{h=0} = \nu_1(\theta),\\
&M = (\cos^4\theta + \sin^2\theta)\nu_1^4(\theta)h^4 + o(h^4).
\end{aligned} \tag{10.6.23}$$

Thus, (10.6.21) and (10.6.23) imply that

$$A_m(h) = \frac{1}{h^{4s+2}} c_m h^{2m} \tilde{g}_m(h), \tag{10.6.24}$$

where $\tilde{g}_m(h)$ are formal series of h,

$$c_m = \int_0^{2\pi} \frac{(1+\sin^2\theta)f_m(\cos^2\theta, \sin\theta)}{(\cos^4\theta + \sin^2\theta)^{s+2}}\; \nu_1^{2m-4s-1}(\theta)d\theta. \tag{10.6.25}$$

(10.2.14), (10.5.1) and (10.6.25) follow that

$$c_m = 2\kappa_m. \tag{10.6.26}$$

By (10.6.20), (10.6.24) and (10.6.26), we have

$$I = \frac{2}{h^{4s+2}} \sum_{m=1}^{\infty} d_m \kappa_m h^{2m} \tilde{g}_m(h). \tag{10.6.27}$$

We see from (10.6.13) and (10.6.27) that

$$\sum_{m=1}^{\infty} (2m-4s-1)\nu_{2m}(-2\pi)h^{2m} g_m(h) = \sum_{m=1}^{\infty} d_m \kappa_m h^{2m} \tilde{g}_m(h). \tag{10.6.28}$$

It follows (10.6.15). Hence, Theorem 10.6.1 holds. □

We see from (10.5.15) and (10.6.17) that if $f_m(x,y) = x^{2m+4}$, then $\kappa_m = \sigma_m$. Thus, we know that Theorem 10.5.2 is a special case of Theorem 10.6.1.

10.7 On the Computation of Quasi-Lyapunov Constants

In this section, we study the computation method of quasi-Lyapunov constants. For the right hand of system (10.2.5), we write that

$$X(x,y) = y + \sum_{k=2}^{\infty} X_k(x,y) = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j,$$
$$Y(x,y) = \sum_{k=2}^{\infty} Y_k(x,y) = \sum_{k+j=2}^{\infty} b_{kj}x^k y^j, \tag{10.7.1}$$

where $X_k(x,y), Y_k(x,y)$ are homogeneous polynomial of degree k of x, y and

$$a_{20} = \mu, \quad b_{20} = 0, \quad b_{11} = 2\mu, \quad b_{30} = -2. \tag{10.7.2}$$

We have

Theorem 10.7.1. *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}, \quad \beta \geqslant 3, \tag{10.7.3}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{m=2}^{\infty} M_k(x,y), \tag{10.7.4}$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=5}^{\infty}\omega_m x^m , \tag{10.7.5}$$

where for all k, $M_k(x,y)$ is homogeneous polynomial of degree k of x, y, $c_{\alpha\beta}, \omega_m$ are all polynomials with rational coefficients of μ and $a_{kj}, b_{kj}, c_{0\beta}$.

Remark 10.7.1. *Because $s\mu = 0$, we see from* (10.7.4) *that*

$$c_{30} = 0, \quad c_{21} = -2\,\frac{s\mu}{s+1} = 0, \quad c_{12} = \frac{a_{11} - 2sb_{02}}{s+1},$$
$$c_{40} = 1 + \frac{2s^2\mu^2}{(s+1)^2} = 1,$$
$$c_{31} = \frac{1}{s+1}a_{30} - \frac{\mu}{3(s+1)^2}a_{11} - \frac{2s+1}{3(s+1)}b_{21},$$
$$c_{22} = \frac{a_{21} - sb_{12} - \mu c_{03}}{s+1} + \frac{(2sb_{02} - a_{11})(2b_{02} + a_{11})}{2(s+1)^2},$$
$$c_{13} = \frac{a_{12} - (2s-1)b_{03} + (2sb_{02} - a_{11})a_{02} + [a_{11} - (3s+1)b_{02}]c_{03}}{s+1},$$
$$\omega_5 = -8\,\frac{s\mu[(s+1)^2 + s^2\mu^2]}{(s+1)^2} = 0. \tag{10.7.6}$$

Thus, when $x = r\cos\theta, y = r^2\sin\theta$, $M(x,y) = x^4 + y^2 + o(r^4)$.

Remark 10.7.2. *By Theorem* 10.5.3, *we can determine a natural number* s *and constant sequence* $\{c_{0\beta}\}$, *such that* $\{\omega_{2k+1}\} = \{0\}$. *Therefore, Theorem* 10.5.3 *implies that if* s *and* $\{c_{0\beta}\}$ *are solutions of the equations* $\{\omega_{2k+1}\} = \{0\}$, *then*

$$\{\lambda_m\} \sim \left\{\frac{\omega_{2m+4}}{2m-4s-1}\right\}. \tag{10.7.7}$$

Equation (10.7.5) is equivalent to

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty} \omega_m x^m. \tag{10.7.8}$$

Clearly, equation (10.7.8) is linear with respect to the unknown function M, so that, we can find the following recursive formulas for the computations of $c_{\alpha\beta}$ and ω_m. Namely, we have

Theorem 10.7.2. *For* $\alpha \geqslant 1, \alpha+\beta \geqslant 3$ *in* (10.7.4) *and* (10.7.5), $c_{\alpha\beta}$ *can be uniquely determined by the recursive formulas*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}\left(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}\right), \tag{10.7.9}$$

when $m \geqslant 5$, ω_m *can be uniquely determined by the recursive formulas*

$$\omega_m = A_{m,0} + B_{m,0}, \tag{10.7.10}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \left[k - (s+1)(\alpha-k+1)\right] a_{kj} c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \left[j - (s+1)(\beta-j+1)\right] b_{kj} c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{10.7.11}$$

Notice that in (10.7.11), *we set*

$$\begin{aligned} &c_{00} = c_{10} = 0 = c_{01} = 0, \\ &c_{20} = c_{11} = 0, \quad c_{02} = 1 \end{aligned} \tag{10.7.12}$$

and when $\alpha < 0$ *or* $\beta < 0$, $c_{\alpha\beta} = 0$.

Proof. We have

$$\begin{aligned} &\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) \\ &= \sum_{m=3}^{\infty}\left[-(s+1)\frac{\partial M_m}{\partial x}y + \Phi_m(x,y) + \Psi_m(x,y)\right], \end{aligned} \tag{10.7.13}$$

where

$$\frac{\partial M_m}{\partial x}y = \sum_{\alpha+\beta=m} \alpha c_{\alpha,\beta} x^{\alpha-1} y^{\beta+1}$$
$$= c_{1,m-1}y^m + 2c_{2,m-2}xy^{m-1} + \cdots + mc_{m0}x^{m-1}y \qquad (10.7.14)$$

and

$$\Phi_m = \sum_{n=2}^{m-1}\left[\frac{\partial X_n}{\partial x}M_{m+1-n} - (s+1)\frac{\partial M_{m+1-n}}{\partial x}X_n\right],$$
$$\Psi_m = \sum_{n=2}^{m-1}\left[\frac{\partial Y_n}{\partial y}M_{m+1-n} - (s+1)\frac{\partial M_{m+1-n}}{\partial y}Y_n\right] \qquad (10.7.15)$$

are homogenous polynomials of degree m of x, y. (10.7.15) implies that

$$\Phi_m = \sum_{n=2}^{m-1} \sum_{\substack{k+j=n\\ \alpha+\beta=m+n-1}} [k-(s+1)\alpha]a_{kj}c_{\alpha\beta}x^{\alpha+k-1}y^{\beta+j},$$
$$\Psi_m = \sum_{n=2}^{m-1} \sum_{\substack{k+j=n\\ \alpha+\beta=m+n-1}} [j-(s+1)\beta]b_{kj}c_{\alpha\beta}x^{\alpha+k}y^{\beta+j-1}. \qquad (10.7.16)$$

Thus,

$$\Phi_m = \sum_{\alpha+\beta=m} A_{\alpha\beta}x^\alpha y^\beta, \quad \Psi_m = \sum_{\alpha+\beta=m} B_{\alpha\beta}x^\alpha y^\beta, \qquad (10.7.17)$$

where $A_{\alpha\beta}, B_{\alpha\beta}$ are given by (10.7.11). Hence, (10.7.13), (10.7.14) and (10.7.17) follow that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right)$$
$$= \sum_{m=3}^{\infty}\left[\sum_{\alpha+\beta=m} -(s+1)\alpha c_{\alpha,\beta}x^{\alpha-1}y^{\beta+1} + \sum_{\alpha+\beta=m}(A_{\alpha\beta}+B_{\alpha\beta})x^\alpha y^\beta\right]. \quad (10.7.18)$$

By using (10.7.8) and (10.7.18), we obtain the conclusion of Theorem 10.7.2. □

10.8 Bifurcations of Limit Cycles Created from a Three-Multiple Nilpotent Singular Point of a Cubic System

In this section, we discuss the following cubic system

$$\frac{dx}{dt} = y - 2xy + (-a_4+a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3,$$
$$\frac{dy}{dt} = -2x^3 + a_1x^2y + y^2 + a_4xy^2 + a_3y^3. \qquad (10.8.1)$$

By Theorem 10.7.1 and Theorem 10.7.2, we can determine a positive integer s and a formal series $M(x,y) = x^4 + y^2 + o(r^4)$, such that (10.7.5) holds. Using the recursive formulas and computer algebra method, for example, Mathematica, we have

Theorem 10.8.1. *For system* (10.8.1), *the first* 8 *quasi-Lyapunov constants of the origin are as follows*:

$$\lambda_1 = \frac{1}{3}a_1,$$

$$\begin{aligned}
\lambda_2 &\sim \frac{2}{5}(a_2 + 3a_3),\\
\lambda_3 &\sim \frac{4}{21}a_7(3a_3 - 5a_6),\\
\lambda_4 &\sim \frac{4}{945}a_6a_7(735 - 105a_4 + 71a_7),\\
\lambda_5 &\sim \frac{8}{121275}a_6a_7(176400 + 18375a_5 + 5460a_7 + 12250a_6^2 - 32a_7^2),\\
\lambda_6 &\sim \frac{32}{7449316875}a_6a_7f_6,\\
\lambda_7 &\sim \frac{32}{895908296236078125}a_6a_7f_7,\\
\lambda_8 &\sim \frac{32}{31270967072166673965472734375}a_6a_7f_8.
\end{aligned} \tag{10.8.2}$$

where

$$\begin{aligned}
f_6 = {} & 30866913000 + 2089303650a_7 - 1188495000a_6^2 + 29397690a_7^2\\
& -15232875a_6^2a_7 - 110996a_7^3,\\
f_7 = {} & -443894563225159200000 - 2155807164550977000a_7\\
& +164713803723315000 0a_6^2 + 114379911724774 50a_7^2\\
& +910916029415875a_7^3 + 22121192499656250a_6^4\\
& -798220526556a_7^4,\\
f_8 = {} & 94233793124416828974515424 00000\\
& +15142987656813199473691128 00000a_7\\
& +82859324997946429848009339000a_7^2\\
& +1864567030459291902188584650a_7^3\\
& +14562086011231729200961815a_7^4\\
& -266619108568395354750 8a_7^5.
\end{aligned} \tag{10.8.3}$$

Lemma 10.8.1. *If the origin of system* (10.8.1) *is a center, then,* $a_6a_7 = 0$.

Proof. By theorem 10.8.1, it only need to prove that equations $f_6 = f_7 = f_8 = 0$ have no real solution. In fat, let

$$\begin{aligned} m_1 = &22(1066395363913341000 + 287995299174040 50a_7 \\ &+1061817239983 5000a_6^2 + 262642872352770a_7^2 \\ &+136092531222375a_6^2a_7 - 991653026468a_7^3), \\ m_2 = &3(64680 + 829a_7)^2, \\ m_3 = &1299748608738765251767207819592280 0000 \\ &-3654689734284190371743146576152750 00a_7 \\ &-6636084295245265658315014527681000a_7^2 \\ &-193072016330746017200389316240550a_7^3 \\ &-158395474078481780708311488885a_7^4 \\ &-44200222391829063402345929 2a_7^5, \\ m_4 = &-829(604792809107550 - 26154872314410a_7 \\ &+10468656247875a_6^2 + 76281002036a_7^2). \end{aligned} \tag{10.8.4}$$

Then

$$\begin{aligned} &m_1f_6 + m_2f_7 = 687241R_1, \\ &\frac{m_3f_6 + m_4R_1}{567035826552566910 00} = -2979915027490377253777500 00a_6^2 - R_2, \end{aligned} \tag{10.8.5}$$

where R_1, R_2 are polynomial of a_7 as follows:

$$\begin{aligned} R_1 = &243072129127249422000000 \\ &+396310816412408898000000a_7 \\ &+225579690981028345 5000a_7^2 + 554223522306195630 00a_7^3 \\ &+524676790165767750a_7^4 + 492626020125225a_7^5 \\ &+1128890456908a_7^6, \\ R_2 = &-5598205484096735678668500000 \\ &-10301583058016133079696500 0a_7 \\ &+13819265426965160101653000a_7^2 \\ &+33888764862229517941995 0a_7^3 \\ &+2780220084868007284 65a_7^4 + 7758198064959108 28a_7^5. \end{aligned} \tag{10.8.6}$$

(10.8.5) and (10.8.6) imply that

$$R_1 = 0, \quad 297991502749037725377750000a_6^2 = -R_2. \tag{10.8.7}$$

if $f_6 = f_7 = 0$.

Because f_8 is also a polynomial of a_7, which has no common factor with R_1. Therefore, Lemma 10.8.1 holds. □

Theorem 10.8.1 and Lemma 10.8.1 follow that

Theorem 10.8.2. *The first* 8 *quasi-Lyapunov constants of system* (10.8.1) *all are zeros if and only if one of the following conditions holds*:

$$a_1 = 0, \quad a_2 = -3a_3, \quad a_7 = 0; \tag{10.8.8}$$

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0, \quad a_6 = 0. \tag{10.8.9}$$

When condition (10.8.8) is satisfied, system (10.8.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y - 2xy - a_4x^2y + a_6y^2 - 3a_3xy^2 + a_5y^3, \\ \frac{dy}{dt} &= -2x^3 + y^2 + a_4xy^2 + a_3y^3. \end{aligned} \tag{10.8.10}$$

When (10.8.9) is satisfied, system (10.8.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y - 2xy + (-a_4 + a_7)x^2y + a_5y^3, \\ \frac{dy}{dt} &= -2x^3 + y^2 + a_4xy^2. \end{aligned} \tag{10.8.11}$$

Obviously, (10.8.10) is a Hamilton system. System (10.8.11) is symmetric with respect to the x-axis. Hence, we have

Theorem 10.8.3. *The origin of system* (10.8.1) *is a center if and only if the first* 8 *quasi- Lyapunov constants are all zero, i.e., one of conditions of Theorem* 10.8.2 *holds.*

By Theorem 10.8.1 and Lemma 10.8.1, we have

Theorem 10.8.4. *The origin of system* (10.8.1) *is a* 8*-order weak focus if and only if*

$$\begin{aligned} &a_1 = 0, \quad a_2 = -3a_3, \quad a_3 = \frac{5}{3}a_6, \quad a_4 = \frac{1}{105}(735 + 71a_7), \\ &a_5 = \frac{-2}{18375}(88200 + 2730a_7 + 6125a_6^2 - 16a_7^2) \end{aligned} \tag{10.8.12}$$

and a_6, a_7 *satisfy* (10.8.7).

Remark 10.8.1. *By using computer to do computations, we find that equation* $R_1 = 0$ *has exact solutions* $a_7 = A_7$ *and* $a_7 = \tilde{A}_7$*, where*

$$A_7 = -13.506311\ldots, \quad \tilde{A}_7 = -28.122170\ldots. \tag{10.8.13}$$

(10.8.7) *follows that*

$$a_6^2|_{a_7=A_7} = 8.42971\ldots, \quad a_6^2|_{a_7=\tilde{A}_7} = -2.85598\ldots. \tag{10.8.14}$$

Thus, the origin of system (10.8.1) *is a* 8*-order weak focus if and only if* (10.8.12) *holds and*

$$a_7 = A_7, \quad 2979915027490377253777500000a_6^2 = -R_2. \tag{10.8.15}$$

Finally, we consider the perturbed system of system (10.8.1) as follows:

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y - 2xy + (-a_4 + a_7)x^2y + a_6y^2 + a_2xy^2 + a_5y^3, \\ \frac{dy}{dt} &= 2\delta y - 2x^3 + a_1x^2y + y^2 + a_4xy^2 + a_3y^3. \end{aligned} \tag{10.8.16}$$

Theorem 10.8.5. *If the origin of system* $(10.8.16)_{\delta=0}$ *is a* 8*-order weak focus, making a small perturbations to the coefficients of system* $(10.8.16)_{\delta=0}$*, then, in a small neighborhood of the origin, there exist at least 8 small amplitude limit cycles of system* (10.8.16)*, which enclosing the origin* $O(0,0)$ (*an elementary node*).

Proof. When one of two conditions in Theorem (10.8.15) holds, we have

$$\begin{aligned} J &= \frac{D(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7)}{D(a_1, a_2, a_3, a_4, a_5, a_6, a_7)} \\ &= \frac{\partial\lambda_1}{\partial a_1}\frac{\partial\lambda_2}{\partial a_2}\frac{\partial\lambda_3}{\partial a_3}\frac{\partial\lambda_4}{\partial a_4}\frac{\partial\lambda_5}{\partial a_5}\begin{vmatrix} \frac{\partial\lambda_6}{\partial a_6} & \frac{\partial\lambda_6}{\partial a_7} \\ \frac{\partial\lambda_7}{\partial a_6} & \frac{\partial\lambda_7}{\partial a_7} \end{vmatrix} \\ &= \frac{524288a_6^3a_7^4J_0}{145850034635282007923510207264053225163895703125}, \end{aligned} \tag{10.8.17}$$

where

$$\begin{aligned} J_0 = &-355568351656266957303825238970788273694748000000 \\ &-574480073401609275376903324138603280114196000a_7 \\ &-32307543851912960186097359148266720399519100a_7^2 \\ &-769340257685895029702330384622007223659380a_7^3 \\ &-643946705700180376373469814745988275831 40a_7^4 \\ &+76862330062195974590661486322086814547a_7^5. \end{aligned} \tag{10.8.18}$$

Because there is no common factor between J_0 and R_1. Thus, when the origin of system $(10.8.16)_{\delta=0}$ is a *8* order weak focus, we have $J \neq 0$. Theorem 10.3.8 implies that Theorem 10.8.5 holds. □

Bibliographical Notes

There are different topological phase portraits in a neighborhood of a nilpotent singular point (see [Zhang Z.F. etc, 1992]). In Section 17-19 of [Amelikin etc, 1982], the authors introduced some study results before 1980s.

In[Amelikin etc, 1982; Takens, 1974; Strozyna etc, 2002; Moussu, 1982; Álvarez etc, 2005; Álvarez etc, 2006], by considering the normal forms of (10.1.1), the authors studied the computation problem of the focal values.

The content of this chapter is taken from the recent papers [Liu Y.R. etc, 2009a; Liu Y.R. etc, 2009b; Liu Y.R. etc, 2011a; Liu Y.R. etc, 2011d].

Bibliography

[Álvarez etc, 2005] Álvarez M.J. and Gasull A., Monodromy and stablility for nipotent critical points, Int. J. Bifur. Chaos. **15**(4), 1253-1265.

[Álvarez etc, 2006] Álvarez M.J. and Gasull A., Cenerating limits cycles from a nipotent critical point via normal forms, J.Math. Anal.Appl. **318**, 271-287.

[Alwash, 1998] Alwash M.A.M., Computing the Poincaré-Liapunov constants, Diff. Equs. Dynam. Sys. **6** (3), 349-361.

[Alwash etc, 1987] Alwash M.A.M.and Lloyd, N.G., Non-autonomous equations related to polynomial two-dimensional systems, Proc. Roy. Soc. Edinburgh Sect. A **105**, 129-152.

[Amelikin etc, 1982] Amelikin B.B., Lukashivich H.A. and Sadovski A.P., Nonlinear Oscillations in Second Order Systems, Minsk, BGY lenin.B. I. Press. (in Russian).

[Andronov etc, 1973] Andronov A.A., Leontovich E.A., Gordon I.I. and Maier A.G., Theory of Bifurcations of Dynamical Systems on a Plane, John Wiley and Sons, New York.

[Annosov etc, 1997] Annosov D.V., Aranson S.K. and Arnold V.I. and others, Ordinary Differential Equations and Smooth Dynamical Systems, Springer, New York.

[Arnold, 1977] Arnold V.I., Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields, Funct. Anal. Appl. **11**, 85-92.

[Arnold, 1983] Arnold V.I., Geometric Methods in Theory of Ordinary Differential Equations, Springer-Verlag, New York.

[Arnold, 1990] Arnold V.I. , Ten problems, Adv. Soviet Math. **1**, 1-8.

[Arnold etc, 1994] Arnold V.I., Afraimovich V.S., Ilyashenko Yu.S. and Shilnikov L.P., Bifurcation Theory, Dynamical systems 5, Springer-Verlag, New York.

[Bamon, 1985] Bamon R., The solution of Dulac's problem for quadratic vector fields, Ann. Acad. Bris. Ciênc. **57**, 111-142.

[Bautin, 1952-1954] Bautin N.N., On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. (N.S.) **30**(72)(1952), 181-196 (in Russian); Transl. Amer. Math. Soc. **100** (1) (1954), 397- 413.

[Bendixson, 1901] Bendixson I., Sur les courbes définies par des équations différentielles, Acta Math. **24**, 1-88.

[Blows etc, 1984a] Blows T.R. and Lloyd N.G., The number of limit cycles of certain polynomial differential equations, Proc. Roy. Soc. Edinburgh Sect. A **98**, 215-239.

[Blows etc, 1984b] Blows T.R. and Lloyd N.G., The number of small-amplitude limit cycles of Liénard equations Math. Proc. Cambridge Philos.Soc. **95**, 359-366.

[Blows etc, 1993] Blows T.R. and Rousseau C., Bifurcation at infinity in polynomial vector fields, J. Diff. Equs. **104**, 215-242.

[Blows etc, 1994] Blows, T. R. and Perko, L. M., Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems, SIAM Rev. **36**, 341-376.

[Bonin etc, 1988] Bonin G. and Legault J., Comparison de la methode des constants de Liapunov et la bifurcation de Hopf, Canad. Math. Bull. **31**(2), 200-209.

[Bruno, 1989] Bruno A.D., Local Methods in Nonlinear Differential Equations, Springer-Verlag, New York.

[Cai S.L. ect, 1987] Cai Suilin and Zhang Pingguang, A quadratic system with a weak saddle, J. Math. Res. Exp. **1**,63-68.

[Cai S.L. etc, 1988] Cai Suilin,Zhang Pingguang , A quadratic system with a weak saddle II, Ann. Diff. Equs. **4**(2), 131-142.

[Cai S.L., 1989] Cai Suilin, Survey of planar quadratic differential systems, Adv. in Math. (China) **18**, 5-21.

[Cairó etc, 1999] Cairó L., Chavrriga, J., Giné J. and Llibre J., A class of reversible cubic systems with an isochronous center, Computers and Mathematics with Applications **38**, 39-53.

[Cair etc, 1997] Cair L. and Llibre J., Phase portraits of planar semi- homogeneous vector fields (I), Nonlinear Anal. **29**(7), 783-811.

[Cair etc, 2000] Cair L. and Llibre J., Phase portraits of planar semi- homogeneous vector fields (II), Nonlinear Anal. **39**(3), 351-363.

[Carbonell etc, 1988] Carbonell M. and Llibre J., Limit cycles of a class of polynomial systems, Proc. Roy. Soc. Edinburgh Sect. A **109**, 187-199.

[Carbonell etc, 1989] Carbonell M., Coll B. and Llibre J., Limit cycles of polynomial systems with homogeneous nonlinearities, J. Math. Anal. Appl. **142**, 573-590.

[Chan H.S.Y. etc, 2001] Chan H.S.Y., Chung K.W. and Qi Dongwen, Some bifurcation diagrams for limit cycles of quadratic differential systems, Int. J. Bifurcation and Chaos, **11** (1), 197-206.

[Chan H.S.Y. etc, 2001] Chan H.S.Y., Chung K.W. and Li Jibin, Bifurcations of limit cycles in a Z_3-equivariant planar vector field of degree 5, Int. J. Bifurcation and Chaos, **11** (8), 2287-2298.

[Chan H.S.Y. etc, 2002] Chan H.S.Y., Chung K.W.and Li Jibin, Bifurcations of Limit Cycles in Z_q-Equivariant Planar Vector Fields of Degree 5, In Proceeding of International Conference on Foundations of Computational Mathematics in honor of Professor Steve Smale's 70th Birthday, World Scientific Press, New Jersey, 61-83.

[Chavarriga, 1994] Chavarriga J., Integrable systems in the plane with center type linear part, Applicationes Mathematicae **22**, 285-309.

[Chavarriga etc, 1995] Chavarriga J. and Giné J., Integrability of cubic with degenerate infinity, in Proceeding of XIV CEDYA, Vic.

[Chavarriga etc, 1996] Chavarriga J. and Giné J., Integrability of a linear center perturbed by fourth degree homogeneous polynomial, Pub. Math. **40**(1), 21-39.

[Chavarriga etc, 1997] Chavarriga J. and Giné J., Integrability of a linear center perturbed by fifth degree homogeneous polynomial, Pub. Math. **41**(2), 335-356.

[Chavarriga etc,1999] Chavarriga J., Giné, J. and Garcí I., A Isochronous centers of a linear center perturbed by fourth degree homogrneous polynomial, Bulletin des Sciences Mathématiques **123** ,77-96.

[Chavarriga etc,2000] Chavarriga J., Giné J. and Garcí I., A Isochronous centers of a linear center perturbed by fifth degree homogrneous polynomial, J.Comput.Appl.Math. **126**, 351-368.

[Chavarriga etc,2001] Chavarriga J., Giné J. and Garcí I., Isochronicity into a family of time-reversible cubic vector fields, Appl. Math. Comput. **121** 129 C 145.

[Chen L.S. etc, 1979] Chen Lansun and Wang Mingshu, On relative locations and the number of limit cycles for quadratic systems, Acta Math. Sinica **22**, 751-758.

[Chen H.B. etc, 2002] Chen Haibo and Luo Jiaowan, Stability and bifurcation of limit cycles of the equator in a class of septic polynomial systems, Mathmatica Applicata (in China) **15**, 22-25.

[Chen H.B. etc , 2003] Chen Haibo and Liu Yirong, Limit cycle of the equator in a quintic polynomial system, Chinese Annals of Mathematics **24A**, 219-224.

[Chen H.B. etc, 2005a] Chen Haibo, Liu Yirong, and Zeng Xianwu, Center conditions and bifurcation of limit cycles at degenerate singular points in a quintic polynomial differential system, Bull. Sci. Math. **129**, 127-138.

[Chen H.B. etc, 2005b] Chen Haibo , Liu Yirong and Zeng Xianwu , Algebraic Recursion Formulas for Quantities of Equator in a Planar Polynomial Differential System, Acta Mathematica Sinica (China) **48**(5), 963-972.

[Chen X.W. etc, 2008] Chen Xingwu, Valery G. Romanovski and Zhang Weinian, Linearizability conditions of time-reversible quartic systems having homogeneous nonlinearities, Nonlinear Analysis Ser.A **69**(5-6), 1525-1539.

[Chen X.W. etc, 2008] Chen Xingwu, Valery G. Romanovski and Zhang Weinian, Nonisochronicity of the center at the origin in polynomial Hamiltonian systems with even degree nonlinearities, Nonlinear Analysis Ser.A **68**(9), 2769-2778.

[Chen X.W. etc, 2009] Chen Xingwu, and Zhang Weinian, Decomposition of algebraic sets and applications to weak centers of cubic systems, J. Comput. Appl. Math. **232**(2), 565-581.

[Chen X.W. etc, 2011] Chen Xingwu, and Zhang Weinian, C^1 linearization for planar contractions, J. Functional Anal. **260**(7), 2043-2063.

[Cherkas, 1968] Cherkas L.A., On the stability of singular cycles, Diff. Equs. **4**, 1012-1017.

[Cherkas, 1976] Cherkas L.A., Number of limit cycles of an autonomous second-order system, Diff. Equs. **5**, 666-668.

[Cherkas, 1981] Cherkas L.A., Structure of a successor function in the neighborhood of a separatrix of a perturbed analytic autonomous system in the plane, Diff. Equs. **17**(3), 469-478.

[Chicone etc, 1982] Chicone C. and Tian Jinghuang, On general properties of quadratic systems, Amer. Math. Monthly **89**, 167-179.

[Chicone etc, 1983] Chicone C. and Schafer S., Separatrix and limit cycles of quadratic systems and Dulac's theorem, Trans. Amer. Math. Soc. **278**, 585-612.

[Chicone etc, 1986] Chicone C. and Sotomayor J., On a class of complete polynomial vector fields in the plane, J. Diff. Equs. **61**, 398-418.

[Chicone, 1986] Chicone C., Limit cycles of a class of polynomial vector fields in the plane, J. Diff. Equs. **96**(2), 355-418.

[Chicone etc, 1989] Chicone, C., Jacobs, M., Bifurcation of critical periods for plane vector fields, Amer. Math. Soc. Trans. **312**, 433-486.

[Chicone, 1990] Chicone C., On bifurcation of limit cycles from centers, Lecture Notes in Math. **1455**, 20-93.

[Chicone etc, 1991] Chicone C. and Jacobs M., Bifurcation of limit cycles from quadratic isochrones, J. Diff. Equs. **91**, 268-326.

[Chicone, 1993] Chicone C., Finiteness theorems for limit cycles, Book review, Bull. Amer. Math. Soc. **28**(1), 123-130.

[Chicone, 1999] Chicone C., Ordinary Differential Equations with Applications, Springer-Verlag, New York.

[Chislenko etc, 1984] Chislenko and Yu S., M-curves of degree 10, J. Soviet Math. **26**, 1689-1699.

[Chow etc, 1982] Chow Shui-Nee and Hale J.K., Methods of Bifurcation Theory, Springer-Verlag, New York.

[Chow etc, 1990] Chow, Shui-Nee, Drachman B. and Wang Duo, Computation of normal form, J. Comput. Appl. Math. **29**, 129-143.

[Chow etc, 1994] Chow, Shui-Nee, Li Chenzhi and Wang Duo, Normal Forms and Bifurcations of Planar Vector Fields, Cambridge University Press.

[Chow etc, 2001] Chow, Shui-Nee, Li Chenzhi and Yi Yingfei , The cyclicity of period annulus of degenerate quadratic Hamiltonian system with elliptic segment, Erg. Theory. Dyn. Systems to appear.

[Christopher, 1994] Christopher C.J., Invariant algebraic curves and conditions for center, Proc. Roy. Soc. Edinburgh Sect. A **124**, 1209-1229.

[Christopher etc, 1995] Christopher C.J. and Lloyd N.G., Polynomial systems: a lower bound for the Hilbert numbers, Proc. Royal Soc. London Ser. A **450**, 219-224.

[Christopher etc, 1995] Christopher C.J., Lloyd N.G. and Pearson J.M., On a Cherkas's method for centre conditions, Nonlinear World **2**, 459-469.

[Christopher ect, 1997] Christopher C.J. and Devlin J., Isochronous centers in planar polynomial systems, SIAM J.Math.Anal. **28**,162-177.

[Christopher etc, 1999] Christopher C.J. and Lynch S., Small-amplitude limit cycles bifurcations for Liénard systems with quadratic or cubic damping or restoring forces, Nonlinearity **12**, 1099-1112.

[Christopher ect, 2003] Christopher C., Mardešić,P., and Rousseau,C., Normalizable, integrable, and linearizable saddle points for complex quadratic systems in C^2, Journal of Dynamical and Control Systems **9**(3), 311-363.

[Christopher ect, 2007] Christopher C. and Li Chengzhi, Limit Cycles of Differential Equations, Birkhäuser Verlag, 2007, Basel-Boston-Berlin.

[Cima etc, 1989] Cima A. and Llibre J., Configurations of fans and nests of limit cycles for polynomial vector fields in the plane, J. Diff. Equs. **82**(1), 71-97.

[Cima etc, 1990a] Cima A. and Llibre J., Algebraic and topological classification of the homogeneous cubic vector fields in the plane, J. Math. Anal. Appl. **147**(2), 420-448.

[Cima etc, 1990b] Cima A. and Llibre J., Bounded polynomial vector fields, Trans. Amer. Math. Soc. **318**(2), 557-579.

[Cima etc, 1995] Cima A. and Gasull A. and Mañosa V., Cyclicity of a family of vector fields, J. Math. Anal. Appl. **196**, 921-937.

[Cima etc, 1997] Cima A. and Gasull A., Mañosa, V. , Algebraic properties of the Liapunov and period constants, Rocky Mountain J. Math. **27**(2), 471-501.

[Coleman, 1983] Coleman C. S., Hilbert's 16th problem: How Many Cycles? In Differential Equation Models, Modules in Applied Mathematics, **1**, 279-297, Springer-Verlag, New York.

[Coll etc, 1997] Coll A., Gasull A. and Prohens R., Differential equations defined by the sum of two quasi homogeneous vector fields, Can. J. Math. **49**, 212-231.

[Coppel, 1988] Coppel W.A., Quadratic systems with a degenerate critical point, Bull. Austral. Math. Soc. **38**, 1-10.

[Coppel, 1991] Coppel W.A., A new class of quadratic systems, J. Diff. Equs. **92**(2), 360-372.

[Coppel, 1966] Coppel W.A., A survey of quadratic systems, J. Diff. Equs. **2**, 293- 304.

[Cushman etc, 1985] Cushman R. and Sanders J., A codimension two bifurcation with a third order Picard-Fuchs equation, J. Diff. Equs. **59**, 243-256.

[Darboux, 1878] Darboux G., Mémoire sur les équations différentielles du premier ordre et du premier degré(Méanges). Bull. Sci.Math., 1878,60-96,123-144,151-200.

[Devin etc, 1998] Devin J., Lloyd N.G. and Pearson J.M., Cubic systems and Abel equation, J. Diff. Equs. **147**, 435-454.

[Ding T.R. etc, 2004] Ding Tongren and Li Chengzhi, Ordinary Differential Equations, Higher Education Press (in Chinese), First Edition 1991, Second Edition 2004, Beijing.

[Drachman etc, 1987] Drachman B., Van Gils S.A. and Zhang Zhifen, Abelian integrals for quadratic vector fields, J. reine ange. Math. **382**, 165-180.

[Du C.X. ect, 2008a] Du Chaoxiong and Liu Yirong, General center conditions and bifurcation of limit cycles for a quasi-symmetric seventh degree system, Computers and Mathematics with Applications **56**, 2957-2969.

[Du C.X. ect, 2008b] Du Chaoxiong, Liu Yirong and Mi Heilong, A class of ninth degree system with four isochronous centers, Computers and Mathematics with Applications **56**, 2609-2620.

[Du C.X. ect, 2009] Du Chaoxiong and Liu Yirong, The problem of general center-focus and bifurcation of limit cycles for a planar system of nine degrees,Journal of Computational and Applied Mathematics **223**, 1043-1057.

[Du C.X. ect, 2010] Du Chaoxiong, Liu Yirong and Chen Haibo,The bifurcation of limit cycles in Z_n-equivariant vector fields, Applied Mathematics and Computation **217**, 2041-2056.

[Du C.X. ect, 2011] Du Chaoxiong, Chen Haibo and Liu Yirong, Center problem and bifurcation behavior for a class of quasi analytic systems, Applied Mathematics and Computation **217**,4665-4675.

[Du N.L. ect, 1994] Du Nailin and Zeng Xianwu , A Recursion Formula for Calculation of Quantities of Focus, Chinese Science Bulletin (China)**39**(19), 1742-1744.

[Dulac, 1923] Dulac H., Sur les cycles limites, Bull. Soc. Math. France, **51**, 45-188.

[Dumortier, 1977] Dumortier F., Singularities of vector fields on the plane, J. Diff. Equs. **23**, 53-106.

[Dumortier, 1986] Dumortier F., A note on finite cyclicity property and Hilbert's 16th problem, Lecture Notes in Math. **1331**, 161-186.

[Dumortier etc, 1987] Dumortier F. and Roussarie R. and Sotomayor J. Generic 3−parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension 3, Ergod. Th. Dynam. Sys. **7**, 375-413.

[Dumortier etc, 1994a] Dumortier F., Roussarie R. and Rousseau C., Hilbert's 16th problem for quadratic vector fields, J. Diff. Equs. **110**, 86-133.

[Dumortier etc, 1994b] Dumortier F., Roussarie R. and Rousseau C., Elementary graphics of cyclicity 1 and 2, Nonlinearity **7**, 1001-1043.

[Dumortier ect, 1996] Dumortier F. and Li Chenzhi, On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations, Nonlinearity **9**, 1489-1500.

[Dumortier etc, 1996] Dumortier F., El Morsalani, M. and Rousseau. C., Hilbert's 16th problem for quadratic vector fields and cyclicity of elementary graphics, Nonlinearity **9**(5), 1209-1261.

[Dumortier etc, 1997a] Dumortier F., Li Chenzhi and Zhang Zhifen, Unfolding of a quadratic integrable system with two centers and two unbounded heteroclinic loops, J. Diff. Equs. **139**, 146 -193.

[Dumortier etc, 1997b] Dumortier F. and Li Chenzhi , Quadratic Liénard equations with quadratic damping, J. Diff. Equs. **139**, 41-49.

[Dumortier etc, 2001a] Dumortier F. and Li Chenzhi, Perturbations from an elliptic Hamiltonian of degree four: (I) saddle loop and two saddle cycle, J. Diff. Equs. **176**, 114-157.

[Dumortier etc, 2001b] Dumortier F. and Li Chenzhi , Perturbations from an elliptic Hamiltonian of degree four: (II) cuspidal loop, J. Diff. Equs. **175**, 209-243.

[Écalle etc, 1987] Écalle J., Martinet J., Moussu R. and Ramis J.P., Non- accumulation des cycles limites, C.R.Acad.Sci. Paris **304**, 375-377, 431-434.

[Écalle, 1990] Écalle J., Finitude des cycles limites et accéléro- sommation de L'application de retor, Lecture Notes in Math. **1455**, 74-159.

[Écalle, 1992] Écalle, J., Introduction aux Functions Analysables et Preuve Constructive de la Conjecture de Dulac, Hermann Publ, Paris.

[Farkas, 1994] Farkas M., Periodic Motion, Springer-Verlag, New York.

[Farr etc, 1989] Farr W.W., Li Chenzhi and Labouriau I.S., Langford, W. F., Degenerate Hopf bifurcation formulas and Hilbert's 16th problem, SIAM J. Math. Anal. **20**(1), 13-30.

[Feng B.Y. ect , 1985] Feng Beiye and Qian Ming , The stability of a saddle point separatrix loop and a criterion for its bifurcation limit cycles, Acta Math. Sinica **28**(1), 53-70 (in Chinese).

[Feng B.Y., 1987] Feng Beiye, Condition for generating limit cycles by bifurcation of the loop of a saddle point separatrix, Acta Math. Sinica New Ser. **3**(4), 373-382.

[Feng B.Y., 1990] Feng Beiye, On the stability of a heteroclinic cycle for the critical case, Acta Math. Sinica **33**(1), 114-134.

[Francoise etc, 1986] Francoise J.P. and Pugh C.C., Keeping track of limit cycles, J. Diff. Equs. **65**(2), 139-157.

[Francoise, 1996] Francoise, J. P., Successive derivative of first return map, application to the study of quadratic vector fields, Ergod. Th. Dynam. Sys. **16**(1), 87-96.

[Francoise, 1997] Francoise J.P., The first derivative of the period function of a plane vector field, Publ. Math. **41**, 127-134.

[Gaiko, 1997] Gaiko V.A., Qualitative theory of two-dimensional polynomial dynamical-systems: problem, approaches, conjectures, Nonlinear Anal. **30**, 1385-1394.

[Gaiko, 2000] Gaiko V.A., On global bifurcations and Hilbert's sixteeth problem, Nonliear Phenom. Compl. Syst. **3**, 11-27.

[Gaiko, 2001] Gaiko V.A., Hilbert's sixteeth problem and global bifurcations of limit cycles, Nonlinear Anal. **47**, 4455-4466.

[Gao B., 2001] Gao Bo and Zhang Weinian, Parametric normal forms of vector fields and their further simplification, Nonlinearity **23**(10), 2539-2557.

[Gao Q.L., 1977] Gao Qiliang, Analytic integration of center and a class of particular center, Acta mathematica sinca (China) **20**(3), 178-194.

[Gasull etc, 1987] Gasull A., Llibre J. and Sotomayor J., Limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$, J. Diff. Equs. **68**, 36-40.

[Gasull etc, 1987] Gasull A., Llibre J. and Sotomayor J., Further considerations on the number of limit cycles of vector fields of the form $X(v) = Av + f(v)Bv$, J. Diff. Equs. **67**(1), 90- 110.

[Gasull etc, 1988] Gasull A. and Llibre J., On the nonsingular quadratic differential equations in the plane, Proc. Amer. Math. Soc. **104**(3), 793-794.

[Gasull etc, 1990] Gasull A. and Llibre J., Limit cycles for a class of Abel equations, SIAM J. Math. Anal. **21**(5), 1235-1244.

[Gasull etc, 1995] Gasull A., Guillamon A. and Mañosa V., Centre and isochronicity conditions for systems with homogeneous nonlinearities, Proceeding of the 2nd Catalan Days on Applied Mathematics, Presses Universitaires de Perpignan, 105-116.

[Gasull etc, 1997a] Gasull A. and Prohens R., Effective computation of the first Liapunov quantities for a planar differential equation, Appl. Math. **24**(3), 243-250.

[Gasull,1997b] Gasull A. and Manosa V., An explicit expression of the first Liapunov and period constants with applications, Journal of Mathematical Analysis and Applications **211**, 190-212.

[Gasull etc, 1998] Gasull A. and Torregrosa J., Center problem for several differential equations via Cherkas's method, J. Math. Anal. Appl. **228**, 322-343.

[Gasull etc, 2001] Gasull A., Llibre J., Mañosa V. and Mañosa F., The focus-center problem for a type of degenerate system, Nonlinearity **13**, 699-729.

[Gasull etc, 2008] Gasull A. and Zhao Yulin, Bifurcation of critical periods from the rigid quadratic isochronous vector field, Bull. Sci. Math. **132**, 292-312.

[Gavrilov, 1998a] Gavrilov L. , Nonoscilation of elliptic integrals related to cubic polynomials with symmetry of order three, Bull. London. Math. Soc. **30**, 267-273.

[Gavrilov, 1998b] Gavrilov L., Petrov modules and zeros of Abelian integrals, Bull. Sci. Math. **122**, 571-584.

[Gavrilov etc, 1990] Gavrilov L. and Horozov E., Limit cycles and zeroes of Abelian integrals satisfying third order Picard-Fuchs equations, Lecture Notes in Math. **1455**, 160-186, Springer-Verlag.

[Gavrilov, 1999] Gavrilov L., Abelian integrals related to Morse polynomials and perturbations of plane Hamiltonian vector fields, Ann. Inst. Fourier **49**, 611-652.

[Gavrilov, 2001] Gavrilov L., The infinitesimal 16th Hilbert problem in the quadratic case, Invent. Math. **143**, 449-497.

[Gavrilov etc, 2002] Gavrilov L. and Iliev I. D., Second order analysis in polynomially perturbed reversible quadratic Hamiltonian systems, Erg. Theory. Dyn. Systems to appear.

[Giacomini etc, 1996] Giacomini H., Llibre J. and Viano M., On the nonexistence, existence and uniqueness of limit cycles, Nonlinearity, **9**, 501-516.

[Giacomini etc, 2000] Giacomini H., Llibre J. and Viano M., On the shape of limit cycles that bifurcate from non-Hamiltonian centers, Nonlinear Anal. **41**, 523-537.

[Giacomini etc, 2001] Giacomini H., Llibre J. and Viano M., The shape of limit cycles that bifurcate from non-Hamiltonian centers, Nonlinear Anal. **43**, 837-859.

[Giné, 2002] Giné J., Conditions for the existence of a center for the Kukless homogeneous systems, Computers and Mathematics with Applications **43**, 1261-1269.

[Giné ect, 2011] Giné J., Kadyrsizova Z., Liu Y.R. and Romanovski Valery G., Linearizability conditions for Lotka-Volterra planar complex quartic systems having homogeneous nonlinearities, Computers and Mathematics with Applications **61**, 1190-1201.

[Gobber etc, 1979] Gobber F., Willamowski K.D., Liapunov approach to multiple Hopf bifurcation, J. Math. Anal. Appl. **71**, 330-350.

[Guckenheimer etc, 1983] Guckenheimer J. and Holmes P.J., Dynamical Systems and Bifurcations of Vector Fields, Springer-Verlag, New York.

[Gudkov, 1969] Gudkov D.A., Position of the circuits of a curve of sixth order, Soviet Math. Dokl. **10**(2), 332-335.

[Gudkov, 1974] Gudkov D.A., The topology of real projective algebraic varieties, Russian Math. Surveys, **29**(4), 3-79.

[Griffiths,1985] Griffiths P., Algebraic Curves, Peking University Press, Beijing(in Chiese).

[Han M.A., 1985] Han Maoan, Uniqueness of limit cycles of quadratic system of type $(III)_{n=0}$ around a focus of order 2, Chin. Ann. Math. **6A**(6), 661-668.

[Han M.A., 1989a] Han Maoan, Existence of limit cycles for quadratic systems with one infinite singular point, Chin. Ann. Math. **10B**(2), 261-271.

[Han M.A., 1989b] Han Maoan, Properties in the large of quadratic systems in the plane, Chin. Ann. Math. **10B**(3), 312-322.

[Han M.A., 1993] Han Maoan, Stability of the equator and bifurcation of limit cycles, World Sci. Publishing, River Edge, NJ, 79-85.

[Han M.A., 1994] Han Maoan, On sufficient conditions for certain two-dimensional systems to have at most two limit cycles, J. Diff. Equs. **107**(2), 231-237.

[Han M.A., 1995] Han Maoan, Uniqueness of limit cycles in codimension two bifurcations, Chin. Sci. Bull. **40**(21), 1766-1771.

[Han M.A., 1997a] Han Maoan, Bifurcations of limit cycles and cusp of order n, Acta Math. Sinica, New Series **13**(1), 64-75.

[Han M.A., 1997b] Han Maoan, Cyclicity of planar homoclinic loops and quadratic integrable systems, Sci. in China Ser. A **40**(12), 1247-1258.

[Han M.A., 1998a] Han Maoan, Bifurcations of limit cycles from a heteroclinic cycle of Hamiltonian systems, Chinese Ann. Math. **19B**(2), 189-196.

[Han M.A., 1998b] Han Maoan, Existence of no or at most two limit cycles surrounding multiple singular points, Acta Math. Appl. Sinica **21**(2), 206-214.

[Han M.A. etc, 1998a] Han Maoan and Jiang K., Normal forms of integrable systems at a resonant saddle, Ann. Diff. Equs. **14**(2), 150-155.

[Han M.A. etc, 1998b] Han Maoan and Ye Yanqian, On the coefficients appearing in the expansion of Melnikov function in homoclinic bifurcations, Ann. Diff. Equs. **14**(2), 156-162.

[Han M.A., 1999] Han Maoan, Liapunov constants and Hopf cyclicity of Liénard systems, Ann. Diff. Equs. **15**(2), 113-126.

[Han M.A. etc, 1999a] Han Maoan, Ye Yanqian and Zhu Deming , Cyclicity of homoclinic loops and degenerate cubic Hamiltonian, Sci. in China Ser. A **42**(6), 607-617.

[Han M.A. etc, 1999b] Han Maoan and Zhang Zhifen, Cyclicity 1 and 2 conditions for a 2-polycycle of integrable systems on the plane, J. Diff. Equs. **155**, 245-261.

[Han M.A., 2000] Han Maoan and Chen Jiang The number of limit cycles bifurcating from a pair of homoclinic loops, Sci. in China Ser. A **30**(5), 401-414.

[Han M.A., 2000] Han Maoan, On Hopf cyclicity of planar systems, J. Math. Anal. Appl. **245**, 404-422.

[Han M.A., 2001] Han Maoan, Existence of at most 1,2,or 3 zeros of a Melnikov function and limit cycles, J. Diff. Equs. **170**, 325-343.

[Han M.A. etc, 2004] Han Maoan, Bi Ping and Xiao Dongmei, Bifurcation of limit cycles and separatrix loops in singular Lienard systems, Chaos, Solitons & Fractals, **20**, 529-546.

[Han M.A. etc, 2007a] Han Maoan and Zhu Huaiping, The loop quantities and bifurcations of homoclinic loops. J. Differential Equations **234**(2), 339-359.

[Han M.A. etc, 2007b] Han Maoan, Zhang Tong-hua and Zang Hong, Bifurcation of limit cycles near equivariant compound cycles. Sci. China Ser. A **50**(4), 503-514.

[Han M.A. etc, 2007c] Han Maoan, Chen Guanrong and Sun Chengjun, On the number of limit cycles in near-Hamiltonian polynomial systems. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17**(6), 2033-2047.

[Han M.A. etc, 2007d] Han Maoan, Wang Zheng and Zang Hong, Limit cycles by Hopf and homoclinic bifurcations for near-Hamiltonian systems. (Chinese) Chinese Ann. Math. Ser. A **28**(5), 679-690; translation in Chinese J. Contemp. Math. **28**(4), 423-434.

[Han M.A. etc, 2007e] Han Maoan, Shang Desheng, Zheng Wang and Yu Pei, Bifurcation of limit cycles in a fourth-order near-Hamiltonian system. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **17**(11), 4117-4144.

[Han M.A. etc, 2008] Han Maoan, Jiang Jiao and Zhu Huaiping, Limit cycle bifurcations in near-Hamiltonian systems by perturbing a nilpotent center. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **18**(10), 3013-3027.

[Han M.A. etc, 2009] Han Maoan, Yang Junmin and Yu Pei, Hopf bifurcations for near-Hamiltonian systems. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19**(12), 4117-4130.

[Han M.A. etc, 2009] On the number of limit cycles of some Li nard systems. Can. Appl. Math. Q. **17**(1), 61-83.

[Han M.A. etc, 2010a] Han Maoan and Zhang Weinian, On Hopf bifurcation in non-smooth planar systems. J. Differential Equations **248**(9), 2399-2416.

[Han M.A. etc, 2010b] Han Maoan and Romanovski Valery G., Estimating the number of limit cycles in polynomials systems. J. Math. Anal. Appl. **368**(2), 491-497.

[Harnack, 1876] Harnack A., *Ü*ber die Vieltheilgkeit der ebenen algebraischen Curven, Math. Ann. **10**, 189-199.

[Hassard etc, 1978] Hassard B.D. and Wan Y.H., Bifurcation formulae derived from center manifold theory, J. Math. Anal. Appl. **63**(1), 297-312.

[Hassard etc, 1981] Hassard B.D., Hazzarinoff N.D. and Wan Y.H., Theory and Applications of Hopf Bifurcation, Cambridge.

[He Z.R. etc, 2009] He Zhirong and Zhang Weinian, Critical periods of a periodic annulus linking to equilibria at infinity in a cubic system, Discr. Contin. Dyn. Syst. Ser. A **24**(3), 841-854.

[Hilbert, 1902] Hilbert D., Mathematical problems (M.Newton,Transl.), Bull. Amer. Math. Soc.**8**, 437-479.

[Horozov etc, 1994] Horozov E. and Iliev I.D., On the number of limit cycles in perturbations of quadratic Hamiltonian systems, Proc. London Math. Soc. **69**, 198-224.

[Horozov etc, 1996] Horozov E. and Iliev I. D., Perturbations of quadratic Hamiltonian systems with symmetry, Ann. Inst. Henri Poincaré, Analyse non lineaire **13**, 17-56.

[Horozov etc, 1998] Horozov E. and Iliev I. D., Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonian, Nonlinearity **11**, 1521-1537.

[Huang J.F. etc, 2004a] Huang Jianfeng and Zhao Yulin, The projective vector fileds of a kind of 3-dimensional quasi-homogeneous on S^2, Nonlinear Analysis **74**, 4088-4104.

[Huang Wentao etc, 2004a] Huang Wentao and Liu Yirong, Bifurcations of limit cycles from infinity for a class of quintic polynomial system, Bull. Sci. Math. **128**, 291-301.

[Huang W.T. etc, 2004b] Huang Wentao and Liu Yirong, A polynomial differential system withnine limit cycles at infinity, Computers and Mathematics with Applications **48**, 577-588.

[Huang W.T. etc, 2005] Huang Wentao and Liu Yirong, Conditions of Infinity to be an Isochronous Center for a class of differential systems, Trends in Mathematics: Differential Equations with Symbolic Computations, Birkhauser Verlag, 37-53.

[Huang W.T. etc, 2006] Huang Wentao and Zhang Li, Limit cycles of infinity in a quintic polynomial system, Natural Science Journal of Xiangtan University **28**(3), 12-16.

[Huang W.T. etc, 2007] Huang Wentao, Liu Yirong and Zhang Weinian, Conditions of infinity to be an isochronous centre for a rational differential system, Mathematical and Computer Modelling **46**, 583-594.

[Huang W.T. etc, 2009] Huang Wentao, Liu Yirong, and Zhu Fanglai, The center-focus problem of a class of polynomial differential systems with degenerate critical points, International Journal of Nonlinear Sciences and Numerical Simulation **10**(9), 1167-1179.

[Iliev, 1996a] Iliev I.D., High-order Melnikov functions for degenerate cubic Hamiltonians, Adv. Diff. Equs. **1**(4), 689-708.

[Iliev, 1996b] Iliev I.D., The cyclicity of the period annulus of the quadratic Hamiltonian triangle, J. Diff. Equs. **128**, 309-326.

[Iliev, 1998] Iliev I.D., Perturbations of quadratic centers, Bull. Sci. Math. **122**, 107-161.

[Iliev, 2000] Iliev, I. D., On the limit cycles obtainable from polynomial perturbations of the Bogdanov-Takens Hamiltonian, Israel J. Math. to appear.

[Ilyashenko, 1969] Ilyashenko Yu.S., The origin of limit cycles under perturbation of the equation $dw/dz = -R_z/R_w$, where $R(z,w)$ is a polynomial, Math. USSR-Sb. **7**, 353-364.

[Ilyashenko, 1982] Ilyashenko Yu.S., Singular points and limit cycles of differential equations in the real and the complex planes, Preprint. Pushchino: Sci.Res. Comput. Cent., Acad. Sci. USSR. 39p.(in Russian)

[Ilyashenko, 1984] Ilyashenko Yu.S., Limit cycles of polynomial vector fields with nondegenerate singular points on the real plane, Funct. Anal. Appl. **18**(3), 32-42.

[Ilyashenko, 1985] Ilyashenko Yu.S., Dulac's memoir Sur les cycles limites and related questions in the local theory of differential equations, Russian Math. Surveys **40**, 41-78.

[Ilyashenko, 1990] Ilyashenko Yu.S., Finiteness theorems for limit cycles, Russian Math. Surveys, **40**, 143-200.

[Ilyashenko etc, 1991] Ilyashenko Yu.S. and Yakovenko S., Finitely smooth normal forms of local families of diffeomorphismes and vector fields, Russian Math. Surveys **46**, 3-19.

[Ilyashenko,1991] Ilyashenko Yu.S. Finiteness for Limit Cycles, Transl.Math.Monographs, **94**, Amer. Math. Soc., Providence, R.I.

[Ilyashenko, 1993] Ilyashenko Yu.S., Local dynamics and nonlocal bifurcations, in Bifurcations and Periodic Orbits of Vector Fields (D.Schlomiuk ed), NATO ASI Series C **408**, 279- 319, Kluwer Academic Publishers, Boston.

[Ilyashenko etc, 1995] Ilyashenko Yu.S. and Yakovenko S., Double exponential estimate for the number of zeros of complete Abelian integrals and rational envelopes of linear ordinary differential equations with an irreducible monodromy group, Inventions Mathematicae **121**, 613-647.

[Ilyashenko etc, 1995] Ilyashenko Yu.S. and Yakovenko S., Concerning the Hilbert 16th Problem, AMS Translations, Ser.2, **165**, Amer.Math. Soc., Providence, R.I.

[Ilyashenko, 1996] Ilyashenko Yu.S. Limit cycles and nonlinear Stokes phenomenon, in The Stokes Phenomenon and Hilbert's 16th Problem, World Scientific Publishing Co., Inc., River Edge, NJ. 7 -23.

[Ilyashenko etc, 1996] Ilyashenko Yu.S. and Yakovenko S., Counting real zeros of analytic functions satisfying linear ordinary differential equations, J. Diff. Equs. **126**, 613-650.

[James etc, 1991] James E.M. and Lloyd N.G., A cubic system with eight small- amplitude limit cycles, IMA J. Appl. Math. **47**(2), 163-171.

[Jebrane etc, 1994] Jebrane J.M., and Zoladek H., Abelian integrals on non- symmetric perturbations of symmetric Hamiltonian vector field, Adv. Appl. Math. **15**, 1-12.

[Jiang Qibao etc ,1999] Jiang Qibao and Han Maoan, Melnikov functions and perturbation of a planar Hamiltonian system, Chin. Ann. of Math. **20B**(2), 233-246.

[Jin X.F. etc ,1990] Jin Xiaofan and Wang Dongming, On the conditions of Kukles for the existence of a centre, Bull. London Math. Soc. **22**, 1-4.

[Joyal, 1988] Joyal P., Generalized Hopf bifurcation and its dual, generalized homoclinic bifurcation, SIAM J. Appl. Math. **48**, 481-496.

[Joyal etc, 1989] Joyal P. and Rousseau C., Saddle quantities and applications, J. Diff. Equs. **78**, 374-399.

[Joyal, 1990] Joyal P., The cusp of order N, J. Diff. Equs. **88**, 1-14.

[Kaloshin, 2001] Kaloshin V.Yu., The Hilbert-Arnold problem and estimate of the cyclicity of polycycles on the plane and in space, Funct. Anal. Appl. **35**(2), 146-147.

[Kertesz etc, 1991] Kertesz V. and Kooij R.E., Degenerate Hopf bifurcation in two dimensions, Nonlinear Anal. **17**, 267-283.

[Khovansky, 1984] Khovansky A.G., Cycles of dynamical systems on a plane and Rolle's theorem, Siberia Math. J. **25**(3), 198-203.

[Khovansky, 1984] Khovansky A.G., Real analytic manifolds with finiteness properties and complex Abelian integrals, Funct. Anal. Appl. **18**, 119-128.

[Korchagin, 1978] Korchagin A.B., New possibilities in Brusotti's method for constructing $M-$curves of order $\geqslant 8$, in Method of the Qualitative Theory of Differential Equations **2**, 149-159, Gorki (in Russian).

[Korchagin, 1989] Korchagin A.B., New $M-$curves of degree 8 and 9, Soviet Math. Dokl. **39**(3), 569-572.

[Korchagin, 1996] Korchagin A.B., Smoothing of 6-fold singular points and constructions of 9th degree M-curves, Amer. Math. Soc. Trans. **173**(2), 141-155.

[Kotova etc, 1995] Kotova A. and Stanzo V., On few-parameter generic families of vector fields on the two-dimensional sphere, in Concerning the Hilbert 16th Problem, Amer. Math. Soc. Transl. Ser.2, **165** Providence, RI, 155-201.

[Leontovich, 1951] Leontovich E.A., The creation of limit cycles from a separatrix, Sov. Math. Daklady, **4**, 641-644.

[Li B.Y. etc, 1992] Li Baoyi and Zhang Zhifen, High order Melnikov function and problem of uniformity in global bifurcation, Annali di Math. Pura. Appl. **IV**, 181-212.

[Li B.Y. etc, 1995] Li Baoyi and Zhang Zhifen, A note on a G.S.Petrov's result about the weakened 16th Hilbert problem, J. Math. Anal. Appl. **190**, 489-516.

[Li B.Y. etc, 1999] Li Baoyi and Zhang Zhifen, The cyclicity of polycycle $S^{(2)}$ of planar integrable systems with small perturbations, Adv. in Math. **17**(1), 29-40.

[Li C.Z., 1983] Li Chenzhi, Two problems of planar quadratic systems, Sci. in China Ser. A **26**, 471-481.

[Li C.Z., 1986] Li Chenzhi Non-existence of limit cycle around a weak focus of order three for any quadratic system, Chin. Ann. Math. **7B**(2), 174-190.

[Li C.Z. etc, 1989] Li Chenzhi and Rousseau C., A system with three limit cycles appearing in Hopf bifurcation and dying in homoclinic bifurcation: The cusp of order 4, J. Diff. Equs. **79**, 132-167.

[Li C.Z. etc, 1995] Li Chenzhi, Llibre J. and Zhang Zhifen, Abelian intrgrals of quadratic Hamiltonian vector fields with a invariant straight line, Publ. Math. **39**(2), 355-366.

[Li C.Z. etc, 1996] Li Chenzhi, Llibre J. and Zhang Zhifen, Weak focus, limit cycles, and bifurcations for bounded quadratic systems, J. Diff. Equs. **115**(1), 193-223.

[Li C.Z. etc, 2000a] Li Chenzhi, Li Weigu, Llibre J. and Zhang Zhifen, On the limit cycles of polynomial systems with homogeneous nonlinearities, Proc. of Edinburgh Math. Soc. **43**, 529-543.

[Li C.Z. etc, 2000b] Li Chenzhi, Li Weigu, Llibre J. and Zhang Zhifen, Linear estimation of the number of zeros of Abelian integrals for quadratic isochronous centers, Nonlinearity **13**, 1775-1800.

[Li C.Z. etc, 2001] Li Chenzhi, Li Weigu, Llibre J. and Zhang Zhifen, Linear estimation of the number of zeros of Abelian integrals for a class of integrable non-Hamiltonian systems, Nonlinear Anal. **47**, 4527-4531.

[Li C.Z., 2001] Li Chenzhi, Li Weigu, Llibre J. and Zhang Zhifen, Polynomial systems: a lower bound for the weakened 16th Hilbert problem, Extracta Mathematicae **16**, 1-7.

[Li C.Z. etc, 2001] Li Chenzhi and Zhang Zenghua, Weak Hilbert 16th problem for $n = 2$, preprint.

[Li C.Z. etc, 2002] Li Chenzhi, Li Weigu, Llibre J. and Zhang Zhifen, Linear estimation of the number of zeros of Abelian integrals for some cubic isochronous centers, J. Diff. Equs., to appear.

[Li C.Z. etc, 2006] Li Chenzhi, Mardesic P. and Roussarie R., Perturbations of symmetric elliptic Hamiltonians of degree four, Journal of Differential Equations, 231(2006),78-91.

[Li C.Z. etc, 2009a] Li Chengzhi, Liu Changjian and Yang Jiazhong, A cubic system with thirteen limit cycles, J. Diff. Equs. **246**, 3609-3619.

[Li C.Z. etc, 2009b] Li Chengzhi and Llibre J., The cyclicity of period annulus of a quadratic reversible Lotka–Volterra system, Nonlinearity **22**, 2971-2979.

[Li C.Z. etc, 2010a] Li Chenzhi and Llibre J., Quadratic perturbations of a quadratic reversible Lotka-Volterra system, Qualitative Theory of Dynamical Systems **9**, 235-248.

[Li C.Z. etc, 2010b] Li Chenzhi and Li Weigu, Weak Hilbert's 16th problem and the related research(in Chinese), Adv. Math. (China) **39**(5), 513-526.

[Li J.B. etc, 1985] Li Jibin and Li Cunfu, Planar cubic Hamiltonian systems and distribution of limit cycles of (E_3), Acta Math. Sinica **28**(4), 509-521.

[Li J.B. etc, 1987a] Li Jibin and Chen Xiaoqiu, Poincaré bifurcations in the quadratic differential system, Kexue Tongbao (English Ed.) **32**(10), 655-660.

[Li J.B. etc, 1987b] Li Jibin and Huang Qimin, Bifurcations of limit cycles forming compound eyes in the cubic system, Chin. Ann. Math. **8B**, 391-403.

[Li J.B. etc, 1988] Li Jibin and Ou Yuehua, Global bifurcations and chaotic behavior in a disturbed quadratic system with two centers, Acta Math. Appl. Sinica **11**(3), 312-323.

[Li J.B. etc , 1989] Li Jibin and Zhao Xiaohua, Rotation symmetry groups of planar Hamiltonian systems, Ann. Diff. Equs. **5**, 25-33.

[Li J.B. etc, 1991] Li Jibin and Wan Shidong, Global bifurcations in a disturbed Hamiltonian vector field approaching a 3:1 resonant Poincaré map, Acta. Math. Appl. Sinica, Eng. Ser. **7**, 80-89.

[Li J.B. etc, 1991a] Li Jibin and Liu Zhengrong, Bifurcation set and limit cycles forming compound eyes in a perturbed Hamiltonian system, Publications Mathmatiques **35**, 487-506.

[Li J.B. etc,1991b] Li Jibin and Lin Yipin, Normal form and critical points of the period of closed orbits for planar autonomous systems, Acta Mathematica Sinica **34**, 490-501 (in Chinese).

[Li J.B. etc, 1992a] Li Jibin and Liu Zhengrong, Bifurcation set and compound eyes in a perturbed cubic Hamiltonian system, In Ordinary and Delay Differential Equations, π Pitman Research Notes in Math. Ser. **272**, 116-128, Longman, England.

[Li J.B. etc, 1992b] Li Jibin and Lin Yipin, Global bifurcations in a perturbed cubic system with Z_2-symmetry, Acta Math. Appl. Sinica (English Ser.) **8**(2), 131-143.

[Li J.B. etc, 1995] Li Jibin and Feng Beiye, Stability, Bifurcations and Chaos, Yunnan Science and Technology Press, Kunming (in Chinese).

[Li J.B. etc, 2002a] Li Jibin, Chan H. S. Y. and Chung, K. W., Bifurcations of limit cycles in a Z_6-equivariant planar vector field of degree 5, Sci. in China Ser. A **45** (7), 817-826.

[Li J.B. etc, 2002b] Li Jibin, Chan H. S. Y. and Chung K. W., Investigations of bifurcations of limit cycles in a Z_2-equivariant planar vector field of degree 5, Int. J. Bifurcation and Chaos, to appear.

[Li J.B. etc, 2002c] Li Jibin, Chan H. S. Y. and Chung K. W., Some lower bounds for $H(n)$ in Hilbert's 16th problem, Qualitative Theory of Dynamical Systems **3**, to appear.

[Li J.B., 2003] Li Jibin Hilbert's 16th problem and Bifurcations of Planar Polynomial vector Fields , Int. J of Bifurcation and Chaos **13**(1): 47-106.

[Li J.B. etc, 2004] Li Jibin and Zhang Mingqiang, Bifurcations of limit cycles of Z_8-equivariant planar vector field of degree 7, J.Dynamics And Differentil Equations **16**(4),1123-1139.

[Li J.B. etc, 2005] Li Jibin and Zhou Hongxian, On the Control of Parameters of Distributions of Limit Cycles for a Z_2-Equivariant Perturbed Planar Hamiltonian Polynomial Vector Field, Int. J. Bifurcation and Chaos **15**(1), 137-155.

[Li J.B. etc, 2006] Li Jibin and Zhang Mingji, Bifurcations of limit cycles in a Z_2-equivariant planar polynomial vector field of degree 7, Int. J. Bifurcation and Chaos **16**, 925-943.

[Li J.B. etc, 2010] Li Jibin and Liu Yirong, New results on the study of Z_q-equivariant planar polynomial vector fields,Qual. Theory Dyn. Syst. **9**,167-219.

[Li W.G., 2000] Li Weigu, Theory and Applications of Normal Forms, Science Press, Beijing (in Chinese).

[Li W.G. ect, 2002] Li Weigu, Zhao Yulin, Li Chengzhi and Zhang Zhifen, Abelian integrals for quadratic centers having almost all their orbits formed by quartics, Nonlinearity **15**, 863-885.

[Liang H.H. ect, 2009] Liang Haihua and Zhao Yulin, On the Period function of reversible quadratic centers with their orbits inside quartics, Nonlinear Analysis: Theory & Applications **71**(2009),5655-5671.

[Liang H.H. ect, 2010] Liang Haihua and Zhao Yulin, Quadratic perturbations of a class of quadratic reversible system with one center, Discrete and Continuous Dynamical Systems **27**(1),325-335.

[Lins etc, 1977] Lins A. De Melo W. and Pugh C.C., On Liénard's equation, Lecture Notes in Math. **597**, 335-357.

[Liu F. ect, 2009] Liu Fei, Chen Cheng and Zhang Xiang, Integrable Hamiltonian systems with positive topological entropy Bull. Sci. Math.**133**, 837-847.

[Liu F. ect, 2011] Liu Fei, Llibre J. and Zhang Xiang, Heteroclinic orbits for a class of Hamiltonian systems on Riemannian manifolds, Discr. Cont. Dynam. Sys. **29**, 1097-1111.

[Liu Y.R., 1987] Liu Yirong, Formula of focal values, conditions of center and center integral for a class of planar cubic system, Chinese Science Bulletin **14**(2) 85-87.

[Liu Y.R., 1988] Liu Yirong, Formulas of focal values, center conditions and center integrals for the system $(E_3^{(3)})$, Chinese Science Bulletin **33**(5), 357-359.

[Liu Y.R. etc, 1989] Liu Yirong and Li Jibin, Theory of values of singular point in complex autonomous differential systems, Sci. in China,**A**(3) ,(in Chinese), 245-255.

[Liu Y.R., 1992] Liu Yirong, Singular point values of (E_n) and some bifurcation problems, Sci. in China Sec. A **30**, 1233-1241.

[Liu Y.R. etc, 1995] Liu Yirong and Wang Wusong, Simple Integration of E_n and Its Application, Hunan Annals of Mathematics **15**(2), 109-112.

[Liu Y.R., 1999] Liu Yirong, Multiplicity of higher order singular point of differential autonomous system, J.Cent.South Univ.Technol. **30**(3), 622-623.

[Liu Y.R., 2001] Liu Yirong, Theory of center-focus in a class of high order singular points and infinity, Sci. in China **31**(1), 37-48.

[Liu Y.R., 2002] Liu Yirong, The Generalized focal values and bifurcation of limits cycles for quasi quadratic systems, Acta Mathematica Sinica **45**(4), 671-682.

[Liu Y.R. etc, 2002a] Liu Yingrong and Chen Haibo, Stability and bifurcations of limit cycles of the equator in a class of cubic polynomial systems, Computers and Mathematics with Applications **44**, 997-1005.

[Liu Y.R. etc, 2002b] Liu Yingrong and Chen Haibo, Formulas of singurlar point quantities and the first 10 saddle quantities for a class of cubic system, Acta Mathematicae Applicatae Sinica **25**(2), 295-302 (in Chinese).

[Liu Y.R. etc, 2003a] Liu Yirong and Huang Wentao, A new method to determine isochronous centerconditions for polynomial differential systems, Bull. Sci. Math. **127**, 133-148.

[Liu Y.R. etc,2003b] Liu Yingrong and Zhao Meichun, Stability and bifurcation of limit cycles of the equator in a class of fifth polynomial systems, Chinese Journal of Contemporary Mathematics **23**(1), 67-72.

[Liu Y.R. etc, 2004] Liu Yirong and Huang Wentao, Center and isochronous center at infinity for differential systems, Bull. Sci. Math. **128**, 77-89.

[Liu Y.R. etc, 2005] Liu yirong and Huang Wentao, A cubic system with twelve small amplitude limit cycles, Bull. Sci. Math. **129**,83-98.

[Liu Y.R. etc, 2006a] Liu Yirong and Huang Wentao, Seven large-amplitude limit cyclea in a cubic polynomial system, Int. J. Bifur. Chaos and Chaos **16**(2), 473-485.

[Liu Y.R. etc, 2006b] Liu Yirong and Li Jibin, Periodic Constants and Time-Angle Difference of Isochronous Centers for Complex Analytic Systems, Int. J. Bifur. Chaos. **16**(12), (2006) 3747-3757.

[Liu Y.R. etc, 2006c] Liu Yirong and Wang Qinlong, Quantities at infinity in translational polynomial vector fields, Bull. Sci.Math.**130**(4) 360-375.

[Liu Y.R. etc, 2008a] Liu Yirong and Li Jibin, center and isochronous center problems for quasi analytic systems, Acta Mathematica Sinica, English Series **24**(9), 1569-1582.

[Liu Y.R. etc 2008b] Liu Yirong, Li Jibin and Huang Wentao, Singular Point Values, Center Problem And Bifurcations Of Limit Cycles Of Two Dimensional Differential Autonomous Systems, Beijing: Science Press.

[Liu Y.R. etc 2009a] Liu Yirong and Li Jibin, New study on the center problem and bifurcations of limit cycles for The Lyapunov system (I), Int. J. Bifur. Chaos. **19**(11), 3791-3801.

[Liu Y.R. etc, 2009b] Liu Yirong and Li Jibin New study on the center problem and bifurcations of limit cycles for The Lyapunov system (II), Int. J. Bifur. Chaos. **19**(9), 3087-3099.

[Liu Y.R. etc, 2009c] Liu Yirong and Li Jibin, Center problem and multiple Hopf Bifurcation for the Z_5-equivariant planar polynomial vector fields of degree 5,Int. J. Bifur. Chaos. **19**(6), 2115-2121.

[Liu Y.R. etc, 2009d] Liu Yirong and Li Jibin, Center problem and multiple Hopf Bifurcation for the Z_6-equivariant planar polynomial vector fields of degree 5, Int. J. Bifur. Chaos. **19**(5), 1741-1749.

[Liu Y.R. etc, 2011a] Liu Yirong and Li Jibin, On the Study of Three-Order Nilpotent Critical Points: Integral Factor Method , Int. J. Bifur. Chaos. **21**(5), 1293-1309.

[Liu Y.R. etc, 2011b] Liu Yirong and Li Jibin, Complete Study on a Bi-Center Problemfor the Z_2 Equivariant Cubic Vector Fields, Acta Mathematica Sinica (English Series) **27**(7), 1379-1394.

[Liu Y.R. etc, 2011c] Liu Yirong and Li Jibin, Z_2 Equivariant Cubic System With 13 Limit Cycles, Acta Mathematicae Applicatae Sinica (English Series),to appear.

[Liu Y.R. etc, 2011d] Liu Yirong and Li Jibin, Bifurcations of limit cycles created by a multiple nilponent critical point of planarndynamical systems, Int. J. Bifur. Chaos. **21**(2), 497-504.

[Liu Z.Q. etc, 1980] Liu Zunquan and Qin Chaobin, Mechanical deduction of formulas of differential equations(I),Science in China **8A**, 812-820.

[Liu Z.Q. etc, 1981] Liu Zunquan and Qin Chaobin Mechanical deduction of formulas of differential equations(II), Cinese Science Bulletin **5**, 257-258.

[Llibre etc, 2001a] Llibre J. and Zhang Xiang, On the number of limit cycles for some perturbed Hamiltonian polynomial systems, Dyn. Contin. Discrete Impuls. Syst. Series A: Math. Anal. **8**, 161-181.

[Llibre etc, 2001b] Llibre J. and Zhang Xiang , Topological phase portraits of planar semi-linear quadratic vector fields, Houston J. Math. **27**(2), 247-296.

bibitem[Llibre etc, 2007]Llibre J. and Zhao Yulin, Algebraic limit cycles in polynomial systems of differential equations J. Phys. A: Math. Theor.**40**, 14207-14222.

bibitem[Llibre etc, 2008]Llibre J. and Zhang Xiang, On the algebraic limit cycles of Li enard systems. Nonlinearity **21**, 2011-2022.

[Llibre etc, 2009a] Llibre J. and Valls C, Classfication of centers, their cyclicity an isochronicity for a class of polynomial differential systems generalizing the linear systems with cubic homogeneous nonlinearties, J.Differential Equations **246**, 2192-2204.

[Llibre etc, 2009b] Llibre J. and Zhang Xiang, Darboux theory of integrability in C_n taking into account the multiplicity, J. Differential Equations **246**, 541-551.

[Llibre etc, 2009c] Llibre J. and Zhang Xiang, Darboux theory of integrability for polynomial vector fields in Rn taking into account the multiplicity at infinity, Bull. Sci. Math. **133**,765-778.

[Llibre etc, 2010] Llibre J. and Zhang Xiang, Rational first integrals in the Darboux theory of integrability in C_n, Bull. Sci. Math. **134**, 189-195.

[Llibre etc, 2011] Llibre J. and Zhang Xiang,On the Hopf-zero bifurcation of the Michelson system, Nonlinear Analysis: Real World Applications **12**,1650-1653.

[Lloyd, 1988] Lloyd N.G., Limit cycles of polynomial systems, some recent developments, In New Directions in Dynamical Systems, Ed. T. Bedford and J. Swift, London Mathematical Society Lecture Notes **127**, 192-234.

[Lloyd etc, 1988] Lloyd N.G. and Lynch, S., Small amplitude limit cycles of certain Lienard systems, Proc. Royal. Soc. London Ser A **418**, 199-208.

[Lloyd etc, 1990a] Lloyd N.G. and Pearson J.M., Reduce and the bifurcation of limit cycles, J. Symbolic Comput. **9**(9), 215-224.

[Lloyd etc, 1990b] Lloyd N.G. and Pearson J.M., Conditions for a centre and the bifurcation of limit cycles in a class of cubic systems, Lecture Notes in Math. **1455**, 231-242.

[Lloyd etc, 1996] Lloyd N.G. and Pearson J.M., Romanovskii, V. A., Computing integrability conditions for a cubic differential system, Computers Math. Applic. **32**(10), 99-107.

[Lloyd etc, 1997] Lloyd N.G., Christopher J., Devlin,J. and Pearson,J.M., Uasmin,N., Quadratic like cubic systems, Differential Equations Dynamical Systems **5**(3-4) , 329-345.

[Lloyd etc,2002] Lloyd N.G. and Pearson J.M., Symmetry in planar dynamical systems, J. Symbolic computation **33**, 21-36, 357-366.

[Loud, 1964] Loud W.S., Behavior of the period of solutions of certain plane autonomous systems near centers, Contributions to Differential Equations **3** , 21-36.

[Luo D.J. etc, 1990] Luo Dingjun and Zhu Deming, The stability of homoclinic loop and the uniqueness for generating limit cycle, Chin. Ann. Math. **11A**(1), 95-103 (in Chinese).

[Luo D.J. etc, 1992] Luo Dingjun, Han Maoan and Zhu Deming, The uniqueness of limit cycles bifurcating from a singular closed orbit (I), Acta Math. Sinica **35**(3), 407-417 (in Chinese).

[Luo D.J. etc, 1997] Luo Dingjun, Wang Xian, Zhu Deming and Han Maoan , Bifurcation Theory and Methods of Dynamical Systems, Advanced Series in Dynamical Systems **15**, World Scientific, Singapore.

[Luo H.Y. etc, 2005] Luo Haiying and Li Jibin, What are the separatrix values named by Leonovich on homoclinic bifurcation, Applied Mathematics and Mechanics **26**(4),457-464.

[Lyapunov, 1947] Lyapunov M.A., Probléme génnéral de la stabilité du mouvement, Ann. of Math. Stud. 17, Princeton University Press.

[Lynch etc, 1999] Lynch S. and Christopher C.J., Limit cycles in highly nonlinear differential equations, J. Sound Vibration **224**(3), 505-517.

[Ma Z.E. etc, 1983] Ma Zhien and Wang Ernian, The stability of a loop formed the separatrix of a saddle point and the condition to produce a limit cycle, Chin. Ann. Math. **4A** (1), 105-110 (in Chinese).

[Mao R. etc, 1996] Mao Rui and Wang Duo, Jumping property of Liapunov values, Sci. in China Ser.A **39**(12), 1280-1287.

[Mardesić, 1990] Mardesić P., The number of limit cycles of polynomial deformations of a Hamiltonian vector field, Ergodic Theory Dynamical Systems **10**, 523-529.

[Mardesić, 1991] Mardesić P., An explicit bound for the multiplicity of zeros of generic Abelian integrals, Nonlinearity **4**, 845-852.

[Mardesić etc, 1995] Mardesic P., Rousseau C. and Toni B., Linearzation of isochronous centers, J.Differential Equations **121**, 67-108.

[Markov, 1996] Markov Ya., Limit cycles of perturbations of a class of quadratic Hamiltonian vector fields, Sezdica Math.J. **22**(2), 91-108.

[Melnikov, 1963] Melnikov V.K., On the stability of the center for time periodic perturbations, Trans. Moscow Math. Soc. **12**, 1-57.

[Mourtada, 1994] Mourtada A., Degenerate and non-trivial hyperbolic polycycles with two vertices, J. Diff. Equs. **113**, 68-83.

[Morsalani etc, 1994] Morsalani M. and Mourtada A., Degenerate and non-trivial 2-hyperbolic polycycles: two independent Ecalle-Roussaie compensators and Khovanskii's theory, Nonlinearity **7**, 1593-1604.

[Moussu,1982] Moussu R., Symétrie et forme normale des centres et foyers dégénérés, Ergodic Theory Dynam. Systems **2**, 241-251.

[Novikov etc, 1995] Novikov D. and Yakovenko S., Simple exponential estimate for the number of real zeros of complete Abelian integrals, Ann. Inst. Fourier **45**, 897-927.

[Otrokov, 1954] Otrokov N.T., On the number of limit cycles of a differential equation in a neighborhood of a singular point, Mat. Sb. **34**, 127-144 (in Russian).

[Pearson etc, 1996] Pearson J.M., Lloyd N.G. and Christopher C.J., Algorithmic derivation of centre conditions, SIAM Rev. **38**, 619-636.

[Perko, 1984] Perko L.M., Limit cycles of quadratic systems in the plane, Rocky Mountain Journal of Math. **14**, 619-645.

[Perko, 1991] Perko L.M., Differential Equations and Dynamical Systems, Springer-Verlag, New York.

[Perko, 1992] Perko L.M., A global analysis of the Bogdanov-Takens system, SIAM J. Appl. Math. **52**(4), 1172-1192.

[Perko, 1995] Perko L.M., Multiple limit cycle bifurcation surfaces and global families of multiple limit cycles, J. Diff. Equs.**122**, 89-113.

[Petrov, 1985] Petrov G.S., Elliptic integrals and their non-oscillation, Funct. Anal. Appl. **20**, 37-40.

[Petrov, 1988] Petrov G.S., The Chebyschev property of elliptic integrals, Funct. Anal. Appl. **22**, 72-73.

[Petrov, 1989] Petrov G.S., Complex zeros of elliptic integral, Funct. Anal. Appl. **23**, 88-89.

[Petrov, 1990] Petrov G.S., Non-oscillations of elliptic integrals, Funct. Anal. Appl. **24**, 45-50.

[Petrovskii, 1938] Petrovskii I.G., On the topology of real plane algebraic curves, Ann. Math.**39**, 189-209.

[Pleshkan,1969] Pleshkan I., A new method of investigating the isochronicity of a system of two differential equations, Differential equations **5**, 796-802.

[Poincaré, 1881-1882] Poincaré H., Mémoire sur les courbes définies par les équations différentielles, J.de Mathématiques (3)**7**(1881),375-422; **8**(1882),251-296; Oeuvers de Henri Poincaré, vol. 1,Gauthier-Villars, Paris, 1951,3-84.

[Poincaré, 1885] Poincaré H., Mémoire sur les courbes définies par les équations différentielles, J.de Math.Pures Appl. (4)**1**(1885),167-244; Oeuvers de Henri Poincaré, vol. 1, Gauthier-Villars, Paris, 1951,95-144.

[Poincaré, 1891] Poincaré H., Sur l'intégration algébrique des équations différentielles,C. R. Acad. Sci. Paris **112**, 761-764.

[Poincaré, 1891-1897] Poincaré H., Sur l'intégration algébrique des équations différentielles du premier ordreet du premier degré I and II, Rendiconti del circolo matematico di Palermo **5**, 161-191; **11**, 193-239.

[Poincaré, 1892] Poincaré H., Méthodes mouvelles de la mécanique célete, **1**, 58.

[Petrovskii etc, 1955] Petrovskii I.G. and Landis E.M., On the number of limit cycles of the equation $dy/dx = P(x, y)/Q(x, y)$ where P and Q are polynomials of the second degree, Mat. Sb. N.S. **37**(79), 209-250; Amer. Math. Soc. Transl. **16**(2),177-221.

[Pontryagin, 1934] Pontryagin L.S., On dynamical systems close to Hamiltonian systems, Zh. EKsp. Teor. Fiz.**4**(8), 234-238.(Russian)

[Pugh, 1975] Pugh C., Hilbert's 16th problem: limit cycles of polynomial vector fields in the plane, Lecture Notes in Math. **468**, Springer, New York.

[Qin Y.X., 1959] Qin Yuanxun, Integral Curves Defined by Differential Equations, Science Press, Beijing (in Chinese).

[Qin Y.X. etc, 1981] Qin Yuanxun and Liu Zunquan Mechanical deduction of formulas of differential equations(III), Cinese Science Bulletin **5**, 388-391.

[Qin Y.X., 1982] Qin Yuanxun, On Hilbert's 16th problem, J. Northwest Univ. **12** (2), 1-15.

[Qin Y.X., 1985] Qin Yuanxun, Integral Surfaces Defined by Differential Equations, Northwest University Press, 52-66.

[Reyn, 1994] Reyn, J., A bibliography of the qualitative theory of quadratic systems of differential equations in the plane, Third edition, Delft University of Technology, Faculty of Technical Mathematics and Informations, Report.

[Romanovskii, 1991] Romanovskii V.G., Properties of the ideal generated by Liapunov values, Diff. Equs. **27**(2), 141-150.

[Romanovskii, 1992] Romanovskii V.G., Basis of an ideal generated by the Liapunov quantities, Diff. Equs. **28**(4), 501-509.

[Romanovskii, 1993] Romanovskii V.G., On a calculation of Liapunov focus quantities in the case of two imaginary roots, Diff. Equs. **29**, 782-787.

[Romanovski ect,2001a] Romanovski V.G. and Robnik M., The center and isochronicity problems for some cubic systems, Journal of Physics A: Mathematical and General **34**, 10267-10292.

[Romanovskii ect, 2001b] Romanovskii V.G. and Suba A., Centers of cubic systems, Ann.of Diff. Equs. **17**(4), 363-370.

[Romanovskii ect,2001c] Romanovski V.G. and Marko R., The centre and isochronicity problems for some cubic system, J.Phys.A: Math.Gen.**34**, 10267-10292.

[Romanovskii ect,2009] Romanovski V.G. and Shafer D.S., The Center and Cyclity Problems, A Computational Algebra Approach, Birkhauser.

[Roussarie, 1986] Roussarie R., On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, Bol. Soc. Brasil. Mat. **17**, 67-101.

[Roussarie, 1988] Roussarie R., A note on finite cyclicity and Hilbert's 16th problem, in Dynamical Systems, Valparaiso 1986 (R. Bamom et al., eds), Lecture Notes in Math. **1331**,161-168, Springer-Verlag, New York.

[Roussarie, 1993] Roussarie R., Techniques in the theory of local bifurcations: cyclicity and desingularization, in Bifurcations and Periodic Orbits of Vector Fields (D.Schlomiuk ed), NATO ASI Series C **408**, 347-382, Kluwer Academic Publishers, London.

[Roussarie, 1998] Roussarie R., Bifurcation of Planar Vector Fields and Hilbert's Sixteenth Problem, Progress in Math. **164**, Birkhauser Verlag, Basel.

[Rousseau ect, 1988] Rousseau C. and Schlomiuk D., Generalized Hopf bifurcations and applications to planar quadratic systems, Ann. Polinici Math. **49**, 1-16.

[Rousseau, 1989] Rousseau C., Codimension 1 and 2 bifurcations of fixed points of diffeomorphisms and periodic solutions of vector fields, Ann. Sci. Math. Quebec **13**, 55-91.

[Rousseau etc, 1991] Rousseau C. and Zoladek H., Zeroes of complete elliptic integrals for 1:2 resonance, J. Diff. Equs. **94**, 42-54.

[Rousseau, 1993] Rousseau C., Bifurcation methods in polynomial systems, in Bifurcations and Periodic Orbits of Vector Fields (D.Schlomiuk ed), NATO ASI Series C **408**, 383-428, Kluwer Academic Publishers, London.

[Rousseau, 1997] Rousseau C., Hilbert's 16th problem for quadratic vector fields and cyclicity of graphics, Nonlinear Anal. **30**(1), 437-445.

[Rousseau etc, 2002] Rousseau C. and Zhu Huaiping, PP-graphics with a nilpotent elliptic singularity in quadratic systems and Hilbert's 16th problem, to appear in J. Diff. Equs.

[Rousseau etc, 2010] Ruan Shigui, Tang Yilei and Zhang Weinian, Versal unfoldings of predator-prey systems with ratio-dependent functional response, J. Diff. Equs. **249**(6), 1410-1435.

[Rujenok, 1987] Rujenok, Limit cycles of two dimensional autonomous systems with cubic nonlinearnterms, Diff.equation **23**, 825-834.(in Russian)

[Rychkov, 1973] Rychkov G.S., The maximum number of limit cycles of the system $\dot{x} = y - a_0x - a_1x^3 - a_2x^5$, $\dot{y} = -x$ is two, Diff. Equs. **11**, 380-391.

[Salih etc,2002] Salih N. and Pons R., Center conditions for a lopsided quartic polynomial vector field, Bull.Sci.Math. **126**, 369-378.

[Schecter, 1987] Schecter S., The saddle-node separatrix loop bifurcation, SIAM J. Math. Anal. **18**, 1142-1156.

[Schlomiuk, 1990] Schlomiuk D., Integraibility of planar quadratic fields,Expo.Math. **8**, 3-25.

[Schlomiuk, 1993a] Schlomiuk D., Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc. **338**, 799-841.

[Schlomiuk, 1993b] Schlomiuk D., Bifurcations and Periodic Orbits of Vector Fields, NATO ASI Series C **408**, Kluwer Academic Publishers, London.

[Schlomiuk, 1993c] Schlomiuk D., Algebraic and Geometric Aspects of the Theory of Polynomial Vector Fields, in Bifurcations and Periodic Orbits of Vector Fields (D.Schlomiuk ed), NATO ASI Series C **408**, 429-467, Kluwer Academic Publishers, London.

[Schlomiuk, 2004] Schlomiuk D., Aspects of planar polynimial vector fields: global vesus local, real versus complex, analytic versus algebraic and geometric, In Y. Ilyashenko and C. Rouseau(eds,), Normal Forms, Bifurcations and Finiteness Problems in Differential Equations, 471-509, Kluwer, Academic Publishers.

[Schlomiuk ect, 2005] Schlomiuk D.,Bolibrukh A.A., Kaloshin V. and Buium A., On Finiteness in Differential Equations and Diophantine Geometry, CRM MonoGraph Series **24**, American Mathematical Society.

[Seidenberg, 1968] Seidenberg A., Reduction of singularities of the differential equation $Ady = Bdx$, Amer. J. Math. **90**, 248-269.

[Shafer etc, 1995] Shafer L.S. and Zegeling A., Bifurcation of limit cycles from quadratic centers, J. Diff. Equs. **122**, 48-70.

[Shao Y. etc, 2007] Shao Yi and Zhao Yulin, The number of zeros of Abelian integrals for a class of plane quadratic systems with two centres (in Chinese), Acta Mathematica Sinica(Chinese Series) **50**, 451-460.

[Shao Y. etc, 2011] Shao Yi and Zhao Yulin, The cyclicity and period function of a class of quadratic reversible Lotka-Volterra system of genus one, J. Math. Anal. & Appl. **377**, 817-827.

[Shen L.R., 1998] Shen Liren, Qualitative Theory of Differential Equations in Complex Domain, Anhui University PressAnhui, 33-34. (in Chinese)

[Shi S.L., 1980] Shi Songling, A concrete example of a quadratic system of the existence of four limit cycles for plane quadratic systems, Sci. Sinica **A 23**, 153-158.

[Shi S.L., 1984] Shi Songling, On the structure of Poincaré-Liapunov constants for the weak focus of polynomial vector fields, J. Diff. Equs. **52**, 52-57.

[Sibirskii, 1965] Sibirskii K.S., The number of limit cycles in the neighborhood of a singular point, Diff. Equs. **1**, 36-47.

[Sibirskii, 1976] Sibirskii K.S., Algebraic Invariants of Differential Equations and Matrices, Kishinev: Shtiintsa (in Russian).

[Simon etc 2000] Simon G. and Pierre T., Integrablity and linearizablity of the Lotka-Voleterra system with a saddte point with rational hyprrbolicitr ratio, Journal of Differential Equations. **184**, 20-47.

[Smale, 1991] Smale S., Dynamics retrospetcive: great problems, attempts that failed, Physica **D 51**, 261-273.

[Smale, 1998] Smale S. , Mathematical problems for the next century, The Math.Intell. **20**(2), 7-15; and In Mathematics: Frontiers and Perspetcives 2000, 271-294.

[Smits, 1998] Smits B., Singular perturbations arising in Hilbert's 16th problem for quadratic vetcor fields, ZAMMZ. Angew. Math. Mech. **78**(2), 133-136.

[Sotomayor etc, 1983] Sotomayor J. and Paterlini R., Quadratic vetcor fields with finitely many periodic orbits, in Geometric Dynamics (J.Palis,Jr.,ed.), Letcure Notes in Math.**1007**, 735-766, Springer-Verlag.

[Strozyna etc,2002] Strozyna E. and Zoladek, H., The analytic normal for the nilpotent singularty, J.Differential Equations. **179**,479-537.

[Sun J.H., 1991] Sun Jianhua, Bifurcation to separatrix loops in supercritical cases, Chin. Ann. Math. **12**(5), 636-643.

[Sun J.H., 1994] Sun Jianhua, The uniqueness bifurcation to separatrix loops in supercritical cases, Chin. Ann. Math. **15B**(4), 493-500.

[Sun J.H. etc, 1991] Sun Jianhua and Shen B.Q., Limit cycles for a class of higher-degree polynomial system, J. Quart. Math. **6**(3), 33-37.

[Takens, 1973] Takens F., Unfolding of certain singularities of vetcor fields:generalized Hopf bifurcations, J. Diff. Equs. **14**, 476-493.

[Takens, 1974] Takens F., Singularities of vetcor fields, Publ. Math. l'IHES **43**, 47- 100.

[Varchenko, 1984] Varchenko A.N., An estimate of the number of zeros of an Abelian integral depending on a parameter and limiting cycles, Funct. Anal. Appl., **18**, 98-108.

[Wan etc, 2006] Wan S., Yu Pei and Li Jibin, Bifurcations of Limit Cycles in a Z_{10}-Equivariant Vetcor Field, Int. J. Bifurcation and Chaos **16**(8), 2309-2324.

[Wang D., 1990a] Wang Duo, An introduction to the normal form theory of ordinary differential equations, Adv. in Math. **19**, 38-71.

[Wang D., 1990b] Wang Duo, Hopf bifurcation at the nonzero foci in 1:4 resonance, Acta Math. Sinica New Ser. **6**, 10-17.

[Wang D., 1990c] Wang Duo , Applications of the representation theory to normal form theory, Sci. in China Ser. **A 33**, 923-933.

[Wang D. etc, 1994a] Wang Duo and Mao Rei , A complex algorithm for computing Liapunov values, Random. Comput. Dynam. **2**, 261-277.

[Wang D. etc, 1994b] Wang Duo and Mao Rei, Improved K-K algorithm for computing Liapunov values, Tatra Mountains Math. Publ. **4**, 235-242.

[Wang J. H. etc, 2011] Wang Jihua and Xiao Dongmei, On the number of limit cycles in small perturbations of a class of hyper-elliptic Hamiltonian systems with one nilpotent saddle, J. Differential Equations. **250**, 2227-2243.

[Wang Q. L. etc, 2007] Wang Qinlong and Liu Yirong, Bifurcation and Isochronicity at Infinity in a Class of Cubic Polynomial Vetcor Fields, Acta Mathematicae Applicatae Sinica, **3**, 451-466.

[Wang Q. L. etc, 2010a] Wang Qinlong, Liu Yirong and Chen Haibo, Hopf bifurcation for a class of three-dimensional nonlinear dynamic systems,Bull. Sci. math. **134**, 786-798.

[Wang Q. L. etc, 2010b] Wang Qinlong and Liu Yirong , Generalized isochronous centers for complex systems, Acta Mathematica Sinica, English Series, **26**(9), 1779-1792.

[Wang X. etc, 1992] Wang Xian and Kooij R.E., Limit cycles in cubic systems with a cusp, SIAM J. Math. Anal. **23**(6), 1609-1622.

[Wang Z.X. etc, 2009] Wang Zhaoxia, Chen Xingwu , and Zhang Weinian, Local bifurcations of critical periods in a generalized 2-D LV system, Appl. Math. Comput. **214**(1), 17-25.

[Wang Z.X. etc, 2010] Wang Zhaoxia, Chen Xingwu, and Zhang Weinian, Non-isochronicity of the center in polynomial Hamiltonian systems, Nonlinear Analysis **73**(1), 228-243.

[Wilson, 1978] Wilson G., Hilbert's sixteenth problem, Topology **17**(1), 53-73.

[Wu K.L., 2010] Wu Kulin and Zhao Yulin, Isochronicity for a class of reversible systems, J. Math. Anal. & Appl. **364** (1), 300-307.

[Xiao D. M. etc, 2000] Xiao Dongmei and Li Wenxia, Limit cycles for competitive three dimensional Lotka-Volterra system, Journal of Differential Equations. **161**, 1-16.

[Xiao D. M. etc, 2003] Xiao Dongmei and Zhang Zhifeng, On the uniqueness and nonexistence of limit cycles for predator-prey systems, Nonlinearity **16**, 1-17.

[Xiao D. M. etc, 2003] Xiao Dongmei and Zhang Zhifen, On the existence and uniqueness of limit cycles for generalized Li nard systems J. Math. Anal. Appl. **343**, 299-309.

[Xiao P., 2005] Xiao Ping, Critical point quantities and integrability conditions for complex planar resonant polynomial differential systems, ph.D thesis, 2005.

[Yakovenko, 1994] Yakovenko S., Complete abelian integrals as rational envelopes, Nonlinearity **7**, 1237-1250.

[Yang X.A., 1991] Yang Xinan, A survey of cubic system, Ann. Diff. Equs. **7**(3), 323- 363.

[Ye Y.Q. etc, 1986] Ye Yanqian and others, Theory of Limit Cycles, Trans. Math. Monographs, **66** (Amer.Math.Soc., Providence, RI).

[Ye Y.Q., 1995] Ye Yanqian, Qualitative Theory of Polynomial Differential Systems, Shanghai Scientific and Technical Publishers, Shanghai (in Chinese).

[Ye Y.Q., 1996] Ye Yanqian, On Hilbert's 16th problem in the case $n = 2$, Ann. Diff. Equs. **12**(1), 110-116.

[Yu P. etc 2004] Yu Pei and Han Maoan, Twelve limit cycles in 3rd-planar system with Z_2 symmetry, Communications on Appl. Pure Anal. **3**, 515-526.

[Yu P. etc 2005a] Yu Pei and Han Maoan, Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vetcor fields, Chaos,Solitons and Fractals **24**, 329-348.

[Yu P. etc 2005b] Yu Pei and Han Maoan, Twelve limit cycles in a cubic case of the 16th Hilbert problem, Int. J. Bifur. Chaos. **15**,2192-2205.

[Yu P. etc 2006] Yu Pei and Han Maoan, Limit cycles in generalized Liénard systems. Chaos, Solitons and Fractals **30**(5), 1048-1068.

[Yu P. etc 2007] Yu Pei and Han Maoan, On limit cycles of the Liénard equation with Z_2 symmetry. Chaos, Solitons and Fractals **31**(3), 617-630.

[Yu P. etc 2009] Yu Pei and Han Maoan, Critical periods of planar revertible vetcor field with third-degree polynomial functions. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **19**(1), 419-433.

[Yu P. etc 2010a] Yu Pei, Han Maoan and Zhang Jizhou, Critical periods of third-order planar Hamiltonian systems. Internat. J. Bifur. Chaos Appl. Sci. Engrg. **20**(7), 2213-2224.

[Yu P. etc 2010b] Yu Pei, Han Manao and Li Jibin, Bifurcation of limit cycles in cubic integrable Z_2-equivariant planar vetcor fields. Qual. Theory Dyn. Syst. **9**(1-2), 221-233.

[Zegeling, 1994] Zegeling A., Separatrix cycles and multiple limit cycles in a class of quadratic systems, J. Diff. Equs. **113**, 355-380.

[Zhang J.Y. etc , 2000] Zhang Jingyan and Feng Beiye, The Geometrical Theory and Bifurcation Problem of Ordinary Differential Equations, Peking University Press, Beijing (the second revised edition, in Chinese).

[Zhang L. etc, 2006] Zhang Li and Huang Wentao, Polynomial differential system with ten limit cycles at infinity, Journal of Chongqing University(Natural Science Edition). **29**(8),146-149.

[Zhang Q. etc, 2006a] Zhang Qi and Liu Yirong, A cubic polynomial system with seven limit cycles at infinity, Applied Mathematics and Computation **177**(1), 319-329.

[Zhang Q. etc, 2006b] Zhang Qi, Liu Yirong and Haibo Chen, Bifurcation at the equator for a class of quintic polynomial differential system, Applied Mathematics and Computation **181**(1), 747-755.

[Zhang Q. etc, 2011] Zhang Qi, Liu Yirong and Chen Haibo, On the equivalence of singular point quantities and the integrability of a fine critical singular point,Nonlinear Analysis: Real World Applications **12**, 2794-2801.

[Zhang W.N. etc,2000] Zhang Weinian, Hou X., Zeng Z., Weak centers and bifurcation of critical periods in reversible cubic systems, Computers and Mathematics with Applications **40**, 771-782.

[Zhang X., 1999a] Zhang Xiang, Existence of limit cycles in a multiply-connetced region, Chin. Ann. Math. **20B**(2), 169-172.

[Zhang X., 1999b] Zhang Xiang, Bifurcation and distribution of limit cycles for quadratic differential systems of type II, J. Diff. Equs. **233**, 508-534.

[Zhang X., 2001] Zhang Xiang, Ergodicity of limit cycles in quadratic system, Nonlinear Analysis **44**, 1-19.

[Zhang X., 2001] Zhang Xiang, The multiplicity of zeros and limit cycles for Abel equations, J. Sys. Sci. Math. Complexity **14**(2), 165-173.

[Zhang X., 2008] Zhang Xiang, Analytic normalization of analytic integrable systems and the embedding flows J. Differential Equations. **244**, 1080-1092.

[Zhang X., 2011] Zhang Xiang, Global structure of quaternion polynomial differential equations, Commun. Math. Phys. **303**, 301-316.

[Zhang Z.F. etc, 1992] Zhang Zhifeng, Din Tongren, Huang Wenzhau and Dong Zhengxi, Qualitative Theory of Differential Equations, Transl. Math. Monographs, **101**, Amer. Math. Soc., Providence, RI.

[Zhang Z.F. etc, 1997a] Zhang Zhifeng and Li Chenzhi, On the number of limit cycles of a class of quadratic Hamiltonian systems under quadratic perturbations, Adv. in Math. **26**(5), 445-460.

[Zhang Z.F. etc, 1997b] Zhang Zhifeng, Li Chenzhi, Zheng Zhiming and Li Weigu, Elementary Theory of Bifurcations of Vetcor Fields, Higher Education Press, Beijing (in Chinese).

[Zhang Z. F. etc, 1997c] Zhang Zhifen, Li Chenzhi and others, Introduction to Bifurcations of Vetcor Fields, Higher Education Press (in Chinese), Beijing.

[Zhao Y.L. etc, 1996] Zhao Yulin, Liang Zhaojun and Lu Gang, The cyclicity of period annulus of the quadratic Hamiltonian systems with non-Morsean point, J. Diff. Equs. **162**, 199-233.

[Zhao Y.L. etc , 1999a] Zhao Yulin and Zhang Zhifen, Linear estimate of the number of zeros of Abelian integrals for a kind of quartic Hamiltonians, J. Diff. Equs. **155**, 73-88.

[Zhao Y.L. etc , 1999b] Zhao Yulin and Zhang Zhifen, Abelian integrals for cubic vetcor fields, Annali di Math. pura ed Appl., LXXVI, 251-272.

[Zhao Y.L. etc , 2000a] Zhao Yulin and Zhang Zhifen, Bifurcations of limit cycles from cubic Hamiltonian systems with a center and a saddle-loop, Publicacions Mateαtiques **44**, 205-235.

[Zhao Y.L. etc , 2000b] Zhao Yulin, Liang Zhaojun and Lu Gang , The cyclicity of the period annulus of the quadratic Hamiltonian system with non-Morsean point, J. Differential Equations **162**, 199-223.

[Zhao Y.L., 2000c] Zhao Yulin, Abelian integrals for a cubic Hamiltonian vetcor fields with cuspidal loop (in Chinese), Acta Mathematica Scientica **20**(2), 229-234.

[Zhao Y.L. etc , 2001a] Zhao Yulin and Zhu Siming, Perturbations of the non-generic quadratic Hamiltonian vetcor fields with hyperbolic segment, Bull. Sci. Math. **125**(2), 109-138.

[Zhao Y.L. etc , 2001b] Zhao Yulin and Zhu Siming, Perturbations of the non-generic quadratic Hamiltonian vetcor fields with hyperbolic segment, Bulletin des Sciences Mathematiques **125**(2), 109-138.

[Zhao Y.L., 2002a] Zhao Yulin, The monotonicity of period function for codimension four quadratic system Q_4, J. Differential Equations **185**(1),370-385.

[Zhao Y.L. etc, 2002b] Zhao Yulin and Zhu Siming, Higher order Melnikov function for a quartic Hamiltonian with cuspidal loop, Discrete and Continuous Dynamical Systems, **8**(4),995-1018.

[Zhao Y.L. etc, 2002c] Zhao Yulin, Li Weigu, Li Chengzhi and Zhang Zhifen, Linear estimate of the number of zeros of Abelian integrals for quadratic centers having all their orbits formed by cubics, Science in China (Series A) **45**(8), 967-974.

[Zhao Y.L. etc, 2002d] Zhao Yulin, Wu Xiaoming and Zhu Siming , Perturbations of a kind of degenerate quadratic Hamiltonian system with saddle-loop, Chinese Annals of Mathematics, **23B**(1), 85-94.

[Zhao Y.L., 2003] Zhao Yulin, Zeros of Abelian integrals for reversible codimension four centers $Q_3^R \cap Q_4$, Israel Journal of Mathematics **136**, 125-143.

[Zhao Y.L., 2005a] Zhao Yulin, On the number of zeros of Abelian integrals for a polynomial Hamiltonian irregular at infinity , J. Differential Equations. **209**(2), 329-364.

[Zhao Y.L. etc , 2005b] Zhao Yulin and Zhang Zhifen, Abelian integrals and period functions for quasihomogeneous Hamiltonian vetcor fields, Applied Mathematics Letters **18**, 563-569.

[Zhao Y.L., 2005c] Zhao Yulin, The period function for quadratic integrable systems with cubic orbits, J. Math. Anal. Appl. **301**(2), 295-312.

[Zhao Y.L., 2005d] Zhao Yulin, On the monotonicity of the period function of a quadratic system, Discrete and Continuous Dynamical Systems **13**(3),795-810.

[Zhao Y.L., 2006] Zhao Yulin, Perturbations of non-Hamiltonian reversible quadratic systems with cubic orbits, Nonlinear Analysis: Theory & Applications, **64**, 2332-2351.

[Zhu D.M., 1987] Zhu Deming , Saddle values and integrability conditions of quadratic differential systems, Chinese Ann. Math. **8B**(4), 466-478.

[Zhu H.P. etc, 2002] Zhu Huaiping and Rousseau C., Finite cyclicity of graphics with a nilpotent singularity of saddle or elliptic type, J. Diff. Equs. **178**, 325-436.

[Zoladek, 1984] Zoladek H., On the versality of certain family of vetcor fields on the plane, Math. USSR Sb. **48**, 463-492 (in Russian).

[Zoladek, 1987] Zoladek H., Bifurcations of certain family of vetcor fields tangent to axes, J. Diff. Equs. **67**, 1-55.

[Zoladek, 1991] Zoladek H., Abelian integrals in unfolding of codimension 3 singular planar vetcor field, Letcure Notes in Math. **1480**, 165-224, Springer-Verlagz.

[Zoladek, 1994] Zoladek H., Quadratic systems with centers and their perturbations, J. Diff. Equs. **109**, 223-273.

[Zoladek, 1994] Zoladek H., On certain generalization of Bautin's theorem, Nonlinearity **7**, 273-280.

[Zoladek, 1995] Zoladek H., The cyclicity of triangles and segments in quadratic systems, J. Diff. Equs. **122**, 137-159.

[Zoladek, 1995] Zoladek H., Eleven small limit cycles in a cubic vetcor field, Nonlinearity **8**, 843-860.

[Zou L. etc, 2007] Zou Lan, Chen Xingwu and Zhang Weinian, Local bifurcations of critical periods for cubic Liénard equations with cubic damping, J. Comput. Appl. Math. **222** (2), 404-410.

Index