Analytical Heat Transfer

Je-Chin Han

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Contents

Preface

Fundamental Heat Transfer is a required course for all mechanical, chemical, nuclear, and aerospace engineering undergraduate students. This senior-level undergraduate course typically covers conduction, convection, and radiation heat transfer. Advanced Heat Transfer courses are also required for most engineering graduate students. These graduate-level courses are typically taught as individual courses named Conduction, Convection, or Radiation. Many universities also offer an Intermediate Heat Transfer or Advanced Heat Transfer course to cover conduction, convection, and radiation for engineering graduate students. For these courses, however, there are not many textbooks available that cover conduction, convection, and radiation at the graduate level.

I have taught an Intermediate Heat Transfer course in the Department of Mechanical Engineering at Texas A&M University since 1980. This book has evolved from a series of my lecture notes for teaching a graduate-level intermediate heat transfer course over the past 30 years. Many MS degree students majoring in thermal and fluids have taken this course as their only graduate-level heat transfer course. And many PhD degree candidates have taken this course to prepare for their heat transfer qualifying examinations as well as to prepare for their advanced-level courses in conduction, convection, or radiation. This book bridges the gap between undergraduate-level basic heat transfer and graduate-level advanced heat transfer as well as serves the need of entry-level graduate students.

Analytical Heat Transfer focuses on how to analyze and solve the classic heat transfer problems in conduction, convection, and radiation in one book. This book emphasizes how to model and how to solve engineering heat transfer problems analytically, rather than simply applying the equations and correlations for engineering problem calculations. This book provides many well-known analytical methods and their solutions such as Bessel functions, separation of variables, similarity method, integral method, and matrix inversion method for entry-level engineering graduate students. It is unique in that it provides (1) detailed step-by-step mathematical formula derivations, (2) analytical solution procedures, and (3) many demonstration examples. This analytical knowledge will equip graduate students with the much-needed capability to read and understand the heat-transfer-related research papers in the open literature and give them a strong analytical background with which to tackle and solve the complex engineering heat transfer problems they will encounter in their professional lives.

This book is intended to cover intermediate heat transfer between the undergraduate and the advanced graduate heat transfer levels. It includes 14 chapters and an Appendix:

There are many excellent undergraduate and graduate heat transfer textbooks available. Although I do not claim any new ideas in this book, I do attempt to present the subject in a systematic and logical manner. I hope this book is a unique compilation and is useful for graduate entry-level heat transfer study.

While preparing this manuscript, I heavily referenced the following books and therefore am deeply appreciative to their authors:

W. Rohsenow and H. Choi, *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.

A. Mills, *Heat Transfer*, Richard D. Irwin, Inc., Boston, MA, 1992.

K. Vincent Wong, *Intermediate Heat Transfer*, Marcel Dekker, Inc., New York, NY, 2003.

F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.

Finally, I would like to sincerely express special thanks to my former student, Dr. Zhihong (Janice) Gao (PhD, 2007). Janice spent a lot of time and effort to input most of the manuscript from my original *Intermediate Heat Transfer Class Notes* in 2005–2007. Without her diligent and persistent contributions, this book would be impossible. In addition, I would like to extend appreciation to my current PhD students, Jiang Lei and Shiou-Jiuan Li for their help in completing the book and drawings.

Je-Chin Han

1

Heat Conduction Equations

1.1 Introduction

1.1.1 Conduction

Conduction is caused by the temperature gradient through a solid material. For example, Figure 1.1 shows that heat is conducted from the high-temperature side to the low-temperature side through a building or a container wall. This is a one-dimensional (1-D) steady-state heat conduction problem if T_1 and T_2 are uniform. According to Fourier's conduction law, the temperature profile is linear through the plane wall.

1.1.1.1 Fourier's Conduction Law

$$q'' = -k\frac{\mathrm{d}T}{\mathrm{d}x} = k\frac{T_1 - T_2}{L} \tag{1.1}$$

and

$$q'' \equiv \frac{q}{A_\mathrm{c}} \quad \text{or} \quad q = q''A_\mathrm{c}$$

where q'' is the heat flux ($\mathrm{W/m^2}$), q the heat rate (W or J/s), k the thermal conductivity of solid material ($\mathrm{W/m\,K}$), A_c the cross-sectional area for conduction, perpendicular to heat flow ($\mathrm{m^2}$), and L the conduction length (m).

One can predict heat rate or heat loss through the plane wall by knowing T_1, T_2, k, L, and A_c. This is the simple 1-D steady-state problem. However, in real-life application, there are many two-dimensional (2-D) or three-dimensional (3-D) steady-state heat conduction problems; there are cases where heat generation occurs in the solid material during heat conduction; and transient heat conduction problems take place in many engineering applications. In addition, some special applications involve heat conduction with moving boundary. All these more complicated heat conduction problems will be discussed in the following chapters.
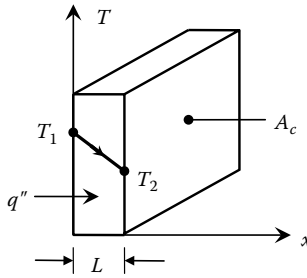
FIGURE 1.1
1-D heat conduction through a building or container wall.

1.1.2   Convection

Convection is caused by fluid flow motion over a solid surface. For example, Figure 1.2 shows that heat is removed from a heated solid surface to cooling fluid. This is a 2-D boundary-layer flow and heat transfer problem. According to Newton, the heat removal rate from the heated surface is proportional to the temperature difference between the heated wall and the cooling fluid. The proportional constant is called heat transfer coefficient; and the same heat rate from the heated surface can be determined by applying Fourier Conduction Law to the cooling fluid.

1.1.2.1   Newton's Cooling Law

$$q'' = -k_{\mathrm{f}} \left. \frac{\mathrm{d}T}{\mathrm{d}y} \right|_{\text{at wall}} = h(T_{\mathrm{s}} - T_{\infty}) \tag{1.2}$$

Also,

$$h = \frac{q''}{T_{\mathrm{s}} - T_{\infty}} = \frac{-k_{\mathrm{f}} \frac{\mathrm{d}T}{\mathrm{d}y} \big|_{y=0}}{T_{\mathrm{s}} - T_{\infty}} \tag{1.3}$$
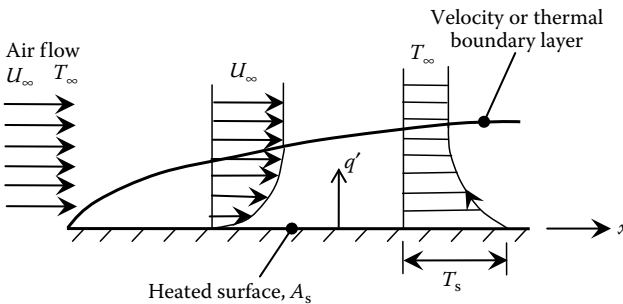
FIGURE 1.2
Velocity and thermal boundary layer.

and

$$q'' \equiv \frac{q}{A_s} \quad \text{or} \quad q = q''A_s$$

where T_s is the surface temperature (°C or K), T_∞ the fluid temperature (°C or K), h the heat transfer coefficient (W/m^2 K), k_f the thermal conductivity of fluid (W/m K), A_s the surface area for convection, exposure to flow (m^2).

It is noted that the heat transfer coefficient depends on fluid properties (such as air or water as coolant), flow conditions (i.e., laminar or turbulent flows), surface configurations (such as flat surface or circular tube), and so on. The heat transfer coefficient can be determined experimentally or analytically. This textbook focuses more on analytical solutions. From Equation 1.3, the heat transfer coefficient can be determined by knowing the temperature profile in the cooling fluid during convection and then taking the cooling fluid temperature gradient near the wall. However, this requires solving 2-D boundary-layer equations and will be the subject of the following chapters. Before solving 2-D boundary-layer equations, one needs heat transfer coefficient as the convection boundary condition (BC) in order to solve the heat conduction problem. Therefore, Table 1.1 provides some typical values of heat transfer coefficient in many convection problems. As can be seen, in general, forced convection has more heat transfer than natural convection; water as a coolant removes much more heat than air; and boiling or condensation, involving phase change, has a much higher heat transfer coefficient than single-phase convection.

1.1.3 Radiation

Radiation is caused by electromagnetic waves from solids, liquid surfaces, or gases. For example, Figure 1.3 shows that heat is radiated from a solid surface

TABLE 1.1

Typical Values of Heat Transfer Coefficient

Type of convection	h, $W/m^2 \cdot K$
Natural convection	
Caused by ΔT : air	5
Caused by ΔT : water	25
Forced convection	
Caused by fan, blower: air	25–250
Caused by pump: water	50–20,000
Boiling or condensation	
Caused by phase change	
Water $\rightleftharpoons$ Steam	10,000–100,000
Freon $\rightleftharpoons$ Vapor	2500–50,000

Any surface at T_s, ε, A_s
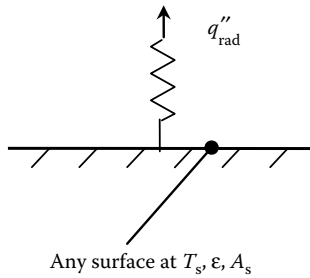
FIGURE 1.3
Radiation from a solid surface.

at a temperature greater than absolute zero. According to Stefan–Boltzmann, radiation heat rate is proportional to the surface's absolute temperature's fourth power, the Stefan–Boltzmann constant, and the surface emissivity. The surface emissivity primarily depends on material, wavelength, and temperature. It is between 0 and 1. In general, the emissivity of metal is much less than nonmetal. Note that radiation from a surface can go through air as well as vacuum environment.

1.1.3.1 Stefan–Boltzmann Law

For real surface,

$$q'' = \varepsilon\sigma T_s^4 \tag{1.4}$$

For ideal (black) surface, $\varepsilon = 1$

$$q'' = \sigma T_s^4$$

and

$$q'' \equiv \frac{q}{A_s} \quad \text{or} \quad q = q''A_s$$

where ε is the emissivity of the real surface, $\varepsilon = 0 - 1$ (ε metal $<$ nonmetal ε), T_s the absolute temperature of the surface, K ($K = {}^\circ C + 273.15$), σ the Stefan–Boltzman constant, $\sigma = 5.67 \times 10^{-8}\,\text{W/m}^2\,\text{K}^4$, and A_s the surface area for radiation (m^2).

1.1.4 Combined Modes of Heat Transfer

In real application, often radiation occurs when conduction or convection takes place. This is called combined modes of heat transfer. For example, Figure 1.4 shows heat transfer between two surfaces involving radiation and convection simultaneously.

Surrounding wall, T_{sur}, ε_{sur}

Air flow

$T_\infty h$

q''_{conv}

$q''_{\text{rad,net}}$

Machine or equipment

Surface A at T_s, ε, A_s
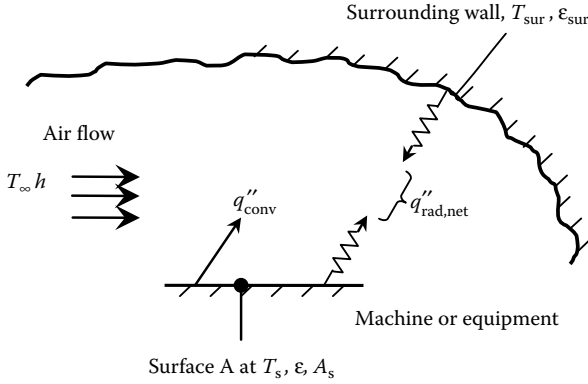
FIGURE 1.4
Heat transfer between two surfaces involving radiation and convection.

Assume: surface $A \ll$ surrounding sky or building wall surface, $T_\infty \neq T_{\text{sur}}$ (or, $T_\infty = T_{\text{sur}}$)

Total heat flux from the surface A is due to convection and radiation, and can be found as

$$q'' = q''_{\text{conv}} + q''_{\text{rad,net}} \tag{1.5}$$

where

$$q''_{\text{conv}} = h(T_s - T_\infty)$$

$$q''_{\text{rad,net}} = \varepsilon \sigma T_s^4 - \alpha \varepsilon_{\text{sur}} \sigma T_{\text{sur}}^4$$

$$= \varepsilon \sigma (T_s^4 - T_{\text{sur}}^4), \quad \text{if } \varepsilon = \alpha, \ \varepsilon_{\text{sur}=1}$$

$$= \varepsilon \sigma (T_s^2 + T_{\text{sur}}^2)(T_s + T_{\text{sur}})(T_s - T_{\text{sur}})$$

$$= h_r(T_s - T_{\text{sur}})$$

Therefore, from Equation 1.5

$$q'' = h(T_s - T_\infty) + h_r(T_s - T_{\text{sur}}) \tag{1.6}$$

where

$$h_r = \varepsilon \sigma (T_s^2 + T_{\text{sur}}^2)(T_s + T_{\text{sur}}) \tag{1.7}$$

Also

$$q'' \equiv \frac{q}{A_s} \quad \text{or} \quad q = q'' A_s$$

where α is the absorptivity, T_{sur} the surrounding wall temperature (°C or K), ε_{sur} the emissivity of the surrounding wall, h_r the radiation heat transfer coefficient (W/m^2 K), and A_s the surface area for radiation (m^2). Total heat transfer rate can be determined by knowing T_s, T_{sur}, h, ε, ε_{sur}, σ, and A_s.

1.2   General Heat Conduction Equations

If the temperature profile inside a solid material $T(x, y, z, t)$ is known, the heat rate q through the solid can be determined as shown in Figure 1.5. As far as thermal stress is concerned, it is equally important to predict the temperature profile in some high-temperature applications. In this chapter, the general heat conduction equations will be derived. The general heat conduction equation can be used to solve various real problems with the appropriate boundary conditions (BCs) and initial condition. The heat conduction can be modeled as 1-D, 2-D, or 3-D depending on the nature of the problem:

    1-D $T(x)$ for steady state or $T(x, t)$ for transient problem

    2-D $T(x, y)$ for steady state or $T(x, y, t)$ for transient problem

    3-D $T(x, y, z)$ for steady state or $T(x, y, z, t)$ for transient problem

To determine the temperature profile, the following should be given:

1. Initial condition and BCs
2. Material thermal conductivity k, density ρ, specific heat C_p, and diffusivity $\alpha = k/\rho C_p$

1.2.1   Derivations of General Heat Conduction Equations

The general form of the conservation of energy in a small control volume of solid material is

$$E_{in} - E_{out} + E_g = E_{st} \tag{1.8}$$

where $E_{in} - E_{out}$ is the net heat conduction, E_g is the heat generation, and E_{st} is the energy stored in the control volume.

Figure 1.6 shows the conservation of energy in a differential control volume in a 3-D Cartesian (rectangular) coordinate. If we consider energy
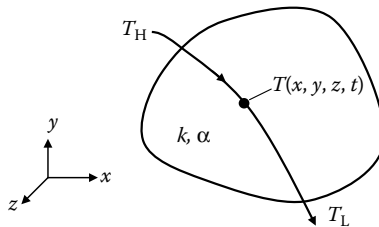
FIGURE 1.5
Heat conduction through a solid medium.
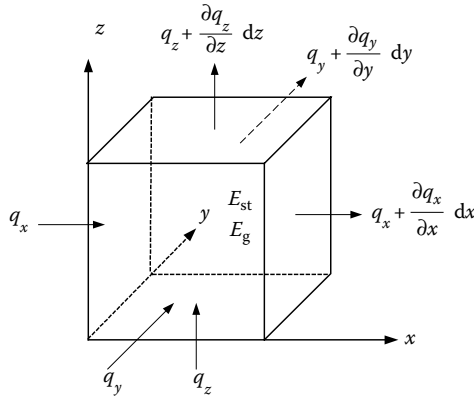
FIGURE 1.6
The volume element for deriving the heat conduction equation.

conservation in a 1-D system (x-direction only),

$$q_x - \left(q_x + \frac{\partial q_x}{\partial x} dx \right) + E_g = E_{st} \tag{1.9}$$

The conduction heat rates can be evaluated from Fourier's Law,

$$q_x = -kA_x \frac{\partial T}{\partial x} \tag{1.10}$$

With control surface area $A_x = dy\,dz$, Equation 1.9 can be written as

$$-\frac{\partial}{\partial x} \left(-k\,dy\,dz \frac{\partial T}{\partial x} \right) dx + E_g = E_{st} \tag{1.11}$$

The thermal energy generation can be represented by

$$E_g = \dot{q}\,dx\,dy\,dz \tag{1.12}$$

where $\dot{q}$ is the energy generation per unit volume $dx\,dy\,dz$.
  The energy storage can be expressed as

$$E_{st} = \frac{\partial(\rho\,dx\,dy\,dz \cdot C_p \cdot T)}{\partial t} \tag{1.13}$$

Substituting Equations 1.12 and 1.13 into Equation 1.11, we have

$$-\frac{\partial}{\partial x} \left(-k \frac{\partial T}{\partial x} \right) dx\,dy\,dz + \dot{q}\,dx\,dy\,dz = \frac{\partial(\rho\,dx\,dy\,dz \cdot C_p \cdot T)}{\partial t} \tag{1.14}$$

Dividing out the dimensions of the small control volume dx dy dz, Equation 1.14 is simplified as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \frac{\partial(\rho C_p T)}{\partial t} \tag{1.15}$$

If we consider energy conservation, in a 3-D system (x-direction, y-direction, z-direction), the heat equation can be written as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \frac{\partial(\rho C_p T)}{\partial t} \tag{1.16}$$

The thermal conductivity k, which is a function of temperature, is hard to determine. If we assume that k, ρ, and C_p are constants, then Equation 1.16 is simplified as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.17}$$

where $\alpha = k/\rho C_p$ is the thermal diffusivity.

In a 3-D cylindrical coordinate system as shown in Figure 1.7a, the heat conduction equation has the form of

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.18}$$

In a 3-D spherical coordinate system as shown in Figure 1.7b, it has the form of

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1.19}$$
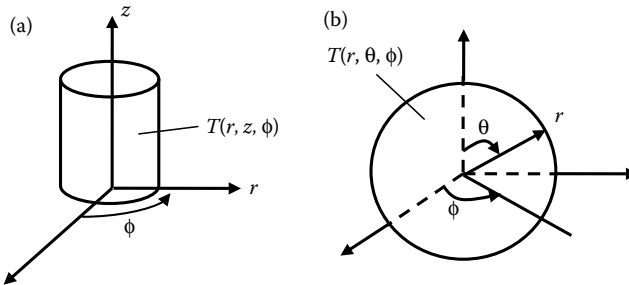
FIGURE 1.7
(a) Cylindrical coordinate system. (b) Spherical coordinate system.

1.3 Boundary and Initial Conditions

The physical conditions existing on the boundary should be known in order to determine the temperature profile in a medium by solving the heat conduction equation. Moreover, the initial condition $T(x, 0) = T_i$ should also be known if the heat transfer is time dependent.

1.3.1 Boundary Conditions

There are three kinds of BCs commonly found in many heat transfer applications [1].

1. Given surface temperature $T(0, t) = T_s$, as shown in Figure 1.8
2. Given surface heat flux, as shown in Figure 1.9a and b
   a. Finite heat flux

$$-k\frac{\partial T(0, t)}{\partial x} = q_s''$$ (1.20)

   b. Adiabatic or insulated surface, which is a special case

$$-k\frac{\partial T(0, t)}{\partial x} = q_s'' = 0$$ (1.21)

3. Given surface convection, as shown in Figure 1.10

$$-k\frac{\partial T(0, t)}{\partial x} = h\left[T_\infty - T(0, t)\right]$$ (1.22)

1.3.2 Initial Conditions

Initial condition, as shown in Figure 1.11, is required for the transient heat transfer problem.

$$T(x, 0) = T_i$$ (1.23)
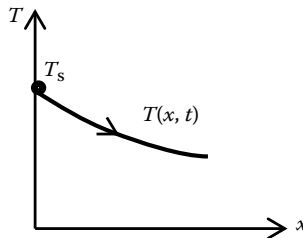
FIGURE 1.8
Boundary conditions—given surface temperature.
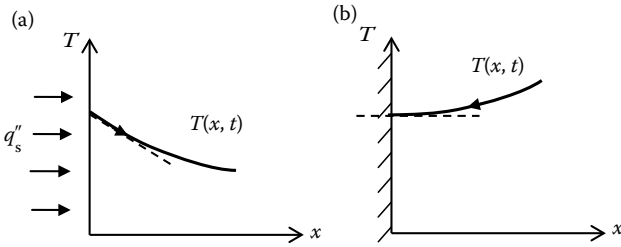
FIGURE 1.9
(a) Finite heat flux. (b) Adiabatic surface.

1.4 Simplified Heat Conduction Equations

1. Steady state $\partial T/\partial t = 0$, no heat generation $\dot{q} = 0$

$$\text{1-D} \quad \frac{\partial^2 T}{\partial x^2} = 0$$

$$\text{2-D} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\text{3-D} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$
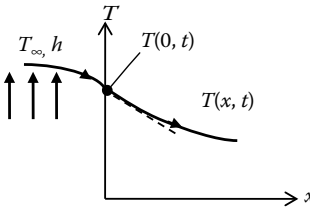
FIGURE 1.10
Convective boundary conditions.
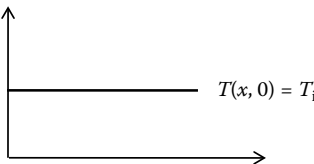
FIGURE 1.11
Initial condition.

2. Steady state, 1-D with heat source—heater application

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0$$

Steady state, 1-D with heat sink—fin application

$$\frac{\partial^2 T}{\partial x^2} - \frac{\dot{q}}{k} = 0$$

3. Transient without heat generation $\dot{q} = 0$

$$1\text{-D} \quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$2\text{-D} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$3\text{-D} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The above-mentioned three kinds of BCs can be applied to the 1-D, 2-D, or –3-D heat conduction problems, respectively. For example, as shown in Figure 1.12,

$$x = 0, \quad -k\frac{\partial T(0, y, t)}{\partial x} = 0 \quad \text{(adiabatic surface)}$$

$$x = a, \quad -k\frac{\partial T(a, y, t)}{\partial x} = h[T(a, y, t) - T_\infty] \quad \text{(surface convection)}$$

$$y = 0, \quad -k\frac{\partial T(x, 0, t)}{\partial y} = q'' \quad \text{(surface heat flux)}$$

$$y = b, \quad T(x, b, t) = T_s \quad \text{(surface temperature)}$$

Remarks

In general, heat conduction problems, regardless of 1-D, 2-D, 3-D, steady or unsteady, can be solved analytically if thermal conductivity is a given constant and thermal BCs are known constants. The problems will be analyzed and solved in Chapters 2 through 4. However, in real-life applications, there are many materials whose thermal conductivities vary with temperature and location, $k(T) \sim k(x, y, z)$. In these cases, the heat conduction equation shown in Equation 1.16 becomes a nonlinear equation and is harder to solve analytically.
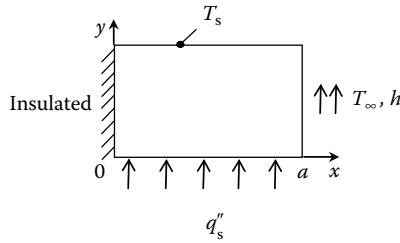
FIGURE 1.12
Heat conduction in 2-D system with various boundary conditions.

In addition, in real engineering applications, it is not easy to determine the precise convection BC shown in Figures 1.10 and 1.12. These require detailed knowledge of complex convection heat transfer to be discussed in Chapters 6 through 10.

PROBLEMS

1.1 Derive Equation 1.18.

1.2 Derive Equation 1.19.

1.3 a. Write the differential equation that expresses transient heat conduction in 3-D (x, y, z coordinates) with constant heat generation and constant conductivity.

   b. Simplify the differential equation in (a) to show steady-state conduction in one dimension, assuming constant conductivity.

   c. If the BCs are: $T = T_1 \ldots$ at $\ldots x = x_1$, $T = T_2 \ldots$ at $\ldots x = x_2$, solve the second-order differential equation to yield a temperature distribution (T). Express the answer (T) in terms of (T_1, T_2, x, x_1, x_2).

   d. Using the Fourier law and the results from (c), develop an expression for the heat rate per unit area, assuming constant conductivity. Express your answer in terms of (k, T_1, T_2, x_1, x_2).

Reference

1. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.

2

1-D Steady-State Heat Conduction

2.1 Conduction through Plane Walls

For 1-D steady-state heat conduction in plane wall shown in Figure 2.1, without heat generation, the heat conduction equation 1.17 can be simplified as

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{2.1}$$

$$\frac{dT}{dx} = C_1$$

Equation 2.1 has the general solution

$$T = c_1 x + c_2 \tag{2.2}$$

with boundary conditions:

$$\text{at } x = 0, \quad T = T_{s1} = c_1 \cdot 0 + c_2 = c_2$$
$$\text{at } x = L, \quad T = T_{s2} = c_1 L + c_2$$

Solve for c_1 and c_2,

$$c_1 = \frac{T_{s2} - T_{s1}}{L}, \quad c_2 = T_{s1},$$

Substituting c_1 and c_2 into Equation 2.2, the temperature distribution is

$$T(x) = T_{s,1} - \frac{T_{s,1} - T_{s,2}}{L} x \tag{2.3}$$

Applying Fourier's Conduction Law, one obtains the heat transfer rate through the plane wall

$$q = -kA \frac{\partial T}{\partial x} = kA \frac{T_{s,1} - T_{s,2}}{L} = \frac{T_{s,1} - T_{s,2}}{(L/KA)} \tag{2.4}$$
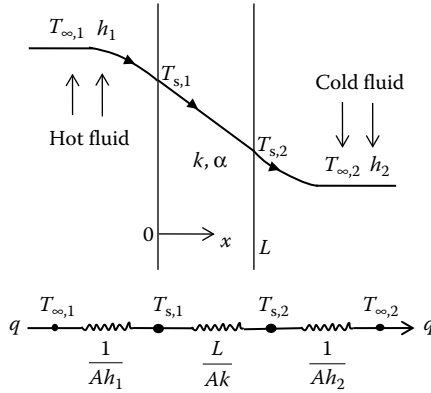
FIGURE 2.1
Conduction through plane wall and thermal–electrical network analogy.

The heat transfer rate divided by the cross-sectional area of the plane wall, that is, heat flux is

$$q'' = \frac{q}{A} = -k\frac{\partial T}{\partial x} = k\frac{T_{s,1} - T_{s,2}}{L} \tag{2.5}$$

At the convective surfaces, from Newton's Cooling Law, the heat transfer rates are

$$q = Ah_1(T_{\infty,1} - T_{s,1}) = \frac{T_{\infty,1} - T_{s,1}}{(1/Ah_1)} \tag{2.6}$$

and

$$q = Ah_2(T_{s,2} - T_{\infty,2}) = \frac{T_{s,2} - T_{\infty,2}}{(1/Ah_2)} \tag{2.7}$$

Applying the analogy between the heat transfer and electrical network, one may define the thermal resistance like the electrical resistance. The *thermal resistance for conduction* in a plane wall is

$$R_{cond} = \frac{L}{kA} \tag{2.8}$$

The *thermal resistance for convection* is then

$$R_{conv} = \frac{1}{Ah} \tag{2.9}$$

The *total thermal resistance* may be expressed as

$$R_{tot} = \frac{1}{Ah_1} + \frac{L}{kA} + \frac{1}{Ah_2} = \frac{1}{UA} \tag{2.10}$$

where U is the overall heat transfer coefficient.

Similar to electric current, the heat rate through the wall is

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{(1/Ah_1) + (L/kA) + (1/Ah_2)} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}} = UA(T_{\infty,1} - T_{\infty,2}) \quad (2.11)$$

Figure 2.2 shows conduction through two plane walls with *contact resistance* between them, the total heat resistance becomes

$$R_{tot} = \frac{1}{Ah_1} + \frac{L_1}{k_1 A} + \frac{R''_{tc}}{A} + \frac{L_2}{k_2 A} + \frac{1}{Ah_2} \quad (2.12)$$

where $R''_{tc} = (T_a - T_b/(q/A)) = $ pre-determined (depends on contact material surface roughness and contact pressure).

2.1.1   Conduction through Circular Tube Walls

1-D steady-state heat conduction, without heat generation, in the radial system shown in Figure 2.3, can be simplified from Equation 1.18

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0 \quad (2.13)$$

$$r\frac{dT}{dr} = c_1$$

The general solution of Equation 2.13 is

$$T(r) = c_1 \ln r + c_2$$

with boundary conditions

$$\text{at } r = r_1, \quad T = T_{s,1} = c_1 \ln r_1 + c_2$$

$$\text{at } r = r_2, \quad T = T_{s,2} = c_1 \ln r_2 + c_2$$
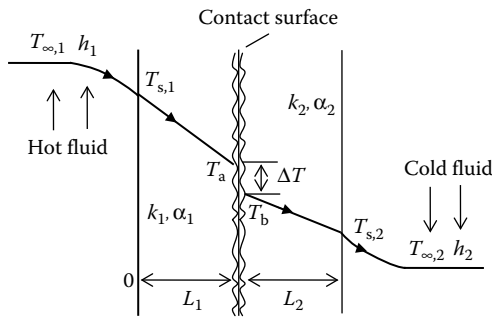
FIGURE 2.2
Temperature drop due to thermal contact resistance between surface a and surface b.
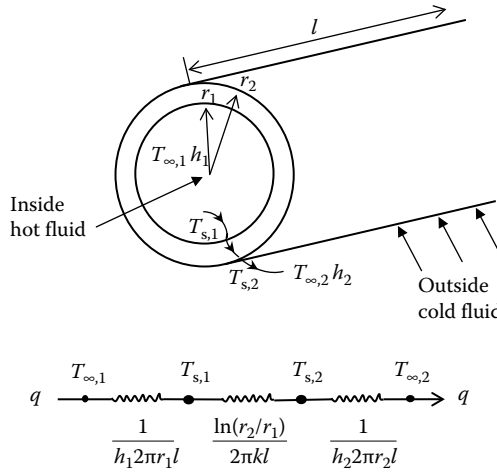
FIGURE 2.3
Conduction through circular tube wall.

Solve for c_1 and c_2, one obtains the temperature distribution

$$T(r) = T_{s,1} - \frac{T_{s,1} - T_{s,2}}{\ln(r_2/r_1)} \ln \frac{r}{r_1} \tag{2.14}$$

The heat transfer rate can be determined from Fourier's Conduction Law as

$$q = -kA\frac{dT}{dr} = -k\,2\pi rl\frac{dT}{dr} = \frac{T_{s,1} - T_{s,2}}{(\ln(r_2/r_1/2\pi kl)} \tag{2.15}$$

At the convective surface, from Newton's Cooling Law, the heat transfer rates are

$$q = A_1 h_1 (T_{\infty,1} - T_{s,1}) = \frac{T_{\infty,1} - T_{s,1}}{(1/A_1 h_1)}$$

and

$$q = A_2 h_2 (T_{s,2} - T_{\infty,2}) = \frac{T_{s,2} - T_{\infty,2}}{(1/A_2 h_2)}$$

where the cross-sectional area for conduction is $A = 2\pi rl$, $A_1 = 2\pi r_1 l$, $A_2 = 2\pi r_2 l$.

Applying the electrical-thermal analogy, the heat transfer rate is expressed as

$$q = \frac{T_{\infty,1} - T_{\infty,2}}{(1/h_1 2\pi r_1 l) + ((\ln r_2/r_1)/2\pi kl) + (1/h_2 2\pi r_2 l)} = \frac{T_{\infty,1} - T_{\infty,2}}{R_{tot}}$$
$$= UA(T_{\infty,1} - T_{\infty,2}) \tag{2.16}$$

where U is the overall heat transfer coefficient, $UA = U_1 A_1 = U_2 A_2$.

If we consider radiation heat loss between the tube outer surface and surrounding wall,

$$q_{\text{radiation}} = A_2 \varepsilon \sigma (T_{s,2}^4 - T_{\text{sur}}^4)$$

$$= A_2 \varepsilon \sigma (T_{s,2}^2 + T_{\text{sur}}^2)(T_{s,2} + T_{\text{sur}})(T_{s,2} - T_{\text{sur}})$$

$$= A_2 h_r (T_{s,2} - T_{\text{sur}})$$

where

$$h_r = \varepsilon \sigma (T_{s,2}^2 + T_{\text{sur}}^2)(T_{s,2} + T_{\text{sur}})$$

and

$$q = q_{\text{conv}} + q_{\text{rad}}$$

$$= A_2 h_2 (T_{s,2} - T_{\infty,2}) + A_2 h_r (T_{s,2} - T_{\text{sur}})$$

If $T_{\text{sur}} = T_{\infty,2}$, then

$$R_{\text{tot}} = \frac{1}{h_1\, 2\pi r_1 l} + \frac{\ln(r_2/r_1)}{2\pi k l} + \frac{1}{(h_2 + h_r)2\pi r_2 l} \tag{2.17}$$

For three concentric cylindrical walls, with radius r_1, r_2, r_3, r_4, respectively, the total heat resistance becomes

$$R_{\text{tot}} = \frac{1}{h_1 2\pi r_1 l} + \frac{\ln r_2/r_1}{2\pi k_1 l} + \frac{\ln r_3/r_2}{2\pi k_2 l} + \frac{\ln r_4/r_3}{2\pi k_3 l} + \frac{1}{h_4 2\pi r_4 l}$$

2.1.2 Critical Radius of Insulation

Consider a tube of insulating material with inside radius r_i at constant temperature T_i. At the outside radius of the insulating tube, r_o, a surface heat transfer coefficient h may be assumed for convection from the outside surface of the insulation to the atmosphere at temperature T_∞. From Equation 2.16 for this case:

$$q = \frac{T_i - T_\infty}{(\ln(r_o/r_i)/2\pi k l) + (1/h2\pi r_o l)}$$

If $l, T_i, T_\infty, h, k,$ and r_i are all assumed to remain constant while r_o varies, the q is a function of r_o alone. As r_o increases, the term $1/h r_o$ decreases but the term $(\ln r_o/r_i)/k$ increases; hence, it is possible that q might have a maximum value. Take derivative of the above equation with respect to r_o; then set $dq/dr_o = 0$ and solve for $(r_o)_{\text{critical}}$, the critical radius for which q is a maximum [1],

$$(r_o)_{\text{critical}} = \frac{k}{h} \tag{2.18}$$
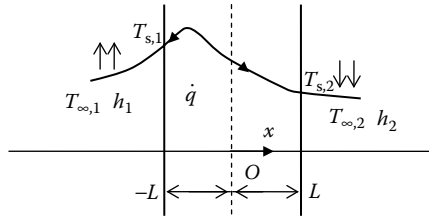
FIGURE 2.4
Flat plate heat conduction with internal heat generation and asymmetrical boundary conditions.

where k is the conductivity of insulation material, h is outside convection coefficient from insulation material. If r_i is less than $(r_o)_{critical}$, q is increased as insulation is added until $r_o = (r_o)_{critical}$. Further increases in r_o cause q to decrease. If, however, r_i is greater than $(r_o)_{critical}$, any addition of insulation with decrease q (heat loss rate).

2.2 Conduction with Heat Generation

1-D steady-state heat conduction equation with heat generation in the plane wall is (from Equation 1.17)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0$$

$$\frac{dT}{dx} = -\frac{\dot{q}}{k}x + c_1$$

(2.19)

The general solution is

$$T = -\frac{\dot{q}}{2k}x^2 + c_1 x + c_2$$

with asymmetrical boundary conditions [2] shown in Figure 2.4,

$$\text{at} \quad x = L, \quad T = T_{s,L} = -\frac{\dot{q}}{2k}L^2 + c_1 L + c_2$$

$$\text{at} \quad x = -L, \quad T = T_{s,1} = -\frac{\dot{q}}{2k}(-L)^2 + c_1(-L) + c_2$$

Solve for c_1 and c_2, one obtains the temperature distribution

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$
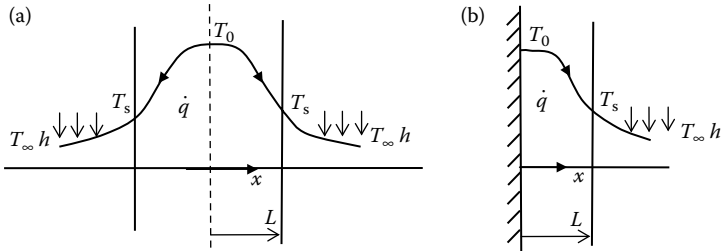
FIGURE 2.5
Flat plate heat conduction with internal heat generation. (a) Symmetric boundary conditions. (b) Adiabatic surface at midplane.

Applying symmetric boundary conditions shown in Figure 2.5a, at $x = L$, $T = T_s$; $x = -L$, $T = T_s$, solve for c_1 and c_2, one obtains the temperature distribution

$$T(x) = T_s + \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) \tag{2.20}$$

At the centerline of the plane wall, the temperature is

$$T_0 = T_s + \frac{\dot{q}L^2}{2k} \tag{2.21}$$

The heat flux to cooling fluid is

$$q'' = -k\left.\frac{dT}{dx}\right|_{x=L} = h(T_s - T_\infty) \tag{2.22}$$

From Equation 2.20, one obtains

$$\left.\frac{dT}{dx}\right|_{x=L} = -\frac{\dot{q}L}{k}$$

From Equation 2.22, $h(T_s - T_\infty) = -k(-(\dot{q}/k)L) = \dot{q}L$.
Therefore, the surface temperature in Equation 2.20 can be determined as

$$T_s = T_\infty + \frac{\dot{q}L}{h} \tag{2.23}$$
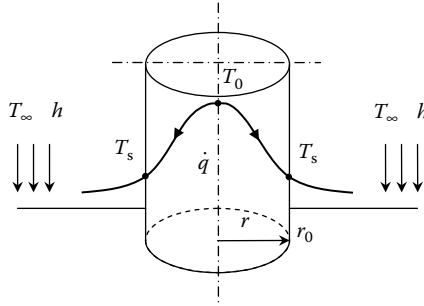
FIGURE 2.6
Cylindrical rod heat conduction with internal heat generation.

In 1-D steady-state cylindrical medium with heat generation, the heat conduction equation is (from Equation 1.18)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{\dot{q}}{k} = 0 \tag{2.24}$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{q}}{k}r$$

The general solution is

$$r\frac{dT}{dr} = -\frac{\dot{q}}{2k}r^2 + c_1$$

$$\frac{dT}{dr} = -\frac{\dot{q}}{2k}r + \frac{c_1}{r}$$

$$T(r) = -\frac{\dot{q}}{4k}r^2 + c_1 \ln r + c_2$$

with boundary conditions as shown in Figure 2.6:

$$\text{at}\quad r = 0,\quad \frac{dT}{dr} = 0 = c_1$$

$$\text{at}\quad r = r_0,\quad T = T_s = -\frac{\dot{q}}{4k}r_0^2 + c_2$$

Solve for c_1 ($c_1 = 0$) and c_2, one obtains the temperature distribution

$$T(r) = T_s + \frac{\dot{q}r_0^2}{4k}\left(1 - \frac{r^2}{r_0^2}\right) \tag{2.25}$$

At the centerline of the cylindrical rod, the temperature is

$$T_0 = T_s + \frac{\dot{q}r_0^2}{4k} \tag{2.26}$$

The heat flux to cooling fluid is

$$q'' = h(T_s - T_\infty) = -k \left. \frac{\mathrm{d}T}{\mathrm{d}r} \right|_{r=r_o} \tag{2.27}$$

From Equation 2.25, one obtains

$$\left. \frac{\mathrm{d}T}{\mathrm{d}r} \right|_{r=r_o} = -\frac{\dot{q}r_o}{2k}$$

From Equation 2.27, $h(T_s - T_\infty) = -k\left(-(\dot{q}r_o/2k)\right) = (\dot{q}r_o/2)$.
    Therefore, the surface temperature in Equation 2.25 can be determined as

$$T_s = T_\infty + \frac{\dot{q}r_o}{2h} \tag{2.28}$$

2.3 Conduction through Fins with Uniform Cross-Sectional Area

From Newton's law of cooling, heat transfer rate can be increased by either increasing temperature difference between surface and fluid, heat transfer coefficient, or surface area. For a given problem, temperature difference between surface and fluid may be fixed, and increasing heat transfer coefficient may result in more pumping power. One popular way to increase heat transfer rate is to increasing surface area by adding fins on the heated surface. This is particularly true when the heat transfer coefficient is relatively low such as air-side heat exchangers (e.g., the car radiators) and air-cooled electronic components. Heat transfer rate can increase dramatically by increasing many times of surface area with many fins. Therefore, heat is conducted from the based surface into fins and dissipated into the cooling fluid. However, temperature drops when heat is conducted through fins due to a finite thermal conductivity of the fins and the convective heat loss to the cooling fluid. This means the fin temperature is not the same as the base surface temperature and the temperature difference between the fin surface and the cooling fluid reduces along the fins. It is our job to determine the fin temperature in order to calculate the heat loss from the fins to the cooling fluid.

In general, the heat transfer rate will increase with the number of fins. But, there is limitation on the number of fins. The heat transfer coefficient will reduce if the fins are too crowded. In addition, heat transfer rate will increase with thin fins with high thermal conductivity. But, there is limitation on the thickness of thin fins due to manufacturing concern. Here we are not interested in optimizing the fin dimensions but in determining the fin temperature
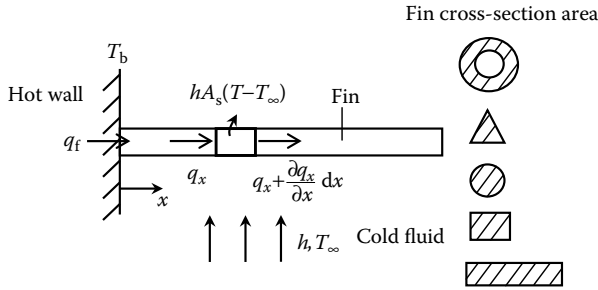
FIGURE 2.7
One-dimensional conduction through thin fins with uniform cross-section area.

at a given fin geometry and working conditions. We assume that heat conduction through the fin is 1-D steady state because the fin is thin. The temperature gradient in the other two dimensions is neglected. We will begin with the constant cross-sectional area fins and then consider the variable cross-sectional area fins. The following is the energy balance of a small control volume of the fin with heat conduction through the fin and heat dissipation into cooling fluid, as shown in Figure 2.7. The result of temperature distributions through fins of different materials can be seen from Figure 2.8.

$$q_x - \left(q_x + \frac{\mathrm{d}q_x}{\mathrm{d}x}\mathrm{d}x \right) - hA_\mathrm{s}(T - T_\infty) = 0 \tag{2.29}$$

where $A_\mathrm{s} = P\,\mathrm{d}x$, and P is perimeter of the fin, and q_x is from Fourier's Conduction Law shown in Equation 1.10.

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(-kA_\mathrm{c}\frac{\mathrm{d}T}{\mathrm{d}x} \right)\mathrm{d}x - hP\,\mathrm{d}x(T - T_\infty) = 0 \tag{2.30}$$
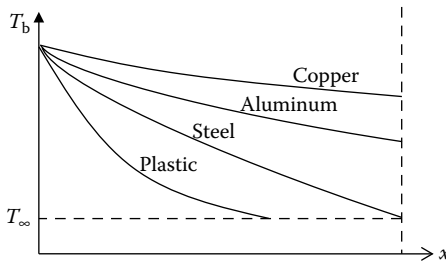
FIGURE 2.8
Temperature distributions through fins of different materials.

If the cross-sectional area is constant, that is, $A_c = $ constant, and k is also constant, Equation 2.30 can be simplified as

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$$\frac{d^2(T - T_\infty)}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

(2.31)

Let $\theta(x) = T(x) - T_\infty$, Equation 2.31 becomes

$$\frac{d^2\theta}{dx^2} - \frac{hP}{kA_c}\theta = 0$$

(2.32)

Let $m^2 = hP/kA_c$, Equation 2.32 becomes

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

(2.33)

The general solution is

$$\theta(x) = c_1 e^{mx} + c_2 e^{-mx}$$

with the following boundary conditions:
At the fin base,
$x = 0, T = T_b$, then $\theta(0) = T_b - T_\infty = \theta_b$
At the fin tip,
$x = L$, there are four possible cases

1. Convection boundary condition $-k(\partial T/\partial x)|_{x=L} = h(T_L - T_\infty)$, then $(\partial\theta(L)/\partial x) = (h/-k)\theta_L$
2. The fin tip is insulated $-k(\partial T/\partial x)|_{x=L} = 0$ or $(\partial\theta(L)/\partial x) = 0$
3. The tip temperature is given as $T|_{x=L} = T_L$ or $\theta(L) = T_L - T_\infty = \theta_L$
4. For a long fin, $L/d > 10 \sim 20$, $T|_{x=L} = T_\infty$, or $\theta(L) = T_\infty - T_\infty = 0$, which is an ideal case.

Applying boundary conditions:

$$x = 0, \quad \theta(0) = \theta_b = c_1 + c_2$$

$$x = L$$

For case 1: $\frac{\partial\theta(L)}{\partial x} = \frac{h}{-k}\theta_L$, that is, $c_1 m e^{mL} + c_2(-m)e^{-mL} = -(h/k)(c_1 e^{mL} + c_2 e^{-mL})$, solve for c_1 and c_2, one obtains the temperature distribution as follows:

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x) + (h/mk)\sinh m(L - x)}{\cosh mL + (h/mk)\sinh mL}$$

(2.34)

The heat transfer rate through the fin base (q_f) is

$$q_f = q_x = -kA_c \left. \frac{dT}{dx} \right|_{x=0} = -kA_c \frac{d\theta(0)}{dx}$$

$$= M \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \tag{2.35}$$

where $M = \sqrt{hPkA_c}\theta_b$

$$\sinh mx = \frac{e^{mx} - e^{-mx}}{2}$$

$$\cosh mx = \frac{e^{mx} + e^{-mx}}{2}$$

$$\frac{d\sinh mx}{dx} = \cosh mx \cdot m \, dx$$

$$\frac{d\cosh mx}{dx} = \sinh mx \cdot m \, dx$$

For case 2: $d\theta/dx|_{x=L} = 0$, that is, $c_1 m e^{mL} - c_2 m e^{-mL} = 0$, solve for c_1 and c_2, one obtains the temperature distribution through the fin base (q_f) as

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL} \tag{2.36}$$

$$q_f = q_x = -kA_c \left. \frac{dT}{dx} \right|_{x=0} = -kA_c \frac{d\theta(0)}{dx} = \sqrt{hPkA_c}\theta_b \tanh mL \tag{2.37}$$

For case 3: $\theta(L) = \theta_L$, that is, $c_1 e^{mL} + c_2 e^{-mL} = \theta_L$, solve for c_1 and c_2, one obtains the temperature distribution and heat transfer through the fin base (q_f) as

$$\frac{\theta(x)}{\theta_b} = \frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL} \tag{2.38}$$

$$q_f = q_x = -kA_c \left. \frac{dT}{dx} \right|_{x=0} = -kA_c \frac{d\theta(0)}{dx} = \sqrt{hPkA_c}\theta_b \frac{\cosh mL - \theta_L/\theta_b}{\sinh mL} \tag{2.39}$$

For case 4: very long fins, $\theta(L) = 0$, then $c_1 = 0$, $c_2 = \theta_b$, we obtained the following temperature distribution and heat transfer rate through the fin base (q_f) as

$$\frac{\theta}{\theta_b} = e^{-mx} \tag{2.40}$$

$$q_f = q_x = -kA_c \frac{d\theta(0)}{dx} = M = \sqrt{hPkA_c}\theta_b \tag{2.41}$$

It should be noted that the above results can be applied to any fins with uniform cross-sectional area. This includes the fins with circular, rectangular, triangular, and other cross sections as shown in Figure 2.7. One should know how to calculate the fin cross-sectional area A_c and fin perimeter P (circumferential length) for a given uniform cross-sectional area fin geometry.

2.3.1   Fin Performance

2.3.1.1   *Fin Effectiveness*

Most often, before adding the fins, we would like to know whether it is worth-while to add fins on the smooth heated surface. In this case, we define the fin effectiveness. The fin effectiveness is defined as the ratio of heat transfer rate through the fin surface to that without the fin (i.e., convection from the fin base area).

$$\eta_\varepsilon = \frac{q_{\text{with fin}}}{q_{\text{without fin}}} \tag{2.42}$$

The fin effectiveness must be greater than unity in order to justify using the fins. Normally, it should be greater than 2 in order to include the material and manufacturing costs. In general, the fin effectiveness is greater than 5 for most of the effective fin applications. For example, for the long fins (case 4 fin tip boundary conditions), the fin effectiveness is

$$\eta_\varepsilon = \frac{\sqrt{hPkA_c}\,\theta_b}{h\theta_b A_c} = \sqrt{\frac{kP}{hA_c}} > 1 \sim 5 \tag{2.43}$$

2.3.1.2   *Fin Efficiency*

We would like to know the fin efficiency after we have decided to add the fins. The fin efficiency is defined in Equation 2.42 as the ratio of heat transfer through the fin surface to that through a perfect conducting fin (an ideal fin with infinite thermal conductivity as super conductors).

$$\eta_f = \frac{q_{\text{fin}}}{q_{\text{max}}} \tag{2.44}$$

By definition, the fin efficiency is between 0 and 1. However, the fin efficiency is around 0.9–0.95 for most of the efficient fin application. For example, for long fins (case 4 fin tip boundary conditions), the efficiency is

$$\eta_f = \frac{\sqrt{hPkA_c}\,\theta_b}{hPL\theta_b} \geq 90\% \tag{2.45}$$

In order to get the higher fin effectiveness and fin efficiency, we need to have a thin fin (larger P/A_c ratio) with larger thermal conductivity k (aluminum or copper) and low working fluid heat transfer coefficient h (air cooling).

$$q = q_{\text{fin}} + q_{\text{non-fin}}$$

$$q = N\eta_f q_{\text{max}} + h(T_b - T_\infty)A_{\text{non-fin}}$$

where N is the number of fins and

$$q_{\text{max}} = h(T_b - T_\infty)A_{s,\text{fin}}$$

$$q_{\text{fin}} = -kA_c \frac{dT}{dx}\bigg|_{x=0}$$

It is important to point out that the temperature distribution through a fin varies depending on the aforementioned fin tip boundary conditions. In general, these temperatures are a decay curve from the fin base to the fin tip as shown in Figure 2.8. These decay curves are the combination of sinh and cosh functions shown above. In addition, the heat transfer rate through the fin depends on the temperature gradient at the fin base and the fin thermal conductivity. For example, the temperature gradient at the fin base is greater for the steel fin than the aluminum fin. However, heat transfer rate through the fin base is higher for an aluminum fin than for a steel fin for the same fin geometry and working fluid conditions. This is because the aluminum fin has a much larger thermal conductivity than the steel fin.

2.3.2   Radiation Effect

If we also consider radiation flux q_r'', the energy balance equation 2.29 can be rewritten as

$$q_x - \left(q_x + \frac{dq_x}{dx}dx\right) - hA_s(T - T_\infty) + A_s q_r'' = 0$$

where $q_r'' = $ radiation gain from solar $= $ constant, or $q_r'' = $ radiation loss $= -\varepsilon\sigma(T^4 - T_{\text{sur}}^4)$. If we consider $q_r'' = $ constant, the solution of above equation can be obtained by Equation 2.34 by setting

$$\theta = T - T_\infty - \frac{q_r''}{h}$$

However, if we consider $q_r'' = -\varepsilon\sigma(T^4 - T_{\text{sur}}^4)$, the above energy balance equation can be rewritten as

$$q_x - \left(q_x + \frac{dq_x}{dx}dx\right) - hA_s(T - T_\infty) - \varepsilon\sigma A_s(T^4 - T_{\text{sur}}^4) = 0$$

If we let $T_\infty = T_{\text{sur}}$, $h_{\text{r}} = \varepsilon\sigma(T^2 + T_\infty^2)(T + T_\infty)$, the above equation can be written as

$$\frac{d^2(T - T_\infty)}{dx^2} - \frac{(h + h_{\text{r}})P}{kA_{\text{c}}}(T - T_\infty) = 0$$

The solution of the above equation can be obtained by numerical integration.

2.4 Conduction through Fins with Variable Cross-Sectional Area: Bessel Function Solutions

Most often, we would like to have a fin with decreasing fin cross-sectional area in the heat conduction direction in order to save material costs. In other situations, fin cross-sectional area increases in the heat conduction direction such as an annulus fin attaching to the circular tube (the so-called fin tube). In these cases, we deal with steady-state 1-D heat conduction through fins with variable cross-sectional area. Bessel function solutions are required to solve temperature distribution through these types of fin problems [2–4]. Figure 2.9 shows the heat conduction through variable cross-sectional area fins and heat dissipation to working fluid.

Consider the energy balance of a small control volume in the fin,

$$q_x - \left(q_x + \frac{dq_x}{dx}dx\right) - hA_{\text{s}}(T - T_\infty) = 0$$

$$-\frac{dq_x}{dx}dx - hA_{\text{s}}(T - T_\infty) = 0$$

where $A_{\text{s}} = P\,dx$, and P is perimeter of the fin, and q_x is from Fourier's Conduction law shown in Equation 1.10.

$$-\frac{d}{dx}\left(-kA_{\text{c}}\frac{dT}{dx}\right)dx - hP\,dx(T - T_\infty) = 0$$

If k is constant, we have

$$\frac{d}{dx}\left(A_{\text{c}}\frac{dT}{dx}\right) - \frac{hP}{k}(T - T_\infty) = 0$$

$$\frac{d}{dx}\left[A_{\text{c}}\frac{d(T - T_\infty)}{dx}\right] - \frac{hP}{k}(T - T_\infty) = 0$$

(2.46)

For example, for an annulus fin with uniform thickness t as shown in Figure 2.9b, the fin cross-sectional area from the centerline of the tube is
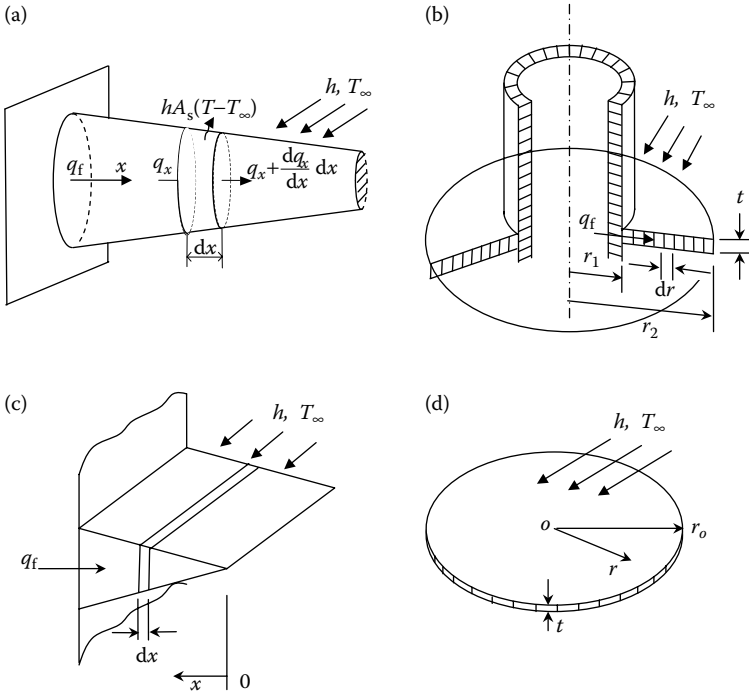
FIGURE 2.9
One-dimensional heat conduction through thin fins with variable cross-section area. (a) Conical
fin; (b) annular fin; (c) taper fin; (d) disk fin.

$A_c = 2\pi r \cdot t$. The fin perimeter including the top and the bottom, $P = 2 \cdot 2\pi r$.
Let $\theta = T - T_\infty$, Equation 2.46 becomes

$$\frac{d}{dr}\left[2\pi rt\frac{d\theta}{dr}\right] - \frac{h \cdot 4\pi r}{k}\theta = 0$$

$$\frac{d}{dr}\left[r\frac{d\theta}{dr}\right] - \frac{2\,hr}{kt}\theta = 0$$

$$r\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - \frac{2\,hr^2}{kt}\theta = 0$$ (2.47)

$$r\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) - m^2r^2\theta = 0$$

where $m^2 = 2h/kt$.

1. For heat generation problem—Bessel function: The solutions of
   Equation 2.47 can be a typical Bessel function with heat generation

as the following format:

$$x\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) + \left(m^2x^2 - v^2\right)\theta = 0 \tag{2.48}$$

$$\theta = a_0 J_v(mx) + a_1 Y_v(mx), \quad v = 0, 1, 2, \ldots \tag{2.49}$$

2. For heat loss problem—Modified Bessel function: The solutions of Equation 2.47 can be the typical modified Bessel function with heat loss as the following format:

$$x\frac{d}{dx}\left(x\frac{d\theta}{dx}\right) - \left(m^2x^2 + v^2\right)\theta = 0 \tag{2.50}$$

$$\theta = a_0 I_v(mx) + a_1 K_v(mx), \quad v = 0, 1, 2, \ldots \tag{2.51}$$

Comparing Equations 2.47 and 2.50, $r = x$, the general solution of Equation 2.47 is the same format as Equation 2.51 with $v = 0$

$$\theta(mr) = a_0 I_0(mr) + a_1 K_0(mr) \tag{2.52}$$

with the following boundary conditions:
At the fin base, $r = r_1$, $T = T_b$, then $\theta = T_b - T_\infty = \theta_b$.
At the fin tip, $r = r_2$, there are four possible cases as discussed before,

1. Convective boundary

$$-k\frac{\partial T}{\partial r}\bigg|_{r=r_2} = h(T_{r2} - T_\infty) \quad \text{or} \quad \frac{\partial\theta(r_2)}{\partial r} = \frac{-h}{k}\theta_{r2}$$

2. The tip fin is insulated

$$-k\frac{\partial T}{\partial r}\bigg|_{r=r_2} = 0 \quad \text{or} \quad \frac{\partial\theta(r_2)}{\partial r} = 0$$

3. Top tip temperature is given

$$T|_{r=r_2} = T_{r2} \quad \text{or} \quad \theta(r_2) = T_{r2} - T_\infty = \theta_{r2}$$

4. For a long fin $r_2/t > 10 \sim 20$

$$T|_{r=r_2} = T_\infty \quad \text{or} \quad \theta(r_2) = T_\infty - T_\infty = 0$$

For case 2, the fin tip is insulated, for example:

$$\text{At} \quad r = r_1, \quad \theta = \theta_b = a_0 I_0(mr_1) + a_1 K_0(mr_1),$$

$$\text{At} \quad r = r_2, \quad \frac{\partial\theta(r_2)}{\partial r} = a_0 \frac{dI_0(mr)}{dr}\bigg|_{r=r_2} + a_1 \frac{dK_0(mr)}{dr}\bigg|_{r=r_2}$$

$$= a_0 m I_1(mr_2) - a_1 m K_1(mr_2) = 0,$$

where applying the properties of I and K,

$$\frac{d}{dr}I_0(mr) = mI_1(mr) \quad \text{and} \quad \frac{d}{dr}K_0(mr) = -mK_1(mr)$$

Solve for a_0 and a_1

$$a_0 = \frac{K_1(mr_2)}{I_0(mr_1)K_1(mr_2) + I_1(mr_2)K_0(mr_1)}\theta_b$$

$$a_1 = \frac{I_1(mr_2)}{I_0(mr_1)K_1(mr_2) + I_1(mr_2)K_0(mr_1)}\theta_b$$

one obtains the temperature distribution as

$$\frac{\theta(r)}{\theta_b} = \frac{T(r) - T_\infty}{T_b - T_\infty} = \frac{I_0(mr)K_1(mr_2) + I_1(mr_2)K_0(mr)}{I_0(mr_1)K_1(mr_2) + I_1(mr_2)K_0(mr_1)} \tag{2.53}$$

Therefore, heat transfer through the fin base can be calculated as

$$q_f = -kA_c\frac{d\theta}{dx}\bigg|_{r_1} = -k(2\pi r_1)(t)\frac{d\theta}{dr}\bigg|_{r=r_1}$$

$$q_f = 2\pi k r_1 tm\frac{I_1(mr_1)K_1(mr_2) - I_1(mr_2)K_1(mr_1)}{I_0(mr_1)K_1(mr_2) + I_1(mr_2)K_0(mr_1)}\theta_b \tag{2.54}$$

And the fin efficiency can be determined as

$$\eta_f = \frac{q_f}{h2\pi(r_2^2 - r_1^2)\theta_b}$$

It is important to point out that temperature decreases from the fin base to the fin tip depending on the specified fin tip boundary conditions. The temperature decay curve again is a combination of Bessel function I_0 and K_0. The characteristics of Bessel functions J, Y, I, and K, and their derivatives are shown in Figure 2.10.

2.4.1 Radiation Effect

If we also consider radiation effect, $q_r'' = \text{constant} = \text{positive value}$, then the above solution can be used by replacing $\theta = T - T_\infty - (q_r''/h)$. However, if we consider $q_r'' = -\varepsilon\sigma(T^4 - T_{sur}^4)$, and $T_\infty = T_{sur}$, $h_r = \varepsilon\sigma(T^2 + T_\infty^2)(T + T_\infty)$, the energy balance equation 2.46 can be written as

$$\frac{d}{dx}\left[A_c\frac{d(T - T_\infty)}{dx}\right] - \frac{(h + h_r)P}{k}(T - T_\infty) = 0$$

The solution of the above equation can be obtained by numerical integration.
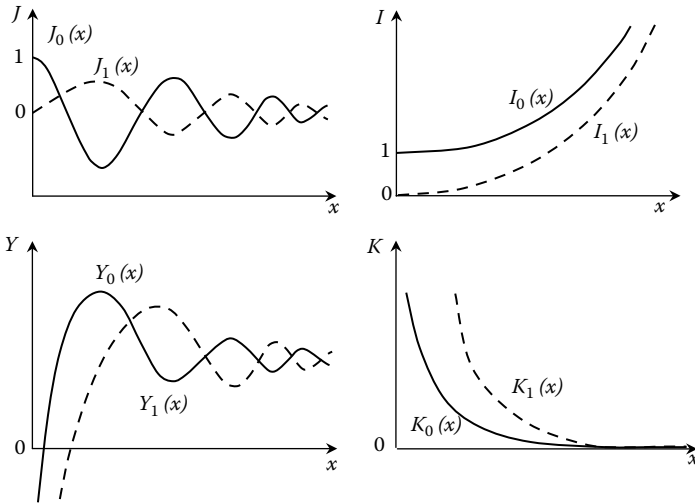
FIGURE 2.10
The characteristics of Bessel functions.

Examples

2.1. A composite cylindrical wall (Figure 2.11) is composed of two materials of thermal conductivity k_A and k_B, which are separated by a very thin, electric resistance heater for which interfacial contact resistances with material A and B $R''_{tc,A}$, $R''_{tc,B}$, respectively. Liquid pumped through the tube is at a temperature $T_{\infty,i}$ and provides a convection coefficient h_i at the inner surface of
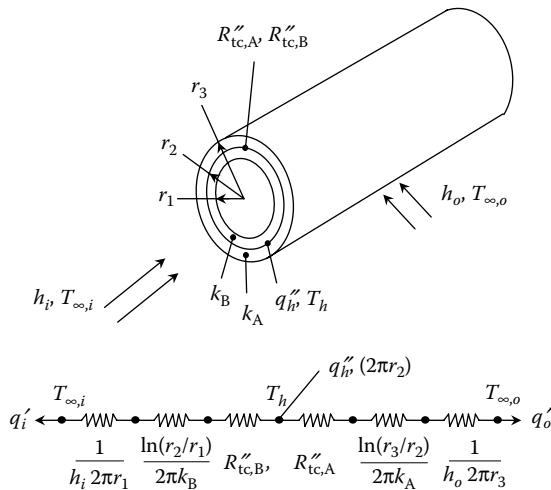
FIGURE 2.11
Thermal circuit of a composite cylindrical wall and all resistance.

the composite. The outer surface is exposed to ambient air, which is at $T_{\infty,o}$ and provides a convection coefficient of h_o. Under steady-state conditions, a uniform heat flux of q_h'' is dissipated by the heater.

a. Sketch the equivalent thermal circuit of the system and express all resistances in terms of relevant variables.

b. Obtain an expression that may be used to determine the heater temperature, T_h.

c. Obtain an expression for the ratio of heat flows to the outer and inner fluids, q_o'/q_i'. How might the variables of the problem be adjusted to minimize this ratio?

SOLUTION

a. See the sketch shown in Figure 2.11.

b. Performing an energy balance for the heater, $\dot{E}_{in} = \dot{E}_{out}$, it follows that

$$q_h''(2\pi r_2) = q_i' + q_o' = \frac{T_h - T_{\infty,i}}{(h_i 2\pi r_1)^{-1} + (\ln(r_2/r_1)/2\pi k_B) + R_{tc,B}''}$$

$$+ \frac{T_h - T_{\infty,o}}{(h_o 2\pi r_3)^{-1} + (\ln(r_3/r_2)/2\pi k_A) + R_{tc,A}''}$$

c. From the circuit,

$$\frac{q_o'}{q_i'} = \frac{(T_h - T_{\infty,o})}{(T_h - T_{\infty,i})} \cdot \frac{(h_i 2\pi r_1)^{-1} + (\ln(r_2/r_1)/2\pi k_B) + R_{tc,B}''}{(h_o 2\pi r_3)^{-1} + (\ln(r_3/r_2)/2\pi k_A) + R_{tc,A}''}$$

2.2. Heat is generated at a rate $\dot{q}$ in a large slab of thickness 2L, as shown in Figure 2.5a. The side surfaces lose heat by convection to a liquid at temperature T_∞. Obtain the steady-state temperature distributions for the following cases:

a. $\dot{q} = \dot{q}_0 \left[1 - (x/L)^2\right]$, with x measured from the centerplane.

b. $\dot{q} = a + b(T - T_\infty)$

SOLUTION

a. $\dot{q} = \dot{q}_0 \left[1 - (x/L)^2\right]$

$$\frac{d^2 T}{dx^2} = -\frac{\dot{q}_0}{k}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

$$\frac{dT}{dx} = -\frac{\dot{q}_0}{k}\left[x - \frac{1}{3}\frac{x^3}{L^2}\right] + C_1$$

$$T = -\frac{\dot{q}_0}{k}\left[\frac{1}{2}x^2 - \frac{1}{12}\frac{x^4}{L^2}\right] + C_1 x + C_2$$

Boundary conditions: $x = 0$, $\dfrac{dT}{dx} = 0$ (symmetry); $x = L$,

$$-k\frac{dT}{dx} = h(T - T_\infty)$$

Substituting the boundary conditions into the above equation to replace C_1 and C_2,

$$T - T_\infty = \frac{\dot{q}_0 L^2}{2k}\left[\frac{5}{6} + \frac{1}{6}\left(\frac{x}{L}\right)^4 - \left(\frac{x}{L}\right)^2 + \frac{4}{3Bi}\right]$$

b. $\dot{q} = a + b(T - T_\infty)$

$$\frac{d^2 T}{dx^2} + \frac{b}{k}\left[T - \left(T_\infty - \frac{a}{b}\right)\right] = 0$$

Solving the above equation,

$$T - \left(T_\infty - \frac{a}{b}\right) = C_1 \cos\left(b/k\right)^{1/2} x + C_2 \sin\left(b/k\right)^{1/2} x$$

Applying boundary conditions,

$$T - \left(T_\infty - \frac{a}{b}\right) = \frac{h\cos\left(b/k\right)^{1/2}\left(a/b\right) x}{h\,\cos\left(b/k\right)^{1/2} L + \left(b/k\right)^{1/2}\sin\left(b/k\right)^{1/2} L}$$

2.3. A long gas turbine blade (Figure 2.12) receives heat from combustion gases by convection and radiation. If reradiation from the blade can be neglected $T_s \ll T_\infty$, determine the temperature distribution along the blade. Assume

a. The blade tip is insulated.

b. The heat transfer coefficient on the tip equals that on the blade sides.

The cross-sectional area of the blade may be taken to be constant.

SOLUTION

If the blade is at a much lower temperature than the combustion gases, radiation emitted by the blade will be much smaller than the absorbed radiation and can be ignored to simplify the problem. An energy balance on an element of fin Δx long gives

$$-kA_c \left.\frac{dT}{dx}\right|_x + kA_c \left.\frac{dT}{dx}\right|_{x+\Delta x} - hP\Delta x(T - T_\infty) + q_{rad}P\Delta x = 0$$

Dividing by Δx and letting $\Delta x \to 0$,

$$\frac{d^2 T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) + \frac{q_{rad}P}{kA_c} = 0$$
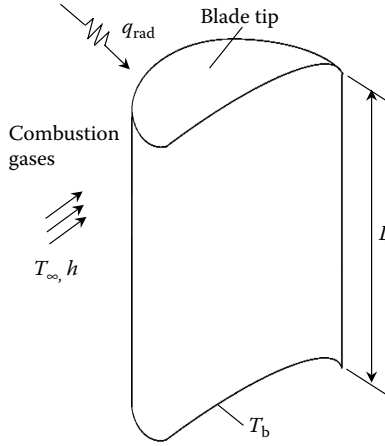
FIGURE 2.12
A turbine blade modeled as a fin with constant cross-sectional area.

Since q_{rad} is a constant, we rewrite this equation as

$$\frac{d^2 T}{dx^2} - \frac{hP}{kA_c}\left[T - \left(T_\infty + \frac{q_{rad}}{h}\right)\right] = 0$$

which defines an "effective" ambient temperature $T'_\infty = T_\infty + q_{rad}/h$. Then the solutions can be obtained by replacing T_∞ with T'_∞.

a. Insulated blade tip.

$$\frac{T - \left(T_\infty + q_{rad}/h\right)}{T_b - \left(T_\infty + q_{rad}/h\right)} = \frac{\cosh m(L - x)}{\cosh mL}$$

b. Tip and side heat transfer coefficient equal.

$$\frac{T - \left(T_\infty + q_{rad}/h\right)}{T_b - \left(T_\infty + q_{rad}/h\right)} = \frac{\cosh m(L - x) + (h/mk)\sinh m(L - x)}{\cosh mL + (h/mk)\sin mL}$$

2.4. The attached Figure shows a straight fin of triangular profile (Figure 2.13). Assume that this is a thin fin with $w \gg t$. Derive the heat conduction equation of fin: determine the temperature distributions in the fin analytically; and determine the fin efficiency.

SOLUTION

From Figure 2.13,

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{h}{k}\frac{dA_s}{dx}(T - T_\infty) = 0 \qquad (2.55)$$

$$\frac{tw}{L}x\frac{d^2 T}{dx^2} + \frac{tw}{L}\frac{dT}{dx} - \frac{2hw}{k}(T - T_\infty) = 0$$
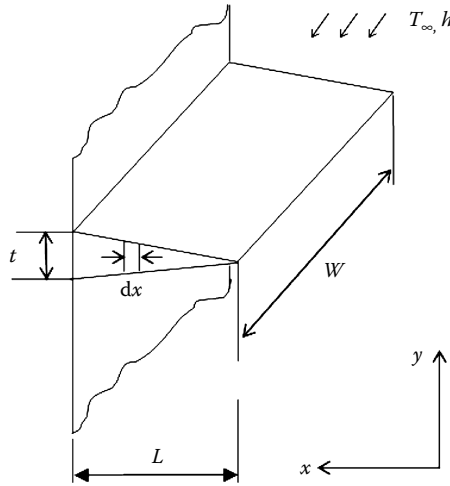
FIGURE 2.13
A triangular straight fin with variable cross-sectional area.

$$x\frac{d^2 T}{dx^2} + \frac{dT}{dx} - \frac{2h}{kt}L(T - T_\infty) = 0$$

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - m^2 Lx\theta = 0 \qquad (2.56)$$

where

$$A_c = w\frac{t}{L}x$$

$$dA_s = 2w\,dx$$

$$\theta \equiv T - T_\infty$$

$$m^2 \equiv \frac{2h}{kt}$$

$$A_s \simeq 2wL$$

But we need $z^2(d^2\theta/dz^2) + z(d\theta/dz) - z^2\theta = 0$, for the modified Bessel function solution

So, $z^2 \sim x$, $z \sim \sqrt{x}$

From the solution form, $I_0\left(2m\sqrt{xL}\right)$, imply $z \sim \sqrt{x} = 2m\sqrt{xL}$

$$\frac{dz}{dx} = 2m\sqrt{L} \cdot \frac{1}{2} \cdot x^{-1/2} = m\sqrt{L}\frac{1}{\sqrt{x}} = \frac{2m^2 L}{z}$$

$$\frac{d\theta}{dx} = \frac{d\theta}{dz} \cdot \frac{dz}{dx}; \quad \frac{d^2\theta}{dx^2} = \frac{d}{dz}\left(\frac{d\theta}{dx}\right) \cdot \frac{dz}{dx};$$

Substituting into Equation 2.56

$$z^2 \frac{d^2\theta}{dz^2} + z\frac{d\theta}{dz} - z^2\theta = 0 \qquad (2.57)$$

The solution to the above equation is

$$\theta = C_1 I_0(z) + C_2 K_0(z)$$

Boundary conditions

$$\text{at} \quad x = 0, \quad K_0 \to \infty, \quad C_2 = 0,$$

$$\text{at} \quad x = L, \quad \theta = \theta_b,$$

$$\theta = \theta_b \cdot \frac{I_0\left(2m\sqrt{xL}\right)}{I_0(2\,mL)}$$

$$q_f = +kA_c \left.\frac{dT}{dx}\right|_{x=L} = \theta_b ktwm \frac{I_1(2\,mL)}{I_0(2\,mL)}$$

$$\eta_f = \frac{q_f}{h\theta_b 2wL} = \frac{1}{mL}\frac{I_1(2\,mL)}{I_0(2\,mL)}$$

2.5. A hollow transistor (Figure 2.14) has a cylindrical cap of radius r_o and height L, and is attached to a base plate at temperature T_b. Show that the heat dissipated is

$$q_f = 2\pi kr_o t\,(T_b - T_\infty)\,m\left[\frac{I_0\,(mr_o)\sinh\,mL + I_1\,(mr_o)\cosh\,mL}{I_0\,(mr_o)\cosh\,mL + I_1\,(mr_o)\sinh\,mL}\right]$$

where the metal thickness is t, $m = (h/kt)^{1/2}$, and the heat transfer coefficient on the sides and top is assumed to be the same, h, and outside temperature, T_∞.
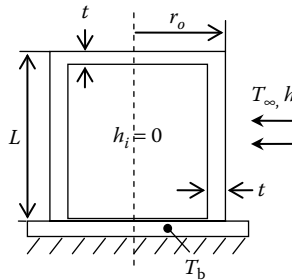
FIGURE 2.14
A hollow transistor modeled as a fin.

SOLUTION

The cap is split into two fins, (1) a straight fin of width $2\pi r_o$ and length L, and (2) a disk fin of radius r_o.

For the straight fin with $T = T_b$ at $x = 0$,

$$T_1 - T_\infty = C_1 \sinh mx + (T_b - T_\infty) \cosh mx; \quad m = (h/kt)^{1/2} \quad (2.58)$$

For the disk with $dT/dr = 0$ at $r = 0$,

$$T_2 - T_\infty = C_2 I_0 (mr); \quad m = (h/kt)^{1/2} \tag{2.59}$$

The constants C_1 and C_2 are determined by matching the temperature and heat flow at the join,

$$T_1 (L) = T_2 (r_o); \quad dT/dx|_{x=L} = - \, dT/dr|_{r=r_o}; \tag{2.60}$$

Since kA_c is the same for both fins at the junction. Substituting Equations 2.58 and 2.59 into Equation 2.60 gives

$$C_2 I_0 (mr_o) = C_1 \sinh mL + (T_b - T_\infty) \cosh mL$$

$$-C_2 I_1 (mr_o) = C_1 \cosh mL + (T_b - T_\infty) \sinh mL$$

Solving,

$$C_1 = - (T_b - T_\infty) \frac{I_0 (mr_o) \sinh mL + I_1 (mr_o) \cosh mL}{I_0 (mr_o) \cosh mL + I_1 (mr_o) \sinh mL}$$

The heat dissipation is the base heat flow of fin 1,

$$q_f = -kA_c \left. \frac{dT_1}{dx} \right|_{x=0} = -kA_c m C_1$$

$$= 2\pi k r_o t (T_b - T_\infty) \frac{I_0 (mr_o) \sinh mL + I_1 (mr_o) \cosh mL}{I_0 (mr_o) \cosh mL + I_1 (mr_o) \sinh mL}$$

Remarks

This chapter deals with 1-D steady-state heat conduction through the plane wall with and without heat generation, cylindrical tube with and without heat generation, and fins with constant and variable cross-sectional area. Although it is a 1-D steady-state conduction problem, there are many engineering applications. For example, heat losses through building walls, heat transfer through tubes, and heat losses through fins. In undergraduate heat transfer, we normally ask you to calculate heat transfer rates through the plane

walls, circular tubes, and fins, with given dimensions, material properties, and thermal boundary conditions. Therefore, you can pick up the right formulas and plug in with these given numbers and obtain the results.

However, in this intermediate heat transfer level, we are more focused on how to solve the heat conduction equation with various thermal boundary conditions and how to obtain the temperature distributions for a given physical problem. For example, how to solve the temperature distributions for the plane walls, circular tubes, with and without heat generation, and fins with constant and variable cross-sectional area, with various thermal boundary conditions. In particular, we have introduced one of very powerful mathematical tools, Bessel function, to solve the fins with variable cross-sectional area with various thermal boundary conditions. This is the only thing new as compared to the undergraduate heat transfer.

PROBLEMS

2.1. The performance of gas turbine engines may be improved by increasing the tolerance of the turbine blades to hot gases emerging from the combustor. One approach to achieving high operating temperatures involves application of a *thermal barrier coating* (TBC) to the exterior surface of a blade, while passing cooling air through the blade. Typically, the blade is made from a high-temperature superalloy, such as Inconel ($k \approx 25\,\text{W/m K}$) while a ceramic, such as zirconia ($k \approx 1.3\,\text{W/m K}$), is used as a TBC.

Consider conditions for which hot gases at $T_{\infty,o} = 1700\,\text{K}$ and cooling air at $T_{\infty,i} = 400\,\text{K}$ provide outer- and inner-surface convection coefficients of $h_o = 1000\,\text{W/m}^2\,\text{K}$ and $h_i = 500\,\text{W/m}^2\,\text{K}$, respectively. If a 0.5-mm-thick zirconia TBC is attached to a 5-mm-thick Inconel blade wall by means of a metallic bonding agent, which provides an interfacial thermal resistance of $R''_{t,c} = 10^{-4}\,\text{m}^2\,\text{K/W}$, can the Inconel be maintained at a temperature that is below its maximum allowable value of 1250 K? Radiation effects may be neglected, and the turbine blade may be approximated as a plane wall. Plot the temperature distribution with and without the TBC. Are there any limits to the thickness of the TBC?

2.2. 1-D heat conduction through a circular tube, as shown in Figure 2.3. Determine heat loss per tube length as the following conditions:

Given:

Steam Inside the Pipe	Air Outside the Pipe	Steel Pipe AISI 1010
$T_{\infty 1} = 250°\text{C}$	$T_{\infty 2} = 20°\text{C}$	$2r_1 = 60\,\text{mm}$
$h_1 = 500\,\text{W/m}^2\text{K}$	$h_2 = 25\,\text{W/m}^2\text{K}$	$2r_2 = 75\,\text{mm}$
		$\varepsilon = 0.8$

Find: $q/L =?$

2.3. Heat is generated at a rate $\dot{q}$ in a long solid cylinder of radius r_0, as shown in Figure 2.6. The cylinder has a thin metal sheath and is immersed in a liquid at temperature T_∞. Heat transfer from the cylinder surface to the liquid can be characterized by a heat transfer coefficient h. Obtain the steady-state temperature distributions for the following cases:

a. $\dot{q} = \dot{q}_0 \left[1 - (r/r_0)^2\right]$.

b. $\dot{q} = a + b(T - T_\infty)$.

2.4. 1-D, a hollow cylindrical copper tube fin with constant cross-sectional area (Figure 2.15). Disk-shaped transistor dissipates 0.2 W during steady state. Good insulation at the base plate.

$L = 15\,\text{mm}$, $r_0 = 7.75\,\text{mm}$, $r_i = 7.5\,\text{mm}$, Assume: $h_1 = 0$ (no cooling)

$t = 0.25\,\text{mm}$

Air cooling: $T_\infty = 25°\text{C}$, $h_2 = 50\,\text{W/m}^2\text{K}$

Find: copper tube temperature distribution.

2.5. A thin metal disk, as shown in Figure 2.9d, is insulated on one side and exposed to a jet of hot air at temperature T_∞ on the other. The convection heat transfer coefficient h can be taken to be constant over the disk. The periphery at $r = R$ is maintained at a uniform temperature T_R.

a. Derive the heat conduction equation of disk.

b. Determine the disk temperature distributions.

c. Do you think the disk temperature is hotter at the center or the periphery? Why?

2.6. Given a relatively thin annular fin with a uniform thickness, as shown in Figure 2.9b, that is affixed to a tube. The inner and outer radii of the fin are r_i and r_0, respectively, and the thickness of the fin is t. The tube surface (i.e., the base of the annular fin) is maintained at a temperature of T_b or $T(r_i) = T_b$. Both the top and the bottom surfaces are exposed to a fluid at T_∞. The convection heat transfer coefficient between the fin surfaces and the fluid is h.

a. Derive the steady-state heat conduction equation of the annular fin, and propose a solution of the annular fin temperature distribution with the associate boundary conditions.
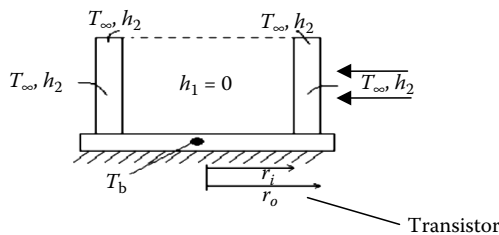
FIGURE 2.15
A hollow cylindrical copper tube fin.

    b. If during the air cooling, the annular fin has also received radiation energy from the surrounding environment, T_{sur}, can you sketch, compare, and comment on the fin temperature profile $T(r)$ with that in (a)?

2.7. Both sides of a very thin metal disk, as shown in Figure 2.9d, are heated by convective hot air at temperature T_∞. The convection heat transfer coefficient h can be taken constant over the disk. The periphery at $r = R$ is maintained at a uniform temperature T_R.

    a. Derive the steady-state heat conduction equation of the disk.

    b. Propose a solution method and the associated boundary conditions that can be used to determine the disk temperature distributions. Sketch the disk temperature profile $T(r)$. Do you think the disk temperature is hotter at the center or the periphery? Why?

    c. If during the air heating, the disk has also emitted a net uniform radiation flux q''_{rad} to the surrounding environment, derive the steady-state heat condition equation of the disk and propose a solution method to determine the disk temperature distributions. Can you sketch, compare, and comment on the disk temperature profile with that in (b)?

2.8. The front surface of a very thin metal disk is cooled by convective air at temperature T_∞ while the back surface is perfectly insulated, as shown in Figure 2.9d. The convection heat transfer coefficient h can be taken to be constant over the front surface of the disk. The periphery at $r = R$ is maintained at a uniform temperature T_R by a heat source.

    a. Derive the steady-state heat conduction equation of disk.

    b. Determine the disk temperature distributions with the associated boundary conditions. Sketch the disk temperature profile $T(r)$.

    c. If during the air cooling, the disk has also emitted a net uniform radiation flux q''_{rad} to the surrounding environment, derive the steady-state heat conduction equation of the disk and determine the disk temperature distributions. Can you sketch, compare, and comment on the disk temperature profile with that in (b)?

2.9. A thin conical pin fin is shown in Figure 2.9a. Determine analytically the temperature profile in the pin fin. Also determine the heat flux through the pin fin base.

Given:

Pin fin tip temperature: $T_R > T_\infty$ Pin fin height: l

Pin fin base diameter: d Pin fin base temperature: T_b

Cooling air at T_∞, h Hot wall at T_b

2.10. The wall of a furnace has a height $L = 1\,\mathrm{m}$ and is at a uniform temperature of 500 K. Three materials of equal thickness $t = 0.1\,\mathrm{m}$, having the properties listed in the table attached, are placed in the order shown in the Figure to insulated the furnace
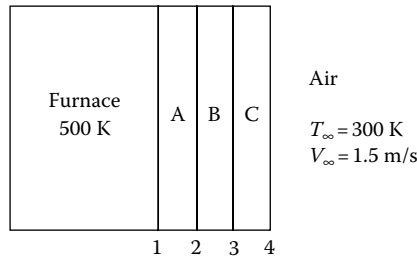
FIGURE 2.16
A furnace composite plane wall model.

wall (Figure 2.16). Air at $T_\infty = 300\,\text{K}$ blows past the outer layer of insulation at a speed of $V_\infty = 1.5\,\text{m/s}$ as shown in the figure.

a. Assuming 1-D, steady conduction through the insulation layers, calculate the heat flux from the wall to the surroundings, q. For this, choose the most appropriate of the following two correlations:

$$\frac{\bar{h}L}{k} = 0.664\,Re_L^{1/2}Pr^{1/3}\,(Re_L < 10^5;\ \text{laminarflow})$$

$$\frac{\bar{h}L}{k} = (0.037Re_L^{0.8} - 850)Pr^{1/3}\,(Re_L > 10^5;\ \text{turbulentflow})$$

where $\bar{h}$ is an average heat transfer coefficient, Re_L is the Reynolds number based on L and V_∞, and Pr is the Prandtl number of air.

b. For the arrangement shown in the figure, calculate the temperatures at surfaces 2, 3, and 4.

c. How should the materials A, B, C be ordered to obtain the steepest temperature gradient possible between surfaces 1 and 2?

d. For this new arrangement, calculate the temperatures at surfaces 2, 3, and 4.

Material	$K(\text{W/m k})$	$\rho(\text{kg/m}^3)$	$\mu(\text{kh/m s})$	$c_p(\text{kJ/kg K})$
A	100			
B	10			
C	1			
Air	0.026	1.177	1.846×10^{-5}	1.006

2.11. A very long copper rod of small diameter is moving in a vacuum with a constant velocity, V (Figure 2.17). The long rod is moving from one constant temperature region, T_0, at $x = 0$, to another temperature region, T_L (at $x = L$). Solve for the steady-state temperature distribution in the rod between $x = 0$ and L. Neglect thermal radiation.
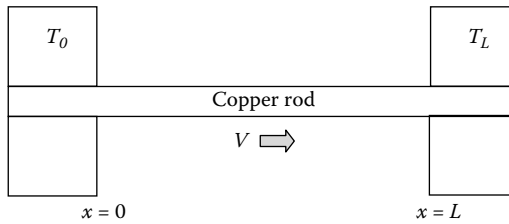
FIGURE 2.17
A moving fin model.

a. Write down the governing equation for axial temperature distribution when $V = 0$.

b. Determine the axial temperature distribution which satisfies the specified boundary conditions when $V = 0$.

c. Write down the governing equation for axial temperature distribution when $V \neq 0$.

d. Determine the axial temperature distribution which satisfies the specified boundary conditions when $V \neq 0$.

(Make any necessary assumption. For example, the thermal conductivity of the copper rod is $k \ldots$)

2.12. A thin long rod extends from the side of a probe that is in outer space. The base temperature of the rod is T_b. The rod has a diameter, D, a length, L, and its surface is at an emissivity, ε. State all relevant assumptions and boundary conditions.

a. Starting with an energy balance on a differential cross section of the pin fin, perform an energy balance on the rod and derive a differential equation that could be used to solve this problem.

b. Sketch on the same plot the temperature distribution along the rod, and compare the heat loss through the rod for the following four cases: (1) The rod is made of aluminum; (2) apply paint on the aluminum rod; (3) the rod is made of steel; (4) apply the paint on the steel rod.

2.13. Given a relatively thin annular fin with uniform thickness that is affixed to a tube. The inner and outer radii of the fin are r_i and r_o, respectively, and the thickness of the fin is w. The tube surface (i.e., the base of the annular fin) is maintained at a temperature of T_b, or $T(r_i) = T_b$. Both the top and bottom surfaces of the fin are exposed to a fluid at T_∞. The convective heat transfer coefficient between the fin surfaces and the fluid is h.

a. Show that, to determine the steady 1-D temperature distribution in the annular fin, $T(r)$, the governing equation may be written in the form of a modified Bessel's equation.

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(c^2 + \frac{n^2}{x^2}\right)y = 0$$

where $y = y(x)$, and c and n are constants.

b. The general solution of the above modified Bessel's equation is $y = C_1 I_n(cx) + C_2 K_n(cx)$, where C_1 and C_2 are constants, and I_n and K_n are the modified Bessel's functions of the first and second kinds of order n, respectively. Assuming that the heat transfer on the outer surface of the fin is negligible [i.e., $dT/dr = 0$ at $r = r_o$], solve the governing equation to obtain the steady 1-D temperature distribution in the annular fin, $T(r)$.

c. You may also determine the steady 1-D temperature distribution in the annular fin numerically using the finite difference method. Give the finite-difference equations for the nodes at $r = r_i$ and $r = r_o$ and for a typical interior node. Please rearrange the equations to give expressions for the temperatures at the nodes.

2.14. An engineer has suggested that a triangular fin would be more effective than a circular fin for a new natural convection heat exchanger. The fins are very long and manufactured from Al 2024-T6. The triangular fin has an equilateral cross section with a base dimension of 1.0 cm and the second fin has a circular cross section with a 0.955 cm diameter. If the base temperature of the fin is maintained at 400°C, which fin will transfer more heat and which fin has a greater effectiveness?

$$[h = 25 \text{ W/m}^2 \text{ K}, k_{2024-T6} = 177 \text{ W/mK}, T_\infty = 25°\text{C}]$$

$$q_{\text{fin}} = (hPkA_c)^{1/2} \cdot \theta_b \quad \varepsilon_{\text{fin}} = q_{\text{fin}}/hA_c\theta_b$$

2.15. A thin conical pin fin is attached to a hot base plate at T_b. The cooling air has temperature T_∞ and convection heat transfer coefficient h. Determine analytically the temperature profile in the pin fin. Also, determine the heat flux through the pin fin base.

2.16. Solve temperature profile for the annulus fin geometry as shown in Figure 2.9b, assume that the fin tip is exposed to convection fluid with same given temperature and convection heat transfer coefficient.

2.17. Solve the temperature profile for the annulus fin geometry as shown in Figure 2.9b, assume that the fin tip is fixed at a given temperature between the fin base and the convection fluid.

2.18. Determine the solutions shown in Equations 2.36 and 2.37.
2.19. Determine the solutions shown in Equations 2.38 and 2.39.
2.20. Determine the solutions shown in Equations 2.40 and 2.41.

References

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3

2-D Steady-State Heat Conduction

3.1 Method of Separation of Variables: Given Temperature BC

Most often heat is conducted in two dimensions instead of one dimension as discussed in Chapter 2. For example, we are interested in determining the temperature distribution in a 2-D rectangular block with appropriate BCs. Once the temperature distribution is known, the associated heat transfer rate can be determined. The following are the steady-state 2-D heat conduction equations without heat generation and the typical BCs with given surface temperatures.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{3.1}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0, \quad \text{if let } \theta = T - T_0 \tag{3.2}$$

Boundary conditions:

$x = 0$, $T = 0$ or $x = 0$, $\theta = 0$ homogeneous BC
$x = a$, $T = 0$ or $x = a$, $\theta = 0$ homogeneous BC
$y = 0$, $T = 0$ or $y = 0$, $\theta = 0$ homogeneous BC
$y = b$, $T = T_s$ or $y = b$, $\theta = T_s - T_0 = \theta_s$ nonhomogeneous BC

Here, we defined a homogeneous BC as $T = 0$, or $\partial T/\partial x = 0$, $\partial T/\partial y = 0$; $\theta = 0$, or $\partial \theta/\partial x = 0$, $\partial \theta/\partial y = 0$, that is, temperature or temperature gradient at a given boundary surface (in the x- or y-direction) equals 0. In contrast, we define a nonhomogeneous BC as $T \neq 0$, $\partial T/\partial x \neq 0$, $\partial T/\partial y \neq 0$; or $\theta \neq 0$, $\partial \theta/\partial x \neq 0$, $\partial \theta/\partial y \neq 0$, that is, temperature or temperature gradient at a given boundary surface (in the x- or y-direction) does not equal 0.

Equations 3.1 and 3.2 can be solved by the method of separation of variable. By means of this, we can separate the temperature from depending on two directions, $T(x, y)$, to one direction each, $T(x)$ and $T(y)$, respectively. The final 2-D temperature distribution is a product of each 1-D temperature solution, that is, $T(x, y) = T(x) \cdot T(y)$. The following outlines the method of separation
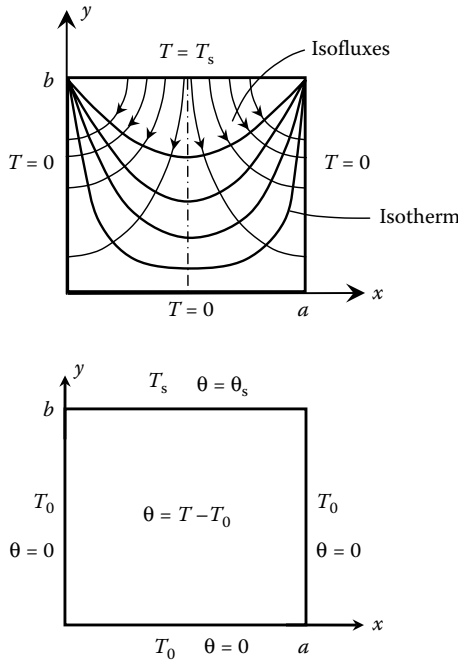
FIGURE 3.1
2-D heat conduction with three homogeneous and one nonhomogeneous boundary conditions.

of variable [1–4]. We need four BCs, two in the x-direction and two in the y-direction, to solve the 2-D heat conduction problem. One important note is that, among four BCs, only one nonhomogenoeus BC is allowed in order to apply the principal of separation of variable method. For a given problem, we need to make sure that only one nonhomogeneous BC exists either in the x- or in the y-direction. The principal of superposition will be used for the problems having two, three, or four nonhomogeneous BCs. We will begin with the simplest case, as shown in Figure 3.1, with given surface temperatures as BCs. Then we will move to more complicated cases with surface heat flux and surface convection BCs as well as the problems required in the principal of superposition.

$$T(x, y) = X(x)Y(y) \tag{3.3}$$

Then take the derivatives

$$\frac{\partial T}{\partial x} = Y\frac{\partial X}{\partial x} = Y\frac{\mathrm{d}X}{\mathrm{d}x}$$

$$\frac{\partial^2 T}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2} = Y\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}$$

$$\frac{\partial T}{\partial y} = X\frac{\partial Y}{\partial y} = X\frac{dY}{dy}$$

$$\frac{\partial^2 T}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2} = X\frac{d^2 Y}{dy^2}$$

Then substitute them into the 2-D conduction equation 3.1

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} = 0$$

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

(3.4)

The equality can hold only if both sides are equal to a constant as each side of Equation 3.4 is a function of an independent variable. The constant can be positive, negative, or zero. However, the positive number of the constant is the only possibility for the BCs in this case. The readers may note that zero and negative constants do not satisfy the BCs. Therefore, we express Equation 3.4 as

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda^2$$

(3.5)

Then we have

$d^2 X/dx^2 + \lambda^2 X = 0 \Rightarrow X = X(x)$, the equation for two homogeneous BCs,

$d^2 Y/dy^2 - \lambda^2 Y = 0 \Rightarrow Y = Y(y)$, the equation for one homogeneous BC,

$X(x) = C_1 \cos \lambda x + C_2 \sin \lambda x$, the solution for equation with two homogeneous BCs,

$Y(y) = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$ or $(Y(y) = C_3 \sinh \lambda y + C_4 \cosh \lambda y)$, the solution for equation with one homogeneous BC.

Solve C_1 and C_2 for the x-direction equation

at $x = 0$, $T = 0$, $\Rightarrow X = 0$, then $C_1 = 0$,
at $x = a$, $T = 0$, $\Rightarrow X = 0$, then $C_2 \sin \lambda a = 0 \Rightarrow \lambda_n a = n\pi, n = 0, 1, 2, 3, \ldots$ and then,

$$\lambda_n = \frac{n\pi}{a}$$

Therefore,

$$X(x) = C_2 \sin \frac{n\pi x}{a}$$

Solve C_3 and C_4 for the y-direction equation at $y = 0$, $T = 0$, $Y = 0 = C_3 + C_4 \Rightarrow C_4 = -C_3$.

$$Y(y) = C_3 e^{-\lambda_n y} - C_3 e^{\lambda_n y} = C_3(e^{-\lambda_n y} - e^{\lambda_n y})$$

With $\sinh(x) = (e^x - e^{-x})/2$, let $C_5 = C_3/2$, the above equation can be written as

$$Y(y) = C_5 \sinh(\lambda_n y)$$

Then solve for the product equation, and let $C_2 C_5 = C_n$.

$$T(x, y) = X(x) \cdot Y(y) = C_2 C_5 \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} = \sum C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

at $y = b$, $T = T_s = \sum C_n \sin(n\pi/a)x \sinh(n\pi/a)b$.
Multiplying both sides by $\sin(n\pi x/a)\, dx$, one obtains

$$\sin \frac{m\pi x}{a} \cdot T_s\, dx = \sum C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} \sin \frac{m\pi x}{a}\, dx$$

C_n can be determined by the integration over x,

$$C_n = \frac{\int T_s \sin(m\pi x/a)\, dx}{\int \sin(n\pi x/a)\, \sinh(n\pi b/a)\, \sin(m\pi x/a)\, dx} \qquad (3.6)$$

$$C_n = 0, \quad \text{if } m \neq n;$$

$$C_n = \frac{\int T_s \sin(n\pi x/a)\, dx}{\int \sin^2(n\pi x/a)\, \sinh(n\pi b/a)\, dx}, \quad \text{if } m = n.$$

How does one perform integration? From the integration table, one obtains

1. $\displaystyle\int_0^a \sin \frac{n\pi x}{a}\, dx = -\frac{a}{n\pi} \cos \frac{n\pi x}{a}\Big|_0^a = \frac{a}{n\pi}(1 - \cos(n\pi))$

$$= \frac{a}{n\pi}[1 - (-1)^n] = \begin{cases} 0, n = \text{even} \\ \dfrac{2a}{n\pi}, n = \text{odd} \end{cases}$$

2. $\displaystyle\int_0^a \sin^2 \frac{n\pi x}{a}\, dx = \frac{a}{2n\pi}\left[\frac{n\pi x}{a} - \frac{1}{2}\sin \frac{2n\pi x}{a}\right]_0^a = \frac{a}{2}$

or

$$\int_0^a \sin^2 \frac{n\pi x}{a}\, dx = \int_0^a \frac{1 - \cos(2n\pi x/a)}{2}\, dx = \frac{a}{2} - \frac{1}{2}\frac{a}{2n\pi}\sin \frac{2n\pi x}{a}\Big|_0^a = \frac{a}{2}$$

Therefore, one obtains

$$C_n = \frac{(2/n\pi)\left[1 - (-1)^n\right]T_s}{\sinh(n\pi b/a)}$$

Finally, the 2-D temperature distribution follows:

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{3.7}$$

$$T(x,y) = \sum_{n=1}^{\infty} \frac{(2/n\pi)\left[1 - (-1)^n\right]T_s}{\sinh(n\pi b/a)} \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{3.8}$$

The second-order partial differential equations (PDEs) $T(x,y)$ are split into two second-order ordinary differential equations (ODEs) $T(x)$ and $T(y)$. The second-order ODE with two homogeneous BCs is the so-called eigenvalue equation. The solution of the eigenfunctions, sine and cosine, depend on the two homogeneous BCs. The eigenvalues, λ, can be determined by one of the two homogeneous BCs (either both in the x-direction, or both in the y-direction). The solution of the other second-order ODE is a decay curve of combining $e(x)$ and $e(-x)$ for an infinite-length problem, or $\sinh(x)$ and $\cosh(x)$ for a finite-length problem. The only nonhomogeneous BC will be used to solve the final unknown coefficient C_n. The integrated value C_n can be determined by performing integration of sin, sin-square or cos, cos-square, depending on the given BCs, by using the characteristics of the orthogonal functions.

If we let $\theta = T - T_0$, follow the same procedure, the 2-D temperature distribution becomes

$$\theta(x,y) = \theta(x)\theta(y) = \sum C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{3.9}$$

$$\theta(x,y) = \sum_{n=1}^{\infty} \frac{(2/n\pi)\left[1 - (-1)^n\right]\theta_s}{\sinh(n\pi b/a)} \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{3.10}$$

3.2 Method of Separation of Variables: Given Heat Flux and Convection BCs

3.2.1 Given Surface Heat Flux BC

The following shows the similar principal of using the separation of variable method to solve the 2-D heat conduction problems with one nonhomogeneous boundary specified as surface heat flux or surface convection condition [2]. As the aforementioned procedure, we need to make all BCs homogeneous
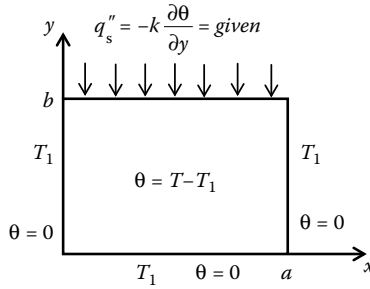
FIGURE 3.2
2-D heat conduction with three homogeneous and one heat flux nonhomogeneous boundary conditions.

but one. For example, in Figure 3.2, the three temperature BCs become homogeneous by setting $\theta = T - T_1$ and the only nonhomogeneous heat flux BC becomes $q_s'' = -k(\partial\theta/\partial y)$. The solution will be the product of sine and cosine in the x-direction (two homogeneous BCs) and sinh cosh in the y-direction (one nonhomogeneous BC). As before, the nonhomogeneous heat flux BC will be used to solve the final unknown integrated value C_n.

Given a long rectangular bar with a constant heat flux along one edge, other edges are isothermal. In order to obtain homogeneous BCs on the three isothermal edges, let $\theta = T - T_1$. Laplace's equation, Equation 3.2, applies to this steady 2-D conduction problem.

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

Boundary conditions:

$$\text{at } x = 0, \quad y = 0, \quad x = a, \quad \theta = 0$$
$$\text{at } y = b, \quad -k(\partial\theta/\partial y) = q_s''$$

The solution is subject to the three homogeneous BCs with θ replacing T,

$$\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

Applying the last BC, nonhomogeneous, to solve for C_n,

$$\frac{\partial\theta}{\partial y}\bigg|_{y=b} = -\frac{q_s''}{k} = \sum_{n=1}^{\infty} C_n \left(\sin\frac{n\pi x}{a}\right)\frac{n\pi}{a} \cosh\frac{n\pi b}{a}$$

$$C_n = \frac{-(q_s''/k)\int_0^a \sin(n\pi x/a)\,dx}{(n\pi/a)\,\cosh(n\pi b/a)\int_0^a \sin^2(n\pi x/a)\,dx}$$
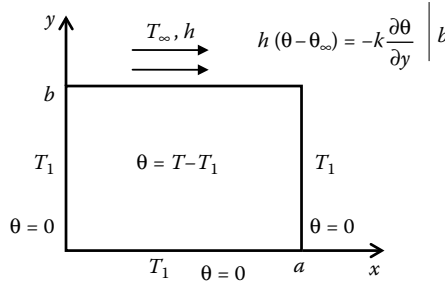
FIGURE 3.3
2-D heat conduction with one convective boundary conditions.

$$= \frac{-(q_s''/k)(a/n\pi)[1 - (-1)^n]}{(n\pi/a)\,\cosh(n\pi b/a)[(a/2)]}$$

$$= \frac{-(q_s''/k)(2/n\pi)[1 - (-1)^n]}{(n\pi/a)\,\cosh(n\pi b/a)}$$

Therefore, we obtain

$$T(x,y) = T_1 - \sum_{n=1}^{\infty} \frac{2q_s''a[1 - (-1)^n]}{kn^2\pi^2\,\cosh(n\pi b/a)} \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \tag{3.11}$$

3.2.2 Given Surface Convection BC

A similar procedure can be applied for the only nonhomogeneous convection BC problem [2] shown in Figure 3.3. Again, the solution will be a product of sine and cosine in the x-direction and sinh and cosh in the y-direction, and the nonhomogeneous convection BC will be used to solve the final unknown integrated value of C_n.

A long rectangular bar with one side cooled by convection and the others maintained at a constant temperature T_1.

Define $\theta = T - T_1$. Laplace's equation applies to this steady 2-D conduction problem.

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

which must be solved subject to BCs

$$x = 0, \quad 0 < y < b: \quad \theta = 0,$$
$$y = 0, \quad 0 < x < a: \quad \theta = 0,$$
$$x = a, \quad 0 < y < b: \quad \theta = 0,$$

$$y = b, \quad 0 < x < a : \begin{cases} h(T - T_\infty) = -k\dfrac{\partial T}{\partial y} \\ h[(T - T_1) - (T_\infty - T_1)] = -k\dfrac{\partial (T - T_1)}{\partial y} \\ h(\theta - \theta_\infty) = -k\dfrac{\partial \theta}{\partial y} \end{cases}$$

The solution is subject to the three homogeneous BCs with θ replacing T,

$$\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}$$

and at $y = b$, applying nonhomogeneous BCs to solve for C_n:

$$\frac{\partial \theta}{\partial y} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \frac{n\pi}{a} \cosh \frac{n\pi b}{a} = -\frac{h}{k}\left[\sum_n C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} - \theta_\infty\right]$$

$$\sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \left(\frac{n\pi}{a} \cosh \frac{n\pi b}{a} + \frac{h}{k} \sinh \frac{n\pi b}{a}\right) = \frac{h}{k}\theta_\infty$$

$$C_n = \frac{(h/k)\theta_\infty \int_0^a \sin(n\pi x/a)\, dx}{((n\pi/a)\, \cosh(n\pi b/a) + (h/k)\, \sinh(n\pi b/a)) \int\limits_0^a \sin^2(n\pi x/a)\, dx}$$

$$= \frac{(h/k)\theta_\infty (a/n\pi)\, [1 - (-1)^n]}{((n\pi/a)\, \cosh(n\pi b/a) + (h/k)\, \sinh(n\pi b/a))\, [(a/2)]}$$

$$= \frac{\theta_\infty (2/n\pi)\, [1 - (-1)^n]}{(\sinh(n\pi b/a) + (n\pi/a)(h/k)\, \cosh(n\pi b/a))}$$

Therefore, we obtain

$$\frac{T - T_1}{T_\infty - T_1} = \sum_{n=1}^{\infty} \frac{(2/n\pi)\, [1 - (-1)^n]\, \sin(n\pi x/a)\, \sinh(n\pi y/a)}{\sinh(n\pi b/a) + (n\pi/a)(h/k)\, \cosh(n\pi b/a)} \tag{3.12}$$

3.3 Principle of Superposition for Nonhomogeneous BCs Superposition

In some applications, we may have all three kinds of surface BCs applied to a given problem. For example, Figure 3.4 shows a 2-D heat conduction problem with given surface temperature, heat flux, and convection BCs,
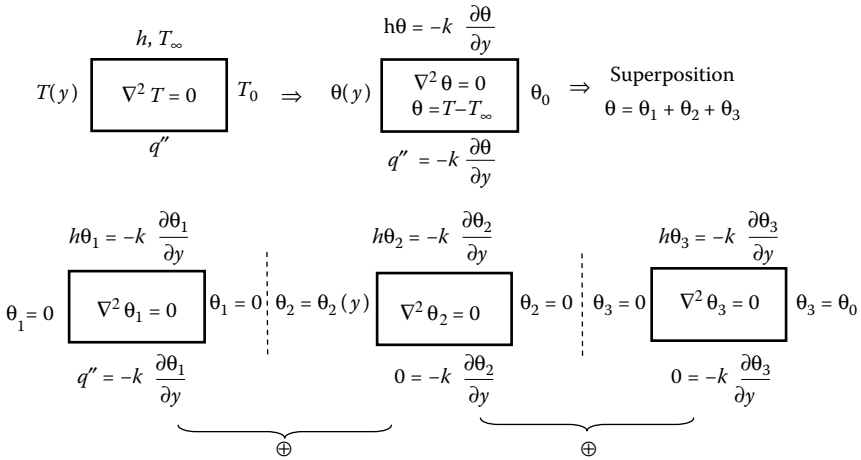
FIGURE 3.4
Principle of superposition for two-dimensional heat conduction with four nonhomogenous boundary conditions.

respectively. The problem becomes involving three, nonhomogeneous BCs after letting $\theta = T - T_\infty$. We have to use the superposition principal, splitting the problem with three nonhomogeneous BCs into three individual problems [1]. For each of the problems, there is only one nonhomogeneous BC. Then the method of separation of variable can be applied. It is important to note that both the heat conduction equations and the associated BCs must satisfy the superposition principal, respectively, that is, $\theta = \theta_1 + \theta_2 + \theta_3$ for both heat conduction equation and four BCs. From the aforementioned discussion, we know how to obtain the solution for θ_1 (two homogeneous x-BCs and one nonhomogeneous at $y = 0$), θ_2 (two homogeneous y-BCs and one nonhomogeneous at $x = 0$), and θ_3 (two homogeneous y-BCs and one nonhomogeneous at $x = b$). Applying superposition, the final temperature distributions is $\theta = \theta_1 + \theta_2 + \theta_3$.

3.3.1 2-D Heat Conduction in Cylindrical Coordinates

Figure 3.5 shows 2-D cylindrical coordinate systems. The following equations can be solved using the method of separation of variables as discussed above. The detailed solutions can be found from any advanced heat conduction textbook.

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} = 0 \quad \Rightarrow \quad T = R(r)\Theta(\phi) \tag{3.13}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2} = 0 \quad \Rightarrow \quad T = R(r)Z(z) \tag{3.14}$$
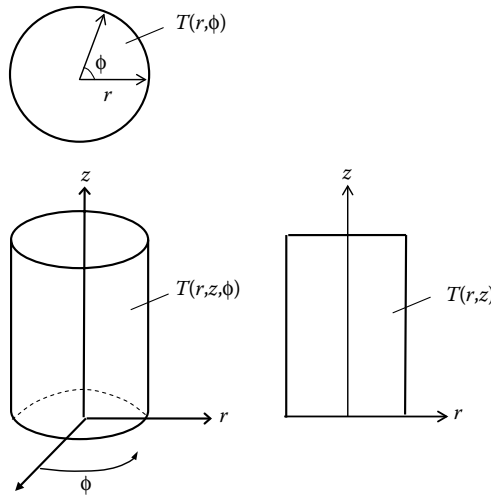
FIGURE 3.5
2-D heat conduction in cylindrical coordinates.

3.4 Principle of Superposition for Multidimensional Heat Conduction and for Nonhomogeneous Equations

3.4.1 3-D Heat Conduction Problem

Sometimes we need to solve the 3-D heat conduction problems in Cartesian (rectangular) coordinates as shown in Figure 3.6. Basically, we are first to convert the 3-D into the 2-D heat conduction problem and then solve the 2-D problem by using the separation of variable method discussed earlier. The following is a brief outline on how to solve this type of problem.

The steady-state 3-D heat conduction equation without heat generation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{3.15}$$
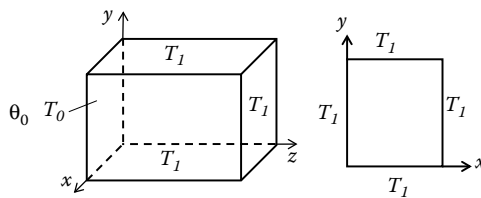
FIGURE 3.6
3-D heat conduction in Cartesian coordinates.

Let $\theta = T - T_1$.

The above governing equation and the associated BCs become

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = 0$$

$x = 0$, $\theta = 0$, or $\partial\theta/\partial x = 0$; $x = a$, $\theta = 0$, two homogeneous BCs,
$y = 0$, $\theta = 0$, or $\partial\theta/\partial y = 0$; $y = b$, $\theta = 0$, two homogeneous BCs,
$z = 0$, $\theta = \theta_0$; $z = c$, $\theta = 0$, one nonhomogeneous BC.

Let $\theta = X(x)Y(y)Z(z)$. Put its derivatives in the above 3-D heat conduction equation and obtain

$$-\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} + \frac{1}{Z}\frac{d^2 Z}{dz^2} = \lambda^2 \qquad (3.16)$$

This implies

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \qquad (3.17)$$

$$-\frac{1}{Y}\frac{d^2 Y}{dy^2} = \frac{1}{Z}\frac{d^2 Z}{dz^2} - \lambda^2 = \mu^2$$

The last equation can be further written as

$$\begin{cases} \dfrac{d^2 Y}{dy^2} + \mu^2 Y = 0 \\[2mm] \dfrac{d^2 Z}{dz^2} - (\lambda^2 + \mu^2)Z = 0 \end{cases} \qquad (3.18)$$

Therefore, we need to solve two eigenvalue equations. The x-direction solution will be sine and cosine, the y-direction solution is sine and cosine, and the z-direction solution is sinh and cosh. The only nonhomogeneous BC in the z-direction will be used to determine the final unknown integrated value C_n.

From the previous discussion, the solutions for X, Y, and Z follow:

$$\begin{cases} X \cong c_1 \cos \lambda x + c_2 \sin \lambda x \\ Y \cong c_3 \cos \mu y + c_4 \sin \mu y \\ Z \cong c_5 e^{-\sqrt{\lambda^2 + \mu^2}\,z} + c_6 e^{\sqrt{\lambda^2 + \mu^2}\,z} \end{cases}$$

3.4.2 Nonhomogeneous Heat Conduction Problem

The problem of steady-state 2-D heat conduction with uniform heat genera-
tion can be split into two problems shown below, $\theta(x, y) = \psi(x, y) + \phi(x)$. We
already know how to solve these two problems.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\dot{q}}{k} = 0 \tag{3.19}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\dot{q}}{k} = 0 \tag{3.20}$$

where

$$\psi = X(x)Y(y) = (c_1 \cos \lambda x + c_2 \sin \lambda x) \cdot (c_3 e^{-\lambda y} + c_4 e^{\lambda y})$$

Examples

3.1 A 2-D rectangular plate is subjected to the following thermal BCs:

$$x = 0, \quad 0 < y < b: \quad T = T_0$$
$$x = a, \quad 0 < y < b: \quad T = T_0$$
$$y = 0, \quad 0 < x < a: \quad T = T_0$$
$$y = b, \quad 0 < x < a: \quad T = cx$$

a. Derive an expression for the steady-state temperature distribution $T(x, y)$.
b. Sketch the isotherms and isofluxes.

SOLUTIONS

a.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \tag{3.21}$$

Let $\theta(x, y) = X(x) \cdot Y(y)$;

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{\partial^2 Y}{\partial y^2} = \lambda^2$$

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0 \tag{3.22}$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 Y = 0 \tag{3.23}$$

$$X = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

$$\theta = \left[C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \right] \left[C_3 e^{-\lambda y} + C_4 e^{\lambda y} \right] \qquad (3.24)$$

Boundary conditions:

i. $x = 0$: $\theta = 0$

ii. $x = a$: $\theta = 0$

iii. $y = 0$: $\theta = 0$

iv. $y = b$: $\theta = cx - T_0$

Applying BC (i) into Equation 3.24, $C_1 = 0$

Applying BC (ii) into Equation 3.24, $C_2 \sinh(\lambda a) = 0$, $\lambda_n = n\pi/a$.

Applying BC (iii) into Equation 3.24, $C_3 = -C_4$.

$$\theta(x, y) = C_2 \cdot C_4 \sin\left(\frac{n\pi x}{a}\right) \left(e^{\lambda_n y} - e^{-\lambda_n y}\right)$$

$$\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right) \qquad (3.25)$$

Applying BC (iv) and using orthogonal functions to evaluate C_n,

$$C_n = \frac{\int_0^a (cx - T_0) \sin(n\pi x/a)\, dx}{\sinh(n\pi b/a) \int\limits_0^a \sin^2(n\pi x/a)\, dx} \qquad (3.26)$$

$$\int\limits_0^a (cx - T_0) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= c\left[\left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{a}\right) - \frac{ax}{n\pi} \cos\left(\frac{n\pi x}{a}\right)\right]_0^a + \frac{T_0 \cdot a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)_0^a.$$

$$\int\limits_0^a (cx - T_0) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{ca^2}{n\pi}(-\cos(n\pi)) + \frac{T_0 \cdot a}{n\pi}(\cos(n\pi) - 1)$$

$$\int\limits_0^a (cx - T_0) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{ca^2}{n\pi}(-1)^{n+1} + \frac{T_0 \cdot a}{n\pi}\left[(-1)^n - 1\right]$$

$$\int\limits_0^a (cx - T_0) \sin\left(\frac{n\pi x}{a}\right) dx = \frac{a}{n\pi}(-1)^n(-ca + T_0) - \frac{T_0 \cdot a}{n\pi}$$
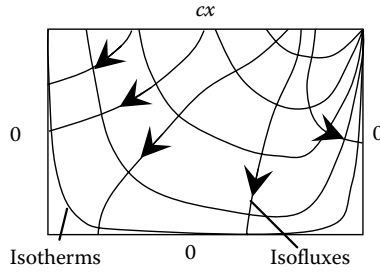
FIGURE 3.7
Sketch for the isotherms and isofluxes.

$$\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \left[\frac{x}{2} - \frac{1}{4n\pi} \sin\left(\frac{2n\pi x}{a}\right)\right]_0^a = \frac{a}{2}$$

$$\Rightarrow C_n = \frac{2(-1)^{n+1}(ca - T_0) - 2T_0}{n\pi \sinh\left((n\pi b/a)\right)}$$

Hence,

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(ca - T_0) - T_0}{n \sinh(n\pi b/a)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

b. See the sketch in Figure 3.7.

3.2. A long rectangular bar $0 \le x \le a, 0 \le y \le b$, shown in Figure 3.8, is heated at $x = 0$ with a uniform heat flux and is insulated at $x = a$ and $y = 0$. The side at $y = b$ loses heat by convection to a fluid at temperature T_∞.

a. Determine the temperature distribution $T(x, y)$.

b. Sketch the isotherm and isoflux.
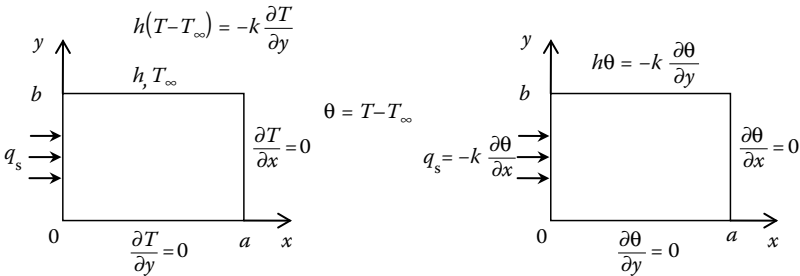
FIGURE 3.8
A long rectangular bar with heat flux as one nonhomogenous boundary condition.

SOLUTIONS

a.

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad (3.27)$$

Let $\theta(x, y) = X(x) \cdot Y(y)$:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{\partial^2 Y}{\partial y^2} = \lambda^2$$

$$\frac{\partial^2 X}{\partial x^2} - \lambda^2 X = 0 \qquad (3.28)$$

$$\frac{\partial^2 Y}{\partial y^2} + \lambda^2 Y = 0 \qquad (3.29)$$

$$X = C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x)$$

$$Y = C_3 \sin(\lambda y) + C_4 \cos(\lambda y)$$

$$\theta = \left[C_1 \sinh(\lambda x) + C_2 \cosh(\lambda x)\right]\left[C_3 \sin(\lambda y) + C_4 \cos(\lambda y)\right] \quad (3.30)$$

Boundary conditions:

i. $x = 0$: $q_s = -k(\partial \theta / \partial x)$

ii. $x = a$: $\partial \theta / \partial x = 0$

iii. $y = 0$: $\partial \theta / \partial y = 0$

iv. $y = b$: $h\theta = -k(\partial \theta / \partial y)$

Applying BC (iii) into Equation 3.30, $C_3 = 0$.

Applying BC (ii) into Equation 3.30, $C_1 \lambda \cosh(\lambda a) + C_2 \lambda \sinh(\lambda a) = 0$, $C_2 = -C_1 \coth(\lambda a)$.

$$\theta = C_n \left[\sinh(\lambda x) - \coth(\lambda a) \cosh(\lambda x)\right] \cos(\lambda y) \qquad (3.31)$$

Applying BC (iv) into Equation 3.31,

$$hC_n \left[\sinh(\lambda x) - \coth(\lambda a) \cosh(\lambda x)\right] \cos(\lambda b)$$

$$= +k\lambda C_n \left[\sinh(\lambda x) - \coth(\lambda a) \cosh(\lambda x)\right] \sin(\lambda b)$$

$$\lambda \tan(\lambda b) = \frac{h}{k}$$

$$\theta = \sum_{n=1}^{\infty} C_n \left[\sinh(\lambda_n x) - \coth(\lambda_n a) \cosh(\lambda_n x)\right] \cos(\lambda_n y) \quad (3.32)$$

where $\lambda_n \tan(\lambda_n b) = (h/k)$.

Applying BC (i) into Equation 3.26,

$$q_s = -k \sum_{n=1}^{\infty} C_n \left\{ \left[\lambda_n \cosh(\lambda_n x) - \coth(\lambda_n a) \, \lambda_n \sinh(\lambda_n x) \right] \cos(\lambda_n y) \right\}_{x=0}$$

$$q_s = -k \sum_{n=1}^{\infty} C_n \lambda_n \cos(\lambda_n y)$$

$$-\frac{q_s}{k\lambda_n} \int_0^b \cos(\lambda_n y) \, dy = \int_0^b C_n \cos^2(\lambda_n y) \, dy$$

$$-\frac{q_s}{k\lambda_n} \frac{1}{\lambda_n} \sin(\lambda_n y)\Big|_0^b = C_n \left[\frac{y}{2} + \frac{2 \sin(\lambda_n y) \cos(\lambda_n y)}{4\lambda_n} \right]_0^b$$

$$C_n = -\frac{2q_s \sin(\lambda_n b)}{k\lambda_n \left(\sin(\lambda_n b) \cos(\lambda_n b) + b\lambda_n \right)}$$

$$\theta = \sum_{n=1}^{\infty} \frac{2q_s \sin(\lambda_n b)}{k\lambda_n \left(\sin(\lambda_n b) \cos(\lambda_n b) + b\lambda_n \right)}$$

$$\times \left[-\sinh(\lambda_n x) + \coth(\lambda_n a) \cosh(\lambda_n x) \right] \cos(\lambda_n y)$$

3.3. A thin rectangular plate, $0 \le x \le a$, $0 \le y \le b$, as shown in Figure 3.9, with negligible heat loss from its sides, has the following BCs:

$x = 0, \ 0 < y < b$: $T = T_1$
$x = a, \ 0 < y < b$: $T = T_2$
$y = 0, \ 0 < x < a$: $q_s'' = 0$ (insulated)
$y = b, \ 0 < x < a$: $T = T_3$
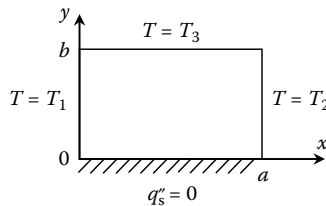
FIGURE 3.9
A thin rectangular plate with two one nonhomogenous boundary condition.

a. Determine the steady-state temperature distribution.

b. Sketch the isotherms and isofluxes.

SOLUTIONS

a.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Let $\theta = T - T_1$, then

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

Let $\theta = \theta_1 + \theta_2$,

$$\frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_1}{\partial y^2} = 0$$

$$\frac{\partial^2 \theta_2}{\partial x^2} + \frac{\partial^2 \theta_2}{\partial y^2} = 0$$

$$\theta_1 = \left[C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \right] \left[C_3 \cosh(\lambda y) + C_4 \sinh(\lambda y) \right]$$

$x = 0, \quad \theta_1 = 0, \quad \Rightarrow C_1 = 0$

$x = a, \quad \theta_1 = 0, \quad \Rightarrow \lambda a = n\pi; \quad \lambda_n = (n\pi/a) \quad n = 1, 2, 3, \ldots$

$y = 0, \quad \partial\theta_1/\partial y = 0; \quad \Rightarrow C_4 = 0$

$$\theta_1 = \sum_{n=1}^{\infty} C_n \cosh(\lambda_n y) \sin(\lambda_n x)$$

$$\theta_1(x, b) = (T_3 - T_1) = \sum_{n=1}^{\infty} C_{n1} \cosh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$C_{n1} = \frac{(T_3 - T_1) \int_0^a \sin\left((n\pi x/a)\right) dx}{\cosh\left((n\pi b/a)\right) \int_0^a \sin^2\left((n\pi x/a)\right) dx}$$

$$= \frac{(T_3 - T_1) a \cdot (1 - \cos(n\pi)/n\pi)}{(a/2) \cosh\left((n\pi b/a)\right)} = \frac{2(T_3 - T_1)}{n\pi} \frac{\left[1 - (-1)^n\right]}{\cosh\left((n\pi b/a)\right)}$$

$$\theta_1 = \sum_{n=1}^{\infty} \frac{2(T_3 - T_1)}{\pi n} \frac{\left[1 - (-1)^n\right]}{\cosh\left((n\pi b/a)\right)} \cdot \cosh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

Solution for θ_2

$$\theta_2 = \left[C_1 \cosh(\lambda x) + C_2 \sinh(\lambda x) \right] \left[C_3 \cos(\lambda y) + C_4 \sin(\lambda y) \right]$$

$$y = 0, \quad \partial\theta/\partial y = 0; \quad \Rightarrow C_4 = 0$$
$$y = b, \quad \theta_2 = 0; \quad \Rightarrow \lambda_n = n\pi/2b$$
$$x = 0, \quad \theta_2 = 0; \quad \Rightarrow C_1 = 0$$

$$\theta_2 = \sum_{n_{odd}}^{\infty} C_{n2} \sinh(\lambda_n x)\cos(\lambda_n y)$$

$$\theta_2(a, y) = (T_2 - T_1) = \sum_{n_{odd}}^{\infty} C_{n2} \sinh(\lambda_n a)\cos(\lambda_n y),$$

$$C_{n2} = \frac{(T_2 - T_1)\int_0^b \cos\left((n\pi y/2b)\right)dy}{\sinh\left((n\pi a/2b)\right)\int_0^b \cos^2(\lambda_n y)dy}$$

$$= \frac{4(T_2 - T_1)\sin\left((n\pi/2)\right)}{n\pi \sinh\left((n\pi a/2b)\right)}$$

where $\sin((n\pi/2)) = (-1)^{(n-1/2)}$ for n odd

$$\theta_2 = \sum_{n_{odd}}^{\infty} \frac{4(T_2 - T_1)(-1)^{(n-1)/2}}{n\pi \sinh\left((n\pi a/2b)\right)} \sinh\left(\frac{n\pi x}{2b}\right)\cos\left(\frac{n\pi y}{2b}\right)$$

Finally,

$$T(x, y) = T_1 + \left\{ \frac{2(T_3 - T_1)}{n\pi} \sum_{n=1}^{\infty} \frac{[1 - (-1)^n]}{\cosh(n\pi b/a)} \cosh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \right.$$

$$\left. + \frac{4(T_2 - T_1)}{n\pi} \sum_{n_{odd}}^{\infty} \frac{(-1)^{(n-1)/2}}{\sinh\left((n\pi a/2b)\right)} \sinh\left(\frac{n\pi x}{2b}\right)\cos\left(\frac{n\pi y}{2b}\right) \right\}$$

3.4. A long rectangular rod $0 \le x \le a, 0 \le y \le b$, as shown in Figure 3.10, has the following thermal BCs:

$$x = 0, \quad 0 < y < b : T = T_o$$

$$x = a, \quad 0 < y < b : T = T_0 + T_1 \sin(\pi y/b)$$

$$y = 0, \quad 0 < x < a : T = T_0$$

$$y = b, \quad 0 < x < a : T = T_0 + T_2 \sin(\pi x/a)$$

a. Determine the steady-state temperature distribution.
b. Sketch the isotherm and isoflux.
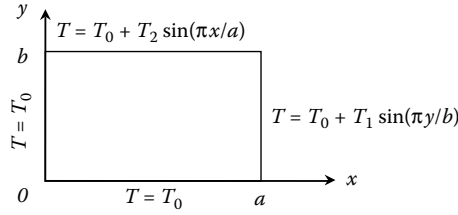
FIGURE 3.10
A long rectangular rod with two nonhomogenous boundary conditions.

SOLUTIONS

a. Let $\theta = T - T_0$, then

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$x = 0; \quad \theta = 0$$
$$x = a; \quad \theta = T_1 \sin \left((\pi y / b) \right)$$
$$y = 0; \quad \theta = 0,$$
$$y = b; \quad \theta = T_2 \sin \left(\frac{\pi x}{a} \right)$$

Let $\theta = \theta_1 + \theta_2$ with θ_1 and θ_2 satisfying the following BCs at

$$x = 0, \quad \theta_1 = 0, \quad \theta_2 = 0$$
$$x = a, \quad \theta_1 = 0, \quad \theta_2 = T_1 \sin \left(\frac{\pi y}{b} \right)$$
$$y = 0, \quad \theta_1 = 0, \quad \theta_2 = 0$$
$$y = b, \quad \theta_1 = T_2 \sin \left(\frac{\pi x}{a} \right) \quad \theta_2 = 0$$

Solutions for θ_1 and θ_2 are obtained as

$$\theta_1 = \sum C_{n1} \sin \left(\frac{n\pi x}{a} \right) \cdot \sinh \left(\frac{n\pi y}{a} \right)$$

$$\theta_2 = \sum C_{n2} \sin \left(\frac{n\pi y}{b} \right) \cdot \sinh \left(\frac{n\pi x}{b} \right)$$

and $T = T_0 + \theta$

$$T = T_0 + \theta_1 + \theta_2$$

Remarks

In this chapter, we have introduced a very powerful mathematical tool, the method of separation of variables, to solve typical 2-D heat conduction problems with various thermal BCs. In the undergraduate-level heat transfer, we normally employ the finite-difference energy balance method to solve the 2-D heat conduction problems with various thermal BCs. The finite-difference numerical methods and solutions will be discussed in Chapter 5. Here we are more focused on the analytical methods and solutions for various 2-D heat conduction problems. In general, all kinds of 2-D heat conduction problems with various BCs can be solved analytically by using superposition of separation of variables.

The most important thing for applying separation of variable is that you have to set up your problem where only one nonhomogeneous BC is allowed. If you have more than one nonhomogeneous BC, you have to employ the superposition principle to split into two or three subproblems in order to use separation of variables. The problems become more complicated if you work with 3-D heat conductions with heat generation and with complex BCs, but they are still workable. However, if you are interested in solving for the 2-D and 3-D cylindrical coordinate systems and for the 2-D and 3-D spherical coordinate systems with complex BCs, they are beyond the intermediate-level heat transfer, you need to look at the advanced heat conduction textbook for solutions.

Another popular method is using the finite-difference method to be discussed in the later chapter. It is particularly true when you deal with complicate BCs such as convection. In real-life engineering applications, the convection heat transfer coefficients normally are varied along the solid surface. This will cause additional complexity for the separation of variables because we normally assume the uniform convection BCs to simplify the problem. This will not cause any complexity at all by using the finite-difference numerical method.

PROBLEMS

3.1. A long rectangular bar $0 \leq x \leq a$, $0 \leq y \leq b$, and a, $b << L$, the bar length, is heated at $y = o$ and $y = b$, respectively, to a uniform temperature T_0 and is insulated at $x = 0$. The side of $x = a$ loses heat by convection to a fluid at temperature T_∞ with a convection coefficient h.
   a. Write down, step by step, a solution method and associated BCs, which can be used to determine the bar steady-state temperature distributions.
   b. Sketch the heat flows and the isothermal profiles in the rectangular bar.
3.2. An infinitely long rod of square cross section (LXL) floats in a fluid. The heat transfer coefficient between the rod and the fluid

is relatively large compared to that between the rod and the ambient air, that is, $h_f \gg h$ or $h_f \cong \infty$. Determine the steady-state temperature distributions in the rod with the associated BCs.

a. Use the analytical approach.

b. Sketch the isotherms and isoflux in the rod, if $T_f < T_\infty$ and $h = a$ constant value.

3.3. A long fin of rectangular cross section ($2LXL$) with a thermal conductivity k is subjected to the BCs is shown on the sketch (the left side is kept at T_0, the right side is perfectly insulated, the upper side is exposed to a constant flux, and the lower side is exposed to a convection air flow). $q'' = $ constant; steam $T_0, h = \infty$; air $h = $ constant, T_∞.

a. Determine the temperature distribution in the fin.

b. Approximately plot the temperature and heat flow profiles in the fin, if $T_0 > T_\infty$.

3.4. Refer to Figure 3.1, and determine the temperature distributions for 2-D heat conduction with the following BCs:

(1)	$x = 0, T = T_0$		(2)	$x = 0, T = T_0$
	$x = a, T = T_0$			$x = a, T = T_0$
	$y = 0, T = T_0$			$y = 0, T = T_s$
	$y = b, T = T_s$			$y = b, T = T_0$
(3)	$x = 0, T = T_0$		(4)	$x = 0, T = T_s$
	$x = a, T = T_s$			$x = a, T = T_0$
	$y = 0, T = T_0$			$y = 0, T = T_0$
	$y = b, T = T_0$			$y = b, T = T_0$

3.5. Refer to Figure 3.2, and determine the temperature distributions for 2-D heat conduction with the following BCs:

(1)	$x = 0, T = T_1$		(2)	$x = 0, T = T_1$
	$x = a, T = T_1$			$x = a, T = T_1$
	$y = 0, T = T_1$			$y = 0, q_s'' = -k(\partial T/\partial y)$
	$y = b, q_s'' = -k(\partial T/\partial y)$			$y = b, T = T_1$
(3)	$x = 0, T = T_1$		(4)	$x = 0, q_s'' = -k(\partial T/\partial x)$
	$x = a, q_s'' = -k(\partial T/\partial x)$			$x = a, T = T_1$
	$y = 0, T = T_1$			$y = 0, T = T_1$
	$y = b, T = T_1$			$y = b, T = T_1$

3.6. Refer to Figure 3.3, and determine the temperature distributions for 2-D heat conduction with the following BCs:

(1) $x = 0, T = T_1$		(2)	$x = 0, T = T_1$
$x = a, T = T_1$			$x = a, T = T_1$
$y = 0, T = T_1$			$y = 0, -k(\partial T/\partial y)$
			$= h(T - T_\infty)$
$y = b, -k(\partial T/\partial y) = h(T - T_\infty)$			$y = b, T = T_1$

(3) $x = 0, T = T_1$ (4) $x = 0, -k(\partial T/\partial x) = h(T - T_\infty)$
     $x = a, -k(\partial T/\partial x)$ $x = a, T = T_1$
                                        $= h(T - T_\infty)$
     $y = 0, T = T_1$ $y = 0, T = T_1$
     $y = b, T = T_1$ $y = b, T = T_1$

3.7. Use the principle of superposition to determine the temperature distribution for 2-D heat conduction with four nonhomogeneous BCs shown in Figure 3.4.

3.8. Refer to Figure 3.6, and determine the temperature distributions for 3-D heat conduction with the following BCs:

(1) $x = 0, T = T_1$ (2) $x = 0, T = T_1$
     $x = a, T = T_1$ $x = a, T = T_1$
     $y = 0, T = T_1$ $y = 0, T = T_1$
     $y = b, T = T_1$ $y = b, T = T_1$
     $z = 0, T = T_o$ $z = 0, T = T_1$
     $z = c, T = T_1$ $z = c, T = T_o$

3.9. Refer to Equation 3.19, and determine the temperature distributions for 2-D heat conduction with uniform heat generation with the following BCs:

(1) $x = 0, -k(\partial T/\partial x) = 0$ (2) $x = 0, T = T_o$
     $x = a, -k(\partial T/\partial x) = h(T - T_\infty)$ $x = a, T = T_o$
     $y = 0, T = T_o$ $y = 0, -k(\partial T/\partial y) = 0$
     $y = b, T = T_o$ $y = b, -k(\partial T/\partial y)$
                                          $= h(T - T_\infty)$

3.10. Obtain an expression for the steady-state temperature distribution $T(x, y)$ in a long square bar of side a. The bar has its two sides and bottom maintained at temperature T_1, while the top side is loosing heat by convection. Consider the surrounding temperature to be T_∞, the heat transfer coefficient at the top wall be denoted by h, and let the thermal conductivity of the bar be equal to k.
   a. Sketch the domain and write the governing equation and BCs for this problem.
   b. Define the temperature $\theta(x, y) = T(x, y) - T_1$, and find a series solution using separation of variables.
   c. Using the BCs write the expression for the coefficients used in the series solution for $\theta(x, y)$, and write an expression.
   d. Write an expression for $(T(x, y) - T_1)/(T_\infty - T_1)$.

3.11. Given a very long and wide fin with a height of $2L$. The base of the fin is maintained at a uniform temperature of T_b. The top and bottom surfaces of the fin are exposed to a fluid whose temperature is T_∞ ($T_\infty < T_b$). The convective heat transfer coefficient between the fin surfaces and the fluid is h.

    a. Sketch the steady 2-D temperature distribution in the fin.

    b. If you were to determine the steady 2-D temperature distribution in the fin using a finite-difference numerical method, you would solve a set of algebraic nodal equations simultaneously for the temperatures at a 2-D array of nodes. Derive the equation for a typical node on one of the surfaces of the fin. Please do not simplify the equation.

    c. Using the method of separation of variables, *derive* an expression for the steady local temperature in the fin, in terms of the thermal conductivity of the fin, k, the convective heat transfer coefficient, h, the half-height of the fin, L, and the base and fluid temperatures, T_b and T_∞.

Note that

$$\int_0^W [\cos^2 (aw)] \, dw = \frac{1}{4}a[2aW + \sin (2aW)]$$

$$\text{and} \quad \int_0^W [\cos(aw) \cdot \cos(bw)] \, dw = 0, \quad \text{when } a \neq b.$$

3.12. A long rectangular rubber pad of width $a = W$ and height $b = 2W$ is a component of a spacecraft structure. Its sides and bottom are bonded to a metal channel at constant temperature T_0, and the temperature distribution along the top of the pad can be approximated as a simple sine curve $T = T_0 + T_m \sin(\pi x/W)$.

    a. Write the differential equation and BCs needed to solve for the temperature distribution in the pad.

    b. Find the solution for the temperature distribution from the differential equation and BCs.

3.13. A long rod of right triangular cross section has the horizontal length "a" at temperature T_1, the vertical length "b" at temperature T_2, and the inclined length perfectly insulated. Obtain an expression for the steady-state temperature distribution $T(x, y)$ in the long rod of triangular cross section as stated. Assume that the thermal conductivity of the material of the rod is constant.

3.14. A long rectangular bar $0 \leq x \leq a$, $0 \leq y \leq b$, and a, $b << L$, the bar length, is heated at $y = 0$ and $y = b$, respectively, to a uniform temperature T_0 and is insulated at $x = 0$. The side of $x = a$ loses heat by convection to a fluid at temperature T_∞ with a convection coefficient of h_∞.

    a. Write down, step by step, a solution method and the associated BCs, which can be used to determine the bar steady-state temperature distributions. You do not need to obtain the final solution of the steady-state temperature distributions.

    b. Sketch the heat flows and the isothermal profiles in the rectangular bar.

3.15. Given a very long and wide fin with a height of $2H$. The base of the fin is maintained at a uniform temperature of T_b. The top and bottom surfaces of the fin are exposed to a fluid whose temperature is T_∞ ($T_\infty < T_b$). The convective heat transfer coefficient between the fin surfaces and the fluid is h.

   a. Derive an expression for the steady 2-D local temperature in the fin, in terms of the thermal conductivity of the fin, k, the convective heat transfer coefficient, h, the half-height of the fin, H, and the base and fluid temperature, T_b and T_∞.

   b. Sketch the steady 2-D temperature and heat flux distribution in the fin.

   Note that

$$\int_0^x \left[\cos^2(ax)\right] dx = \frac{1}{4}a\left[2ax + \sin(2ax)\right] \quad \text{and}$$

$$\int_0^x \left[\cos(ax) \cdot \cos(bx)\right] dx = 0 \quad \text{when } a \neq b.$$

References

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2. A. Mills, *Heat Transfer*, Richard D. Irwin, Inc., Boston, MA, 1992.
3. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.
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4

Transient Heat Conduction

The temperature in a solid material changes with location as well as with time, and this is the so-called transient heat conduction problem. We may have 1-D, 2-D, or 3-D transient heat conduction problem depending on the real applications. However, some problems can be modeled as zero-dimensional (0-D) because the temperature in a solid material uniformly changes only with time and does not depend on location. This is a special case of the transient heat conduction problem. The finite-length solid material of 1-D, 2-D, or 3-D transient problem can be solved by separation of variable method and the 0-D transient problem can be solved by the lumped capacitance method. Figure 4.1 shows typical finite-length solid materials of the 1-D transient problem for the slab (or the plane wall), cylinder, and sphere coordinates. The semiinfinite solid material of the 1-D transient problem can be solved by the similarity method, the Laplace transform method, or the approximate integral method.

The unsteady 3-D heat conduction equation with heat generation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.1}$$

The simplified case of the unsteady 1-D heat conduction equation without heat generation becomes

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{4.2}$$

We need both the initial and two BCs in order to solve the temperature depending on (x, t) as sketched in Figure 4.2. The solution of separation of variable method is

$$T = T(x, t)$$

Initial condition:

$$t = 0, \quad T(x, 0) = T_i.$$

(a)   (b)   (c)

$$\frac{\theta}{\theta_0} = fn\left(B_i, F_o, \frac{x}{L}\right); \qquad \frac{\theta}{\theta_0} = fn\left(B_i, F_o, \frac{r}{r_o}\right); \qquad \frac{\theta}{\theta_0} = fn\left(B_i, F_o, \frac{r}{r_o}\right)$$

Thick slab   Cylinder   Sphere
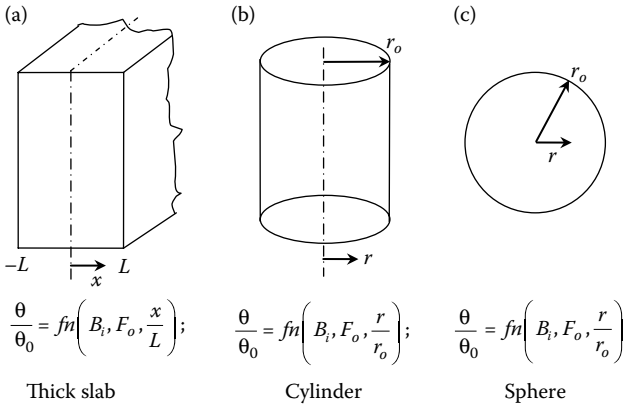
FIGURE 4.1
1-D transient heat conduction and the characteristic length.

Boundary conditions:

$$x = 0, \quad \frac{dT}{dx} = 0$$

$$x = L, \quad T(L, t) = T_s \quad \text{or} \quad -k\frac{\partial T(L, t)}{\partial x} = h[T(L, t) - T_\infty]$$

4.1 Method of Lumped Capacitance for 0-D Problems

In real applications, many transient heat conductions in a solid material can be modeled as a 0-D problem. The important assumption is that the entire material temperature changes only with time. If that is the case, those solid geometries shown in Figure 4.1 can be solved by the following lumped
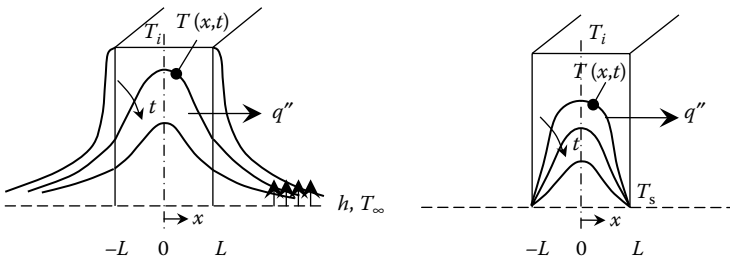
FIGURE 4.2
1-D transient problems for a slab or a plane wall.
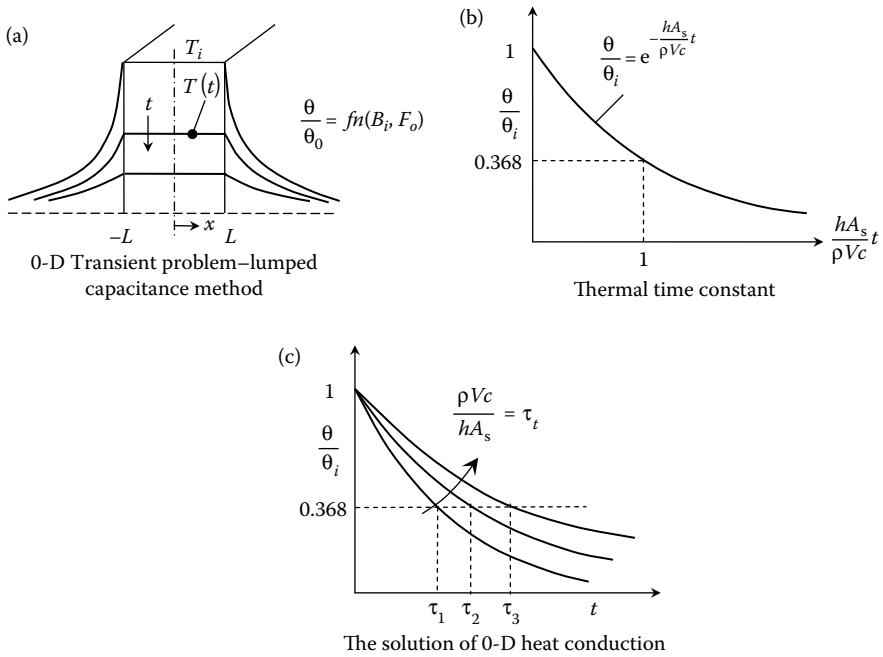
FIGURE 4.3
Method of lumped capacitance.

capacitance method. Note that the lumped method can be applied to any irregular geometry as long as the assumption of the entire material temperature uniformly changing with time is valid during the transient. Therefore, we do not need to solve for 1-D, 2-D, or 3-D transient conduction equations.

Consider the energy balance on the solid material during the cooling (or heating) process as shown in Figure 4.3:

$$\frac{d(\rho VCT)}{dt} = -hA_s(T - T_\infty) \tag{4.3}$$

Let $\theta = T - T_\infty$ then

$$\rho VC\frac{d\theta}{dt} = -hA_s\theta$$

$$\frac{d\theta}{dt} = -\frac{hA_s}{\rho VC}\theta$$

$$\int d\theta = \int -\frac{hA_s}{\rho VC}\theta\, dt$$

$$\frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = e^{-(hA_s/\rho VC)t} = e^{-(hL_c/k)\cdot(\alpha t/L_c^2)} = e^{-Bi\cdot Fo} = f(Bi, Fo) \tag{4.4}$$

where $Bi = (hL_c/k)$, $Fo = (\alpha t/L_c^2)$, $L_c = (V/A_s) = $ (volume/surface area).

The above temperature decay solution is plotted in Figure 4.3. Therefore, the solid temperature can be predicted with time for a given material with certain geometry under the cooling or heating condition. The material with a smaller thermal time constant ($\tau_t = (\rho Vc/hA_s)$) can quickly reach the environment temperature.

Now, the question is under what condition the lumped capacitance solution can be used. The answer is that the Biot (Bi) number must be less than 0.1. Therefore, to use 0-D solution, the condition $Bi = (hL_c/k) < 0.1$ must be satisfied. The Biot number is defined as the ratio of surface convection (h, the convection heat transfer coefficient from the solid surface; L_c, the characteristic length of the solid material) to solid conduction (k, the thermal conductivity of the solid material). The smaller Bi number implies a small-sized material with high-conductivity exposure to a low convection cooling or heating fluid. For the case of smaller Bi, the temperature inside the solid material changes uniformly (independent of location) with environmental cooling or heating during the transient. Of course, $Bi = 0.1$ implies that we may have 10% error by using the lumped solution. The smaller Bi is better for using the lumped solution. Another point is that the solution can be applied to any geometry if the condition of $Bi < 0.1$ is valid. For a given solid geometry, the characteristic length $L_c = $ (volume/surface area) $= (V/A_s)$. For example, as shown in Figure 4.1, the characteristic length $L_c = L$ is for a 2L-thick slab (plane wall),

$$L_c = \frac{1}{2}R_o \text{ for the cylinder} \quad \text{and} \quad L_c = \frac{1}{3}R_o \text{ for the sphere coordinate.}$$

4.1.1 Radiation Effect

If we also consider radiation flux q_r'' and internal heat generation $\dot{q}$, the energy balance equation 4.3 can be rewritten as

$$\frac{d(\rho VCT)}{dt} = -hA_s(T - T_\infty) + \dot{q}V + A_s q_r''$$

where $q_r'' = $ radiation gain from solar flux $=$ constant, or $q_r'' = $ radiation loss $= \varepsilon\sigma(T^4 - T_{sur}^4)$; $\dot{q} = $ heat generation $= I^2R$ due to electric current and resistance heating $=$ constant.

If we consider $q_r'' = $ constant and $\dot{q} = $ constant, the solution of the above equation can be obtained by Equation 4.4 by setting

$$\theta = \left[(T - T_\infty) - \left(\frac{\dot{q}V + A_s q_r''}{hA_s} \right) \right]$$

However, if we consider $\dot{q} = 0$, but $q_r'' = -\varepsilon\sigma(T^4 - T_{sur}^4)$, the energy balance equation can be rewritten as

$$\frac{d(\rho VCT)}{dt} = -hA_s(T - T_\infty) - \varepsilon\sigma A_s(T^4 - T_{sur}^4)$$

If let $T_\infty = T_{sur}$, $h_r = \varepsilon\sigma(T^2 + T_\infty^2)(T + T_\infty)$, the above equation can be written as

$$\frac{d(T - T_\infty)}{dt} + \frac{(h + h_r)A_s}{\rho VC}(T - T_\infty) = 0$$

The solution of the above equation can be obtained by numerical integration.

4.2 Method of Separation of Variables for 1-D and for Multidimensional Transient Conduction Problems

4.2.1 1-D Transient Heat Conduction in a Slab

The solution of the 1-D transient conduction problem for a slab (plane wall) is expected as $T(x, t)$. The separation of variable method used for the 2-D steady-state heat conduction problem can be applied here if we consider $T(x, t)$ similar to $T(x, y)$. In other words, we separate the temperature $T(x, y)$ into the product of $T(x) \cdot T(y)$ for the 2-D steady state and $T(x, t)$ into $T(x) \cdot T(t)$ for the 1-D transient, respectively. Then we can follow the similar procedure as before in order to solve the 1-D transient problem [1].

For a 1-D plane wall transient problem, as shown in Figure 4.4a, with the convection BC,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Let $\theta = T - T_\infty$ and $\theta(x, t) = X(x)\tau(t)$, then,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{4.5}$$

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0$$

$$\frac{d\tau}{dt} + \lambda^2\alpha\tau = 0$$

$$X = c_1 \sin \lambda x + c_2 \cos \lambda x$$

$$\tau = c_3 e^{-\lambda^2\alpha t}$$

(a)

(b)

Convective boundary condition

Constant surface
temperature boundary condition
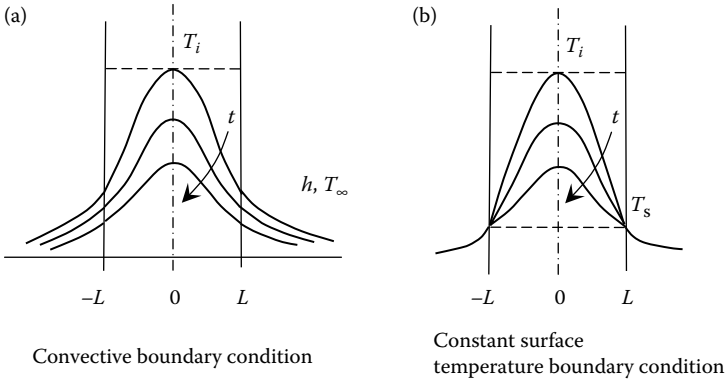
FIGURE 4.4
1-D transient heat conduction.

Initial condition:

$$\theta(x,0) = \theta_i = T_i - T_\infty$$

Boundary conditions:

$$\begin{cases} \dfrac{\partial \theta(0,t)}{\partial x} = 0 \\ -k\dfrac{\partial \theta(L,t)}{\partial x} = h\theta(L,t) \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ -kc_2(-\sin \lambda L)\lambda = hc_2 \cos \lambda L \end{cases}$$

$$k\sin(\lambda L) \cdot \lambda = h\cos(\lambda L)$$

$$\lambda_n = \frac{h}{k}\cot(\lambda_n L),$$

$$\lambda_n L = \frac{hL}{k}\cot(\lambda_n L) = Bi\cot(\lambda_n L).$$

λ_n is determined by the convection BC.

$$\theta = c_2 \cos \lambda_n x \cdot c_3 e^{-\lambda_n^2 \alpha t} = c_n \cos(\lambda_n x)e^{-\lambda_n^2 \alpha t},$$

where $c_n = c_2 c_3$.
   Applying the initial condition

$$\theta_i = c_n \cos(\lambda_n x)e^0,$$

$$\theta_i \cos(\lambda_n x) = c_n \cos^2(\lambda_n x),$$

$$c_n = \frac{\theta_i \int_0^L \cos(\lambda_n x)\,dx}{\int_0^L \cos^2(\lambda_n x)\,dx} = \frac{\theta_i 2\sin \lambda_n}{\lambda_n + \sin \lambda_n \cos \lambda_n},$$

where $\lambda_n L = \lambda_n$

$$\Rightarrow \theta = T - T_\infty = \sum c_n \cos(\lambda_n x) \, e^{-\lambda_n^2 \alpha t},$$

$$\Rightarrow \frac{\theta}{\theta_i} = \frac{T - T_\infty}{T_i - T_\infty} = \sum \frac{2 \sin \lambda_n \cos \lambda_n (x/L)}{\lambda_n + \sin \lambda_n \cos \lambda_n} \cdot e^{-\lambda_n^2 Fo}. \tag{4.6}$$

In addition to $T(x, t)$, the ratio of the total energy transferred from the wall over the time t is

$$\frac{Q}{Q_o} = \frac{\int \rho C_p [T_i - T(x, t)] \, dV}{\rho C_p V (T_i - T_\infty)} \tag{4.7}$$

$$= \frac{1}{V} \int \left(1 - \frac{\theta}{\theta_i} \right) dV$$

$$= 1 - \sum \frac{2 \sin \lambda_n}{\lambda_n + \sin \lambda_n \cos \lambda_n} \cdot \frac{\sin \lambda_n}{\lambda_n} \cdot e^{-\lambda_n^2 Fo} \tag{4.8}$$

where $Fo = (\alpha t / L^2)$.

Similarly, for the case of a given surface temperature as BC shown in Figure 4.4b,

$$\text{at } x = 0, \quad \frac{\partial T(0, t)}{\partial x} = 0$$

$$\text{at } x = L, \quad T(L, t) = T_s$$

The solution can then be obtained as

$$\frac{\theta}{\theta_i} = \frac{T - T_s}{T_i - T_s} = \sum \frac{2(-1)^n}{\left(n + \frac{1}{2} \right) \pi} \cos \left(n + \frac{1}{2} \right) \pi \frac{x}{L} e^{-\left(n + \frac{1}{2} \right)^2 \pi^2 Fo} \tag{4.9}$$

4.2.2 Multidimensional Transient Heat Conduction in a Slab (2-D or 3-D)

The governing equation for multidimensional heat conduction, as shown in Figure 4.5, is

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t} \tag{4.10}$$

$$\theta(x, y, z, t) = \theta_x(x, t) \cdot \theta_y(y, t) \cdot \theta_z(z, t) \tag{4.11}$$

$$\begin{cases} \dfrac{\partial^2 \theta_x}{\partial x^2} = \dfrac{1}{\alpha} \dfrac{\partial \theta_x}{\partial t} \\[2mm] \dfrac{\partial^2 \theta_y}{\partial y^2} = \dfrac{1}{\alpha} \dfrac{\partial \theta_y}{\partial t} \\[2mm] \dfrac{\partial^2 \theta_z}{\partial z^2} = \dfrac{1}{\alpha} \dfrac{\partial \theta_z}{\partial t} \end{cases}$$
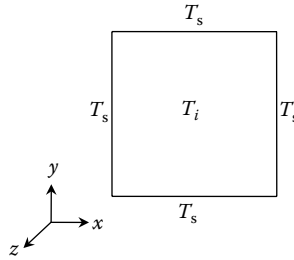
FIGURE 4.5
Multidimensional transient heat conduction.

where $\theta_x(x,t)$, $\theta_y(x,t)$, and $\theta_z(x,t)$ can be solved by the aforementioned method of separation of variables with a given surface temperature or convection BCs.

4.2.3 1-D Transient Heat Conduction in a Rectangle with Heat Generation

The governing equation for 1-D transient heat conduction with heat generation is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{4.12}$$

The temperature profile can be solved by separation of variables. For example, to use the separation of variables method, we define

$$\theta = \theta_1(x) + \theta_2(x,t) \tag{4.13}$$

$$\frac{d^2\theta_1}{dx^2} + \frac{\dot{q}}{k} = 0$$

where θ_1 can be solved by 1-D heat conduction with internal heat generation and θ_2 can be solved by the aforementioned method of separation of variables with a given convection or surface temperature BCs.

For 3-D transient heat conduction with heat generation, as shown in Figure 4.6, the following equations can be used to solve temperature profiles $T(x,y,z,t)$ or $\theta(x,y,z,t)$:

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{4.14}$$

$$\theta = \theta_1(x) + \theta_1(y) + \theta_1(z) + \theta_{2x}(x,t) \cdot \theta_{2y}(y,t) \cdot \theta_{2z}(z,t) \tag{4.15}$$
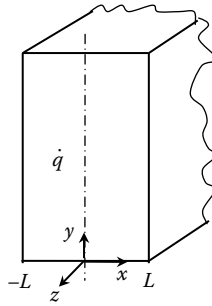
FIGURE 4.6
Multidimensional transient heat conduction with heat generation.

where

$$\frac{\partial^2 \theta_1}{\partial x^2} + \frac{\dot{q}}{k} = 0$$

$$\frac{\partial^2 \theta_{2x}}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta_{2x}}{\partial t}$$

$$\frac{\partial^2 \theta_{2y}}{\partial y^2} = \frac{1}{\alpha} \frac{\partial \theta_{2y}}{\partial t}$$

$$\frac{\partial^2 \theta_{2z}}{\partial z^2} = \frac{1}{\alpha} \frac{\partial \theta_{2z}}{\partial t}$$

A general form of the solution of 1-D transient heat conduction with the convection BC applicable to a slab, a cylinder, and a sphere, as shown in Figure 4.1, can be written [2] as

$$\frac{\theta}{\theta_i} = \sum_{n=1}^{\infty} C_n f(\lambda_n \eta)\, e^{-\lambda_n^2 Fo} \tag{4.16}$$

$$\frac{Q}{Q_o} = 1 - \sum_{n=1}^{\infty} C_n F(\lambda_n)\, e^{-\lambda_n^2 Fo} \tag{4.17}$$

Geometry	$\theta(x, t)$	C_n	$f(\lambda_n \eta)$	$F(\lambda_n)$
Slab	$(\partial^2\theta/\partial x^2)$ $= (1/\alpha)(\partial\theta/\partial t)$	$(2\sin\lambda_n/(\lambda_n$ $+ \sin\lambda_n \cos\lambda_n))$	$\cos(\lambda_n(x/L))$	$\sin\lambda_n/\lambda_n$
Cylinder	$(1/r)(\partial/\partial r)(r(\partial\theta/\partial r))$ $= (1/r)(\partial\theta/\partial t)$	$(2J_1(\lambda_n)/(\lambda_n[J_0^2(\lambda_n)$ $+J_1^2(\lambda_n)]))$	$J_0(\lambda_n(r/r_o))$	$(2J_1(\lambda_n)/\lambda_n)$
Sphere	$(1/r^2)(\partial/\partial r)(r^2(\partial\theta/\partial r))$ $= (1/r)(\partial\theta/\partial t)$	$(2[\sin\lambda_n - \lambda_n\cos\lambda_n])/$ $(\lambda_n - \sin\lambda_n\cos\lambda_n)$	$(\sin(\lambda_n(r/r_o))/$ $(\lambda_n(r/r_o))$	$3(\sin\lambda_n$ $-\lambda_n\cos\lambda_n)/\lambda_n^3$

With the eigenvalues as

$$Bi \cos \lambda_n - \lambda_n \sin \lambda_n = 0 \quad \text{for the slab, } \eta = \frac{x}{L}, \quad Bi = \frac{hL}{k}, \quad Fo = \frac{\alpha t}{L^2}$$

$$\lambda_n J_1(\lambda_n) - Bi J_0(\lambda_n) = 0 \quad \text{for the cylinder, } \eta = \frac{r}{r_o}, \quad Bi = \frac{hr_o}{k}, \quad Fo = \frac{\alpha t}{r_o^2}$$

$$\lambda_n \cos \lambda_n + (Bi - 1) \sin \lambda_n = 0 \quad \text{for the sphere, } \eta = \frac{r}{r_o}, \quad Bi = \frac{hr_o}{k}, \quad Fo = \frac{\alpha t}{r_o^2}$$

4.3   1-D Transient Heat Conduction in a Semiinfinite Solid Material

4.3.1   Similarity Method for Semiinfinite Solid Material

The semiinfinite solid is characterized by a single identifiable surface. If a sudden change in temperature occurs at this surface as shown in Figure 4.7, then transient 1-D conduction will take place within the solid. The similarity method can be employed to solve this kind of problem [3]. The 1-D transient heat conduction equation in a semiinfinite solid without heat generation is
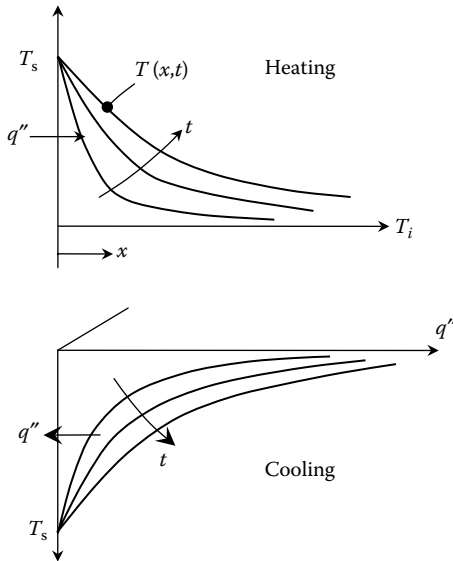
FIGURE 4.7
1-D transient heat conduction in a semiinfinite solid material.

given by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

The surface BC is

$$T(0,t) = T_s$$

And the interior BC is prescribed by

$$T(\infty, t) = T_i$$

The initial condition is

$$T(x,0) = T_i$$

By applying the similarity method, we may transform the PDE, which involves two independent variables (x and t), to an ODE expressed in terms of a single similarity variable (η).

As $x \sim \sqrt{\alpha t}$ ($\sqrt{\alpha t}$ is the diffusion length), we define the similarity variable $\eta = (x/\sqrt{4\alpha t})$; therefore $T(x,t) = T(\eta)$,

$$x = \eta\sqrt{4\alpha t} \tag{4.18}$$

Let $\theta = ((T - T_i)/(T_s - T_i))$.
The 1-D transient heat equation can be expressed as

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

Applying the similarity variable into θ, that is, $\theta(x,t) = \theta(\eta)$
Therefore,

$$\frac{\partial \theta}{\partial t} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial t} = \frac{d\theta}{d\eta}\left[\frac{-x}{2t\sqrt{4\alpha t}}\right]$$

$$\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\eta}\frac{\partial \eta}{\partial x} = \frac{d\theta}{d\eta}\left(\frac{1}{\sqrt{4\alpha t}}\right)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial \theta}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{d}{d\eta}\left[\frac{\partial \theta}{\partial \eta}\left(\frac{1}{\sqrt{4\alpha t}}\right)\right]\left(\frac{1}{\sqrt{4\alpha t}}\right) = \frac{1}{4\alpha t}\frac{\partial^2 \theta}{\partial \eta^2}$$

Inserting the above terms into the heat equation, we obtain

$$\frac{1}{4\alpha t}\frac{d^2\theta}{d\eta^2} = \frac{1}{\alpha}\frac{-x}{2t(4\alpha t)^{1/2}}\frac{d\theta}{d\eta}$$

Rearranging it, we obtain

$$\frac{\alpha}{4\alpha t}\frac{d^2\theta}{d\eta^2} = \frac{-x}{2t(4\alpha t)^{1/2}}\frac{d\theta}{d\eta}$$

$$\frac{d^2\theta}{d\eta^2} = \frac{-2x}{\sqrt{4\alpha t}}\frac{d\theta}{d\eta} = -2\eta\frac{d\theta}{d\eta} \qquad (4.19)$$

For the case of given constant surface temperature,

$$x = 0, \quad \eta = 0, \quad \theta(0) = 1$$
$$x = \infty, \quad \eta = \infty, \quad \theta(\infty) = 0$$

Let $P \equiv (d\theta/d\eta)$; rearranging the equation and integrating it, we obtain

$$\frac{dP}{d\eta} = -2\eta P \quad \Rightarrow \quad P \equiv \frac{d\theta}{d\eta} = c_1 e^{-\eta^2}$$

$$\begin{cases} \dfrac{dP}{P} = -2\eta\,d\eta \\[2mm] \displaystyle\int \dfrac{dP}{P} = \int -2\eta\,d\eta \\[2mm] \ln P = -\eta^2 + c \end{cases}$$

$$\theta = c_1 \int_0^\eta e^{-\eta^2}d\eta + c_2$$

From the BC,

$$\eta = 0, \quad \theta = 1 \quad \Rightarrow \quad c_2 = 1$$

From the BC,

$$\eta = \infty, \quad \theta = 0$$

$$0 = c_1 \int_0^\infty e^{-u^2}du + 1 = c_1\frac{\sqrt{\pi}}{2} + 1 \quad \Rightarrow \quad c_1 = -\frac{2}{\sqrt{\pi}}$$

Inserting c_1 and c_2 into the integral, we obtain the following results as sketched in Figure 4.8.

$$\theta = \frac{T - T_i}{T_s - T_i} = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2}du = 1 - erf(\eta) = erfc\left(\frac{x}{\sqrt{4\alpha t}}\right) = 1 - erf\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$$(4.20)$$
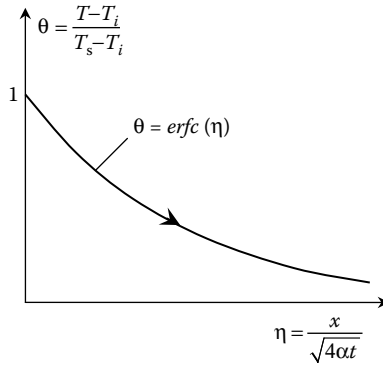
FIGURE 4.8
Solution of 1-D transient heat.

and

$$q_s = -k \left.\frac{\partial T}{\partial x}\right|_{x=0} = -k(T_s - T_i) \left.\frac{\partial \theta}{\partial x}\right|_{x=0}$$

$$= -k(T_s - T_i)\frac{\partial}{\partial x}\left[1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-u^2} du\right]_{x=0}$$

$$= -k(T_s - T_i)\left[-\frac{2}{\sqrt{\pi}}e^{-\eta^2}\frac{1}{\sqrt{4\alpha t}}\right]_{\eta=0}$$

$$= \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}$$

(4.21)

4.3.2 Laplace Transform Method for Semiinfinite Solid Material

The 1-D transient conduction problem in a semiinfinite solid material can be solved by the Laplace transform method [1]. The heat diffusion equation for 1-D transient conduction is of the form

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Case 1: Constant surface temperature BC

$$T(0, t) = T_s$$

$$T(\infty, t) = T_i$$

Initial condition:

$$T(x, 0) = T_i.$$

Let $\theta = T - T_i$, therefore,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha} \frac{\partial \theta}{\partial t}$$

$$\theta(0, t) = \theta_s = T_s - T_i$$

$$\theta(x, 0) = 0$$

$$\theta(\infty, t) = 0$$

Defining the Laplace transform of a given temperature,

$$L(T) = \overline{T} = \int_0^\infty T e^{-st} dt \tag{4.22}$$

$$L(T_i) = \frac{T_i}{s}$$

$$L(0) = 0$$

$$L\left(\frac{\partial T}{\partial t}\right) = s\overline{T} - \overline{T}_i$$

$$L\left(\frac{\partial^n T}{\partial x^n}\right) = \frac{\partial^n \overline{T}}{\partial x^n}$$

$$T(x, t) = \overline{T}(x, s)$$

Applying the Laplace transform to 1-D transient heat conduction,

$$L\left(\frac{\partial^2 T}{\partial x^2}\right) = L\left(\frac{1}{\alpha} \frac{\partial T}{\partial t}\right) \tag{4.23}$$

$$\frac{\partial^2 \overline{\theta}}{\partial x^2} = \frac{1}{\alpha}[s\overline{\theta} - \overline{\theta}(x, 0)] \tag{4.24}$$

From the initial condition, $\overline{\theta}(x, 0) = 0$.

$$\frac{d^2 \overline{\theta}}{dx^2} - \frac{s}{\alpha}\overline{\theta} = 0 \Rightarrow \overline{\theta} = c_1 e^{-\sqrt{s/\alpha}x} + c_2 e^{\sqrt{s/\alpha}x}$$

Applying the Laplace transform to the BCs,

$$\begin{cases} \overline{\theta}(\infty, s) = 0, & c_2 = 0 \\ \overline{\theta}(0, s) = \dfrac{\theta_s}{s}, & c_1 = \dfrac{\theta_s}{s} \end{cases}$$
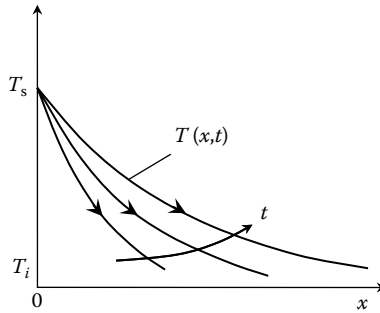
FIGURE 4.9
Solution of 1-D transient heat conduction with given surface temperature boundary condition.

So

$$\bar{\theta} = \frac{\theta_s}{s} e^{-\sqrt{s/\alpha}\,x}$$

Applying the inverse Laplace transform from a given table, we obtain the following results as sketched in Figure 4.9.

$$\frac{\theta}{\theta_s} = \frac{T - T_i}{T_s - T_i} = erfc\frac{x}{\sqrt{4\alpha t}} \qquad (4.25)$$

Case 2: Constant surface heat flux BC:

$$T(x, 0) = T_i$$

$$T(\infty, t) = T_i$$

$$-k\frac{\partial T(0, t)}{\partial x} = q_s''$$

Let $\theta = T - T_i$, then

$$\theta(x, 0) = 0$$

$$\theta(\infty, 0) = 0$$

$$-k\frac{\partial \theta(0, t)}{\partial x} = q_s''$$

The Laplace transform solution is

$$\bar{\theta} = \frac{q_s''}{k}\frac{1}{s\sqrt{s/\alpha}} - e^{-\sqrt{s/\alpha}\,x}$$

FIGURE 4.10
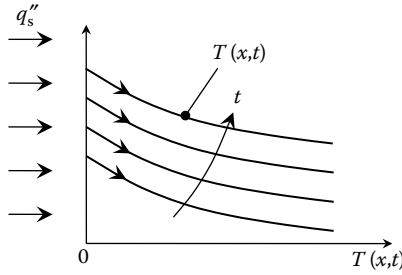Solution of 1-D transient heat conduction with constant surface heat flux boundary condition.

Applying the inverse Laplace transform from a given table, we obtain the following results as sketched in Figure 4.10.

$$T - T_i = \frac{q_s''}{k}\left[\frac{\sqrt{4\alpha t}}{\sqrt{\pi}}e^{-(x^2/4\alpha t)} - x \cdot erfc\left(\frac{x}{\sqrt{4\alpha t}} \right) \right] \qquad (4.26)$$

$$\text{at } x = 0, \quad T_s - T_i = \frac{q_s''}{k}\frac{\sqrt{4\alpha t}}{\sqrt{\pi}} \qquad (4.27)$$

Case 3: Convective surface BC:

$$T(x,0) = T_i$$

$$T(\infty,t) = T_i$$

$$-k\frac{\partial T(0,t)}{\partial x} = h[T_\infty - T(0,t)]$$

Let $\theta = T - T_i$

$$\theta(x,0) = 0$$

$$\theta(\infty,0) = 0$$

$$-k\frac{\partial \theta(0,t)}{\partial x} = h[\theta_\infty - \theta(0,t)]$$

Applying the inverse Laplace transform from a given table, we obtain the following results as sketched in Figure 4.11.

$$\frac{\theta}{\theta_\infty} = \frac{T - T_i}{T_\infty - T_i} = \left[erfc\left(\frac{x}{\sqrt{4\alpha t}} \right) - e^{(x(h/k)+\alpha(h^2/k^2)t)}erfc\left(\frac{x}{\sqrt{4\alpha t}} + \frac{h}{k}\sqrt{\alpha t} \right) \right]$$
$$(4.28)$$

$$q'' = -k\frac{\partial T(0,t)}{\partial x} = h[T_\infty - T(0,t)] \qquad (4.29)$$
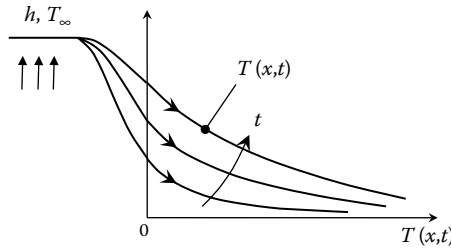
FIGURE 4.11
Solution of 1-D transient heat conduction convective surface boundary condition.

4.3.3 Approximate Integral Method for Semiinfinite Solid Material

Let $\theta = T - T_s$,

$$\theta_i = T_i - T_s, \quad t = 0$$

$$\theta(0, t) = 0, \quad \frac{\partial \theta(\delta, t)}{\partial x} = 0, \quad t > 0$$

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \tag{4.30}$$

$$\int_0^\delta \frac{\partial \theta}{\partial t} \, dx = \int_0^\delta \alpha \frac{\partial^2 \theta}{\partial x^2} \, dx$$

$$= \alpha \left[\frac{\partial \theta}{\partial x} \right]_{x=\delta} - \alpha \left[\frac{\partial \theta}{\partial x} \right]_{x=0}$$

where

$$\int_0^\delta \frac{\partial \theta}{\partial t} \, dx \equiv \frac{d}{dt} \int_0^\delta \theta \, dx - \frac{d\delta}{dt} \theta_{x=\delta} + \frac{d\delta}{dt} \theta_{x=0} = \frac{d}{dt} \int_0^\delta \theta \, dx - \frac{d\delta}{dt} \cdot \theta_i + 0$$

$$\therefore \frac{d}{dt} \int_0^\delta \theta \, dx - \frac{d\delta}{dt} \cdot \theta_i = \alpha \left(\frac{\partial \theta}{\partial x} \right)_{x=\delta} - \alpha \left(\frac{\partial \theta}{\partial x} \right)_{x=0}$$

$$\Rightarrow \frac{d}{dt} \left[\int_0^\delta \theta \, dx - \theta_i \delta \right] = -\alpha \left(\frac{\partial \theta}{\partial x} \right)_{x=0}$$

Assume a temperature profile is a second-order polynomial,

$$\theta(x, t) = a + bx + cx^2$$
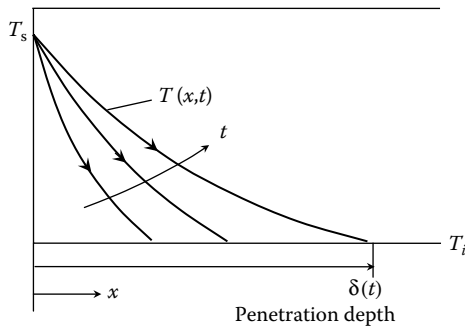
FIGURE 4.12
Solution of 1-D transient conduction for semiinfinite solid material using approximate integral method.

subject to BC:

$$\frac{\theta}{\theta_i} = 2\frac{x}{\delta} - \left(\frac{x}{\delta}\right)^2 \equiv 2\eta - \eta^2$$

$$\eta = \frac{x}{\delta}, \quad dx = \delta\,d\eta$$

Then

$$\frac{d}{dt}\left[\int_0^1 \theta_i(2\eta - \eta^2)\delta\,d\eta - \theta_i\delta\right] = -\frac{2\alpha\theta_i}{\delta}$$

$$\Rightarrow \frac{d\delta^2}{dt} - 12\alpha = 0 \quad \text{at } t = 0, \quad \delta(0) = 0$$

$$\Rightarrow \delta = \sqrt{12\alpha t}$$

From the approximate integral method, we obtain the following results as sketched in Figure 4.12:

$$\frac{\theta}{\theta_i} = 2\frac{x}{\delta} - \left(\frac{x}{\delta}\right)^2 = \frac{2x}{\sqrt{12\alpha t}} - \frac{x^2}{12\alpha t} \tag{4.31}$$

4.4   Heat Conduction with Moving Boundaries

There are many engineering application problems involving heat conduction with moving boundaries such as freezing or melting for solar storage systems. Other examples are related to high-temperature droplet evaporation and ablation applications. The problems can be solved by using the similarity method or the integral approximate method [4].
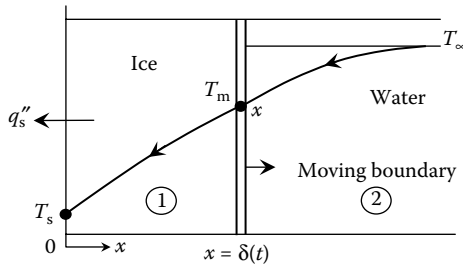
FIGURE 4.13
Heat conduction with moving boundary problem—freezing.

4.4.1 Freezing and Solidification Problems Using the Similarity Method

Freezing—Neumann solution (exact solution) [4], as sketched in Figure 4.13:
Governing equation:

$$\theta = T - T_m \tag{4.32}$$

$$\frac{\partial^2 \theta_1}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial \theta_1}{\partial t}$$

$$\frac{\partial^2 \theta_2}{\partial x^2} = \frac{1}{\alpha_1} \frac{\partial \theta_2}{\partial t}$$

Boundary conditions (4.33):

$$x = 0, \quad \theta_1 = \theta_s \tag{4.33a}$$

$$x = \infty, \quad \theta_2 = \theta_\infty \tag{4.33b}$$

$$x = \delta(t), \quad \theta_1 = 0 \tag{4.33c}$$

$$\theta_2 = 0 \tag{4.33d}$$

$$k_1 \frac{\partial \theta_1}{\partial x} - k_2 \frac{\partial \theta_2}{\partial x} = L\rho_1 \frac{d\delta}{dt} \tag{4.33e}$$

where L represents the latent heat of melting.
   Initial conditions:

$$t = 0, \quad \delta = 0, \quad T = T_\infty$$

   Assume $\rho_1 = \rho_2$.
   Neumann applied the similarity method:

$$\theta_1 = c_1 + c_2 erf \frac{x}{\sqrt{4\alpha_1 t}} \tag{4.33f}$$

$$\theta_2 = c_1' + c_2' erf \frac{x}{\sqrt{4\alpha_2 t}} \tag{4.33g}$$

At $x = \delta$, $\delta \sim \sqrt{t}$ to obtain $\theta_1 = 0$, $\delta = b\sqrt{t}$.

Insert Equations 4.33a through 4.33d into Equations 4.33f and 4.33g

$$c_1 = \theta_s$$

$$c_2 = \frac{-\theta_s}{erf\,(b/\sqrt{4\alpha_1})}$$

$$c_2' = \frac{\theta_\infty}{erfc(b/\sqrt{4\alpha_2})}$$

$$c_1' = \theta_\infty - \frac{\theta_\infty}{erfc(b/\sqrt{4\alpha_2})}$$

From Equation 4.33e,

$$\frac{-k_1\theta_s \exp(-b^2/4\alpha_1)}{\sqrt{\pi\alpha_1}erf\,(b/\sqrt{4\alpha_1})} - \frac{k_2\theta_\infty \exp(-b^2/4\alpha_2)}{\sqrt{\pi\alpha_2}erfc(b/\sqrt{4\alpha_2})} = L\rho_1\frac{b}{2} \qquad (4.33\text{h})$$

$$\because \frac{\partial}{\partial x}\left[erf\,\frac{x}{\sqrt{4\alpha_1 t}}\right] = \frac{2}{\sqrt{\pi}}\frac{\partial}{\partial x}\int_0^\eta e^{-\eta^2}\,d\eta$$

$$= \frac{2}{\sqrt{\pi}}\frac{\partial\eta}{\partial x}\frac{d}{d\eta}\int_0^\eta e^{-\eta^2}\,d\eta$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{4\alpha_1 t}}\exp\left(\frac{-x^2}{4\alpha_1 t}\right)$$

$\therefore b$ can be obtained from Equation 4.33h:

$$\delta = b\sqrt{t}$$

θ_1 can be obtained from Equation 4.33f, θ_2 can be obtained from Equation 4.33g

$$q_1 = k_1\left.\frac{\partial\theta_1}{\partial x}\right|_0 \quad \text{can be determined}$$

Special case: slow freezing, as sketched in Figure 4.14, approximately linear.
$\Rightarrow \theta_\infty = 0$:

$$\frac{\theta_1}{\theta_s} = 1 - \frac{erf\,(x/\sqrt{4\alpha_1 t})}{erf\,(b/\sqrt{4\alpha_1})} \qquad (4.34)$$

b can be obtained

$$\delta = b\sqrt{t}$$
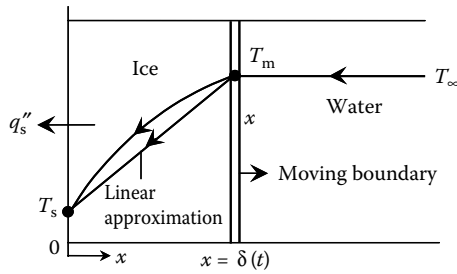
FIGURE 4.14
Slow freezing: $T_\infty \cong T_m$.

4.4.2 Melting and Ablation Problems Using the Approximate Integral Method

Melting—integral technique by Goodman (1958) [4], as sketched in Figure 4.15:

$$\delta(0) = 0$$

Let $\theta = T - T_m$,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \tag{4.35}$$

Initial condition: $\theta(x, 0) = 0$.
BCs for slow melting (Figure 4.16):

$$\begin{cases} \theta(0, t) = T_s - T_m = \theta_s \\ \theta(\delta, t) = 0 \\ -k\left.\frac{\partial \theta}{\partial x}\right|_\delta = L\rho\frac{d\delta}{dt} \end{cases} \tag{4.36}$$
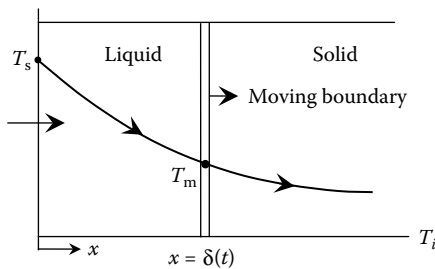
FIGURE 4.15
Heat conduction with moving boundary problems.
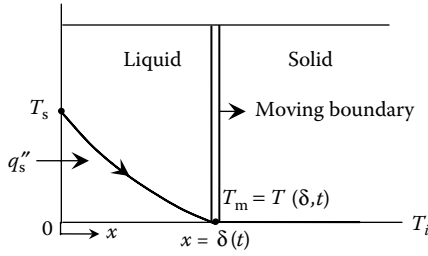
FIGURE 4.16
Slow melting: $T_m \equiv T_i$.

where L is the latent heat of melting.

$$\frac{d}{dt}\left[\int_0^\delta \theta\,dx - \theta(\delta,t)\,\delta\right] = \alpha\left[\left(\frac{\partial\theta}{\partial x}\right)_\delta - \left(\frac{\partial\theta}{\partial x}\right)_0\right]$$

$$\frac{d}{dt}\int_0^\delta \theta\,dx = -\alpha\left[\rho L\frac{d\delta}{dt} + \left(\frac{\partial\theta}{\partial x}\right)_0\right]$$

Assume $\theta = c_1(x-\delta) + c_2(x-\delta)^2$ (4.37)

Boundary conditions:

$$x = 0, \quad \theta_s = -c_1\delta + c_2\delta^2 \tag{4.37a}$$

$$x = \delta, \quad \left.\frac{d\theta}{dt}\right|_\delta = 0 = \left(\frac{\partial\theta}{\partial x}\right)_\delta\frac{d\delta}{dt} + \left(\frac{\partial\theta}{\partial t}\right)_\delta, \text{ that is, } \theta(\delta,t) = 0$$

$$= -\left(\frac{\partial\theta}{\partial x}\right)_\delta^2\frac{k}{\rho L} + \left(\frac{\partial\theta}{\partial t}\right)_\delta$$

$$\therefore \alpha\left(\frac{\partial^2\theta}{\partial x^2}\right)_\delta = \left(\frac{\partial\theta}{\partial x}\right)_\delta^2\frac{k}{\rho L}$$

$$\alpha\cdot 2c_2 = c_1^2\frac{k}{\rho L}. \tag{4.37b}$$

From Equations 4.37a and 4.37b, we obtain

$$c_1 = \frac{\alpha\rho L}{\delta k}[1 - \sqrt{1+\mu}]$$

$$c_2 = \frac{\alpha\delta + \theta_s}{\delta^2}$$

where $\mu = (2\theta_s C_p/L)$ is the Stefan number.
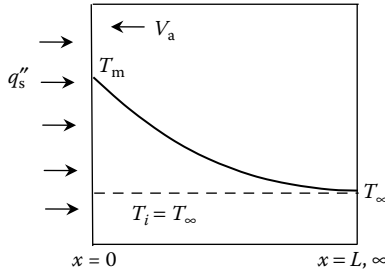
FIGURE 4.17
Ablation at surface of flat wall.

Then, inserting Equation 4.37 into Equation 4.35, we obtain

$$\delta \frac{d\delta}{dt} = \frac{6\alpha \left[1 - \sqrt{1+\mu} + \mu\right]}{5 + \mu + \sqrt{1+\mu}}$$

$$\delta = b\sqrt{t}$$

$$b = 2\sqrt{3\alpha} \left[\frac{1 - \sqrt{1+\mu} + \mu}{5 + \mu + \sqrt{1+\mu}}\right]^{1/2}$$

Linear approximation for small μ:

$$\delta \sim b\sqrt{t} \sim \sqrt{2\alpha\mu t}$$

4.4.2.1 Ablation

Ablating heat shields have been successful in satellite and missile reentry to the earth's atmosphere as a means of protecting the surface from aerodynamic heating. In this application, the high heat flux generated at the surface first causes an initial transient temperature rise until the surface reaches the melting temperature, T_m. Ablation (melting of the surface) begins and follows a second short transient period, and then a steady-state ablation velocity is reached. The melted material is assumed to run off immediately. The problem can be simplified to 1-D transient heat conduction with moving boundary due to ablation. Figure 4.17 shows ablation at the surface of the flat plate with an imaging ablation velocity, V_a, moving to the left [5]. The heat conduction equation for this reference frame is Equation 4.38 with an added enthalpy flux term associated with the moving velocity V_a:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \rho c V_a \frac{\partial T}{\partial x} = \rho c \frac{\partial T}{\partial t} \tag{4.38}$$

The problem can be solved by three stages: (1) the initial transient before the surfaces reach T_m, (2) the second transient period during ablation, and

(3) after the steady-state ablation velocity has been reached, the temperature distribution in the material is steady. The initial transient problem has been solved before (i.e., $V_a = 0$, given the surface heat flux BC). The second transient problem, Equation 4.38, can be solved by the finite-difference method. For the steady-state problem with constant ablation velocity, Equation 4.38 becomes

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = -\rho c V_a \frac{\partial T}{\partial x} \tag{4.39}$$

With the accompanying low thermal conductivity material (such as glasses and plastics) the temperature gradient at the surface is very steep so that $x = L$ may be considered as $x = \infty$. Proper BCs are as follows:

$$x = 0, \quad T = T_m$$

$$x = \infty, \quad T = T_\infty = T_i$$

$$x = \infty, \quad \frac{\partial T}{\partial x} = 0 \tag{4.40}$$

For constant properties k, ρ, c, and for the case of constant surface heat flux q_s'', Equation 4.39 is solved by integrating twice and evaluating the integration constants with Equation 4.40. Let $\theta = (dT/dx)$, then

$$\frac{d}{dx}\theta = -\frac{V_a}{\alpha}\theta$$

$$\frac{d\theta}{\theta} = -\frac{V_a}{\alpha}dx$$

$$\int\frac{d\theta}{\theta} = \int -\frac{V_a}{\alpha}dx$$

$$\ln\theta = -\frac{V_a}{\alpha}x + C$$

$$\theta = \frac{dT}{dx} = C_1 e^{-(V_a/\alpha)x} + C_2$$

where at $x = \infty$, $(dT/dx) = 0 = \theta$, $\therefore C_2 = 0$.
Then $(dT/dx) = C_1 e^{-(V_a/\alpha)x}$,

$$\int dT = \int C_1 e^{-(V_a/\alpha)x}\,dx$$

$$T = -C_1\frac{\alpha}{V_a}e^{-(V_a/\alpha)x} + C_3$$

$$\text{at } x = \infty, \quad T = T_\infty = C_3$$

$$\text{at } x = 0, \quad T = T_m = -C_1 \frac{\alpha}{V_a} + C_3$$

$$-C_1 = (T_m - C_3) \frac{V_a}{\alpha} = (T_m - T_\infty) \frac{V_a}{\alpha}$$

Therefore,

$$T = (T_m - T_\infty) e^{-(V_a/\alpha)x} + T_\infty$$

$$\frac{T - T_\infty}{T_m - T_\infty} = e^{-(V_a/\alpha)x} \tag{4.41}$$

Applying energy balance on the surface in order to determine the ablation velocity V_a (assuming L as the heat of ablation of the material),

$$q_s'' - \rho L V_a = -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = \rho C V_a (T_m - T_\infty) \tag{4.42}$$

$$V_a = \frac{q_s''}{\rho L + \rho C(T_m - T_\infty)} \tag{4.43}$$

The total heat conducted into the solid material evaluated with the temperature distribution, Equation 4.41, is

$$q_c'' = \rho C \int_0^\infty (T - T_\infty) \, dx = \frac{k(T_m - T_\infty)}{V_a} \tag{4.44}$$

The total heat transferred to the surface in time t is $q_s'' \cdot t$. Then for this period of time t, the fraction of the total heat transferred which was conducted into the solid material is obtained by substituting Equation 4.43 into Equation 4.44:

$$\frac{q_c''}{q_s''} = \frac{k(T_m - T_\infty)[\rho L + \rho C(T_m - T_\infty)]}{q_s'' \cdot q_s'' \cdot t} \tag{4.45}$$

Comparing Equations 4.43 and 4.45, a large magnitude of $[\rho L + \rho C(T_m - T_\infty)]$ is desirable to reduce the amount of material ablated, but a small magnitude is desirable to reduce the fraction of q_s'' which is conducted into the solid material. A compromise is necessary.

Examples

4.1. Solve transient temperature profiles of a convectively cooled cylinder, as shown in Figure 4.1b, by separation of variables.

SOLUTION

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} \quad \text{or} \quad \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

Boundary conditions:

   i. $r = 0,\quad \left.\frac{\partial \theta}{\partial r}\right|_{r=0} = 0$

   ii. $r = r_o,\quad -k\left.\frac{\partial \theta}{\partial r}\right|_{r=r_o} = h\theta_{r_o}$

Separation of variables:

$$\theta = R(r)\tau(t)$$

$$\frac{\partial \tau}{\partial t} + \lambda^2 \tau = 0; \quad \tau = C_3\, e^{-\alpha\lambda^2 t}$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \lambda^2 R = 0$$

$$R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

Applying BCs

$$\left.\frac{\partial R}{\partial r}\right|_{r=0} = -C_1\lambda J_1(0) + C_2\lambda Y_1(0) = 0, \quad J_1(0) = 0, \quad \Rightarrow C_2 = 0$$

$$-k\left.\frac{\partial R}{\partial r}\right|_{r=r_o} = kC_1\lambda J_1(\lambda r_o) = h\theta_R, \quad \lambda J_1(\lambda r_o) = \frac{h}{k}J_0(\lambda r_o), \quad \lambda_n = \lambda r_o$$

$$\lambda_n J_1(\lambda_n) - Bi J_0(\lambda_n) = 0$$

where $Bi = (hr_o/k)$,

$$\theta = \sum_{n=1}^{\infty} C_n e^{-(\alpha/r_o^2)\lambda_n^2 t} J_0\left(\lambda_n \frac{r}{r_o}\right)$$

at $t = 0,\quad \theta = 1$

$$1 = \sum_{n=1}^{\infty} C_n J_0\left(\lambda_n \frac{r}{r_o}\right)$$

$$C_n = \frac{\int_0^{r_o} r J_0(\lambda_n(r/r_o))\,dr}{\int_0^{r_o} r J_0^2(\lambda_n(r/r_o))\,dr} = \frac{(r_o^2/\lambda_n)J_1(\lambda_n)}{(r_o^2/2)[J_0^2(\lambda_n) + J_1^2(\lambda_n)]}$$

$$\theta = \sum_{n=1}^{\infty} \frac{2J_1(\lambda_n)J_0(\lambda_n(r/r_o))}{\lambda_n\left[J_0^2(\lambda_n) + J_1^2(\lambda_n)\right]} \cdot e^{-\lambda_n^2 F_0}$$

where $F_0 = (\alpha t/r_o^2)$.

4.2. Solve transient temperature profiles of a convectively cooled sphere, as shown in Figure 4.1c, by separation of variables.

SOLUTION

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\theta) = \frac{1}{\alpha}\frac{\partial\theta}{\partial t}$$

Boundary conditions:

i. $r = r_o \quad -k\frac{\partial\theta}{\partial r} = h\theta_{r_o}$

ii. $t = 0 \quad \theta = 1$

Let $U(r, t) = r\theta(r, t)$,

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$$

$$\text{BCs}\begin{cases} U = 0 & \text{at } r = 0 \\ \dfrac{\partial U}{\partial r} + \left(\dfrac{h}{k} - \dfrac{1}{r_o}\right)U = 0 & \text{at } r = r_o \end{cases}$$

$$U = r \quad \text{for } t = 0$$

$$U = R(r)\tau(t)$$

$$\frac{\partial\tau}{\partial t} + \alpha\lambda^2\tau = 0; \quad \Rightarrow \quad \tau(t) = C_3\,e^{-\alpha\lambda^2 t}$$

$$\frac{\partial^2 R}{\partial r^2} + \lambda^2 R(r) = 0$$

$$R(r) = C_1\sin(\lambda r) + C_2\cos(\lambda r)$$

$$\text{at } r = 0, \quad U = 0, \quad \Rightarrow \quad C_2 = 0$$

$$C_1\lambda\cos(\lambda r_o) + \left(\frac{h}{k} - \frac{1}{r_o}\right)C_1\sin(\lambda r_o) = 0$$

$$\lambda_n = \lambda r_o$$

$$\lambda_n\cos(\lambda_n) + (B_i - 1)\sin(\lambda_n) = 0$$

where $B_i = (hr_o/k)$,

$$U = \sum_{n=1}^{\infty} C_n\sin\left(\lambda_n\frac{r}{r_o}\right)e^{-\lambda_n^2 F_0}$$

where $F_0 = (\alpha t/r_o^2)$.

At $t = 0$, $\theta = 1$, $U = r$,

$$C_n = \frac{\int_0^{r_0} r\sin(\lambda_n(r/r_0))\,dr}{\int_0^{r_0} \sin^2(\lambda_n(r/r_0))\,dr}$$

$$C_n = \frac{2(\sin(\lambda_n) - \lambda_n\cos(\lambda_n))}{\lambda_n - \sin(\lambda_n)\cos(\lambda_n)}$$

$$\theta = \sum_{n=1}^{\infty} \frac{2\,[\sin(\lambda_n) - \lambda_n\cos(\lambda_n)]}{\lambda_n - \sin(\lambda_n)\cos(\lambda_n)} \cdot \frac{\sin(\lambda_n(r/r_0))}{\lambda_n(r/r_0)} \cdot e^{-\lambda_n^2 F_0}$$

4.3. Solve transient temperature profiles in a semiinfinite solid body using the Laplace transform for the following BCs of *constant surface heat flux*.

   If at time $t = 0$ the surface is suddenly exposed to a constant heat flux q_s''—for example, by radiation from a high-temperature source—the resulting temperature response is

$$T - T_i = \frac{q_s''}{k}\left[\left(\frac{4\alpha t}{\pi}\right)^{1/2} e^{-x^2/4\alpha t} - x\,\mathrm{erfc}\frac{x}{(4\alpha t)^{1/2}}\right]$$

SOLUTION

1-D transient:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Initial condition:

$$t = 0; \quad T = T_i.$$

Boundary conditions:

i. $x = 0$; $\quad -k\frac{\partial T}{\partial x} = q_s''$
ii. $x \to \infty$; $\quad T = T_i$

Let $\theta = T - T_i$,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

Initial condition:

$$t = 0; \quad \theta = 0.$$

Boundary conditions:

i. $x = 0$; $\quad -k\frac{\partial \theta}{\partial x} = q_s''$
ii. $x \to \infty$; $\quad \theta = 0$

Applying the Laplace transform,

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} - \frac{s}{\alpha}\tilde{\theta} = 0 \tag{1}$$

Boundary conditions:

i. $x = 0;$ $-k\frac{\partial \tilde{\theta}}{\partial x} = \frac{q_s''}{s}$

ii. $x \to \infty;$ $\tilde{\theta} = 0.$

Solving,

$$\tilde{\theta} = C_1\, e^{-\sqrt{s/\alpha}\,x} + C_2\, e^{\sqrt{s/\alpha}\,x} \tag{2}$$

Applying BC (ii), $C_2 = 0.$
Applying BC (i), $C_1 = (q_s''/s)(1/k\sqrt{s/\alpha}).$
Substituting this into above (2),

$$\tilde{\theta} = \frac{q_s''}{s}\frac{1}{k\sqrt{s/\alpha}}e^{-\sqrt{s/\alpha}\,x} \tag{3}$$

Rearranging,

$$\tilde{\theta} = \frac{q_s''/k\sqrt{\alpha}}{s^{3/2}}e^{-(x/\sqrt{\alpha})\sqrt{s}}$$

Applying the Laplace inverse,

$$\theta = \frac{q_s''}{k}\sqrt{\alpha}\left[2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{x}{\sqrt{\alpha}}erfc\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$

$$T - T_i = \frac{q_s''}{k}\left[\left(\frac{4\alpha t}{\pi}\right)^{1/2}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\,erfc\frac{x}{(4\alpha t)^{1/2}}\right]$$

4.4. Solve transient temperature profiles in a semiinfinite solid body using the Laplace transform for the following BCs of *convective heat transfer to the surface.*

If at time $t = 0$ the surface is suddenly exposed to a fluid at temperature T_∞, with a convective heat transfer coefficient h, the resulting temperature response is

$$\frac{T - T_i}{T_\infty - T_i} = erfc\frac{x}{(4\alpha t)^{1/2}} - e^{hx/k+(h/k)^2\alpha t}erfc\left(\frac{x}{(4\alpha t)^{1/2}} + \frac{h}{k}(\alpha t)^{1/2}\right)$$

SOLUTION

1-D transient

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Initial condition:

$$t = 0; \quad T = T_i.$$

Boundary conditions:

   i. $x = 0; \quad -k\frac{\partial \theta}{\partial x} = h(\theta_\infty - \theta_0)$

   ii. $x \to \infty; \quad T = T_i$

Let $\theta = T - T_i$,

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial \theta}{\partial t}$$

Initial condition:

$$t = 0; \quad \theta = 0$$

Boundary conditions:

   i. $x = 0; \quad -k\frac{\partial \theta}{\partial x} = h(\theta_\infty - \theta_0)$

   ii. $x \to \infty; \quad \theta = 0$

Applying the Laplace transform,

$$\frac{\partial^2 \tilde{\theta}}{\partial x^2} - \frac{s}{\alpha}\tilde{\theta} = 0 \tag{1}$$

Boundary conditions:

   i. $x = 0; \quad -k\frac{d\tilde{\theta}(0,t)}{dx} = h\left(\frac{\theta_\infty}{s} - \tilde{\theta}(0, t)\right)$

   ii. $x \to \infty; \quad \tilde{\theta} = 0$

By solving,

$$\tilde{\theta} = C_1\, e^{-\sqrt{s/\alpha}x} + C_2\, e^{\sqrt{s/\alpha}x} \tag{2}$$

Applying BC (ii), $C_2 = 0$.
Applying BC (i),

$$C_1 = \frac{(h/k)\theta_\infty}{((h/k) + (\sqrt{s/\alpha}))s}$$

Substituting this into above (2),

$$\tilde{\theta} = \frac{(h/k)\theta_\infty}{((h/k)+(\sqrt{s/\alpha}))s}e^{-\sqrt{s/\alpha}x} \tag{3}$$

By rearranging,

$$\tilde{\theta} = \theta_\infty\frac{(h/k)\sqrt{\alpha}}{((h/k)\sqrt{\alpha}+\sqrt{s})s}e^{-(x/\sqrt{\alpha})\sqrt{s}}$$

Applying the Laplace inverse,

$$\frac{\theta}{\theta_\infty} = -e^{(h/k)x} e^{(h^2\alpha/k^2)t} \, erfc \left(\frac{h}{k}\sqrt{\alpha t} + \frac{x}{\sqrt{4\alpha t}} \right) + erfc \left(\frac{x}{\sqrt{4\alpha t}} \right)$$

$$\frac{T - T_i}{T_\infty - T_i} = erfc \frac{x}{(4\alpha t)^{1/2}} - e^{hx/k + (h/k)^2 \alpha t} \, erfc \left(\frac{x}{(4\alpha t)^{1/2}} + \frac{h}{k}(\alpha t)^{1/2} \right)$$

Remarks

There are many engineering problems involving 0-D, 1-D, 2-D, or 3-D transient heat conduction with various thermal BCs. In the undergraduate-level heat transfer, we have focused mainly on how to apply the lumped capacitance solutions to solve relative simple engineering problems. For 1-D and multidimensional transient conduction problems, we normally do not go through the detailed mathematical equations and solutions. Instead, students are expected to apply these formulas to solve many engineering relevant problems by giving solid material geometry with appropriate thermal properties and thermal BCs.

In the intermediate-level heat transfer, Chapter 4 we have introduced several very powerful mathematical tools such as similarity method, Laplace transform method, and integral approximate method, in addition to the separation of variables method already mentioned in Chapter 3. Specifically, the separation of variables method is convenient to solve the transient conduction problems with finite-length dimensions such as the plate, cylinder, and sphere with various thermal BCs. However, the similarity method, Laplace transform method, or integral approximate method is more appropriate to solve the transient conduction problems with semiinfinite solid material for various thermal BCs.

1-D transient heat conduction with moving boundaries belongs to advanced heat conduction material. (Readers can skip these topics.)

PROBLEMS

4.1. The wall of a rocket nozzle is of thickness $L = 25$ mm and is made from a high-alloy steel for which $\rho = 8000$ kg/m^3, $c = 500$ J/kg K, and $k = 25$ W/m K. During a test firing, the wall is initially at $T_i = 25°$C and its inner surface is exposed to hot combustion gases for which $h = 500$ W/m^{2}K and $T_\infty = 1750°$C. The firing time is limited by the nozzle inner-wall temperature when it reaches 1500°C. The outer surface is well insulated.

a. Write the transient heat conduction equation, the associated BCs, and determine the nozzle wall temperature distributions.

(*Note*: The diameter of the nozzle is much larger than its thickness. No need to perform integration of orthogonal functions if you run out of time.)

b. Sketch several nozzle wall temperature profiles during transient heating.

c. To increase the firing time, changing the wall thickness L is considered. Should L be increased or decreased? Why? The value of the firing time could also be increased by selecting a wall material with different thermophysical properties. Should materials of larger or smaller values of ρ, c, and k be chosen?

4.2. A one-side-insulated metal plane wall with a thickness of L is initially at temperature T_i and suddenly the other side is heated by forced convection water at temperature T_∞ with a convection heat transfer coefficient h.

a. Outline, step by step, the procedures and the associated initial and BCs that may be used to solve the temperature distributions in the plane wall. You do not need to solve the transient temperature distribution.

b. Sketch the temperature profiles in the plane wall during the heating process. Also, estimate the surface temperature at the final steady-state condition.

4.3. A long metal plane wall with a thickness of $2L$ is initially at temperature T_i and suddenly both sides are heated by convection fluid flow at temperature T_∞ with a convection heat transfer coefficient h. Outline the procedures that may be used to solve the temperature distributions in the plane wall and sketch the temperature profiles in the plane wall during the heating process for two different cases.

a. Fluid flow is natural convection air.

b. Fluid flow is forced convection water.

c. Also, estimate the surface temperature at the final steady-state condition. Which fluid flow will reach the steady temperature faster? Why? Make appropriate assumptions in order to justify your answers.

4.4. A large flat plate (with a thickness of $2L$) initially at T_i is suddenly plunged into a liquid bath at T_∞. Derive an expression for the instantaneous temperature distribution in the plate, if,

a. The heat transfer coefficient between the two surfaces of the plate and the liquid, h, is given as constant and finite.

b. The heat transfer coefficient h between the two surfaces of the plate and the liquid is very large so that the temperatures on the two surfaces of the plate may be assumed to change abruptly to the temperature of the liquid (i.e., $T(L,t) = T_\infty$, for $t > 0$).

c. Sketch the instantaneous temperature distribution in the plate for both (a) and (b) if $T_\infty > T_i$.

4.5. A semiinfinite solid initially at a uniform temperature T_i and suddenly exposed at its surface to a constant heat flux q''.

a. Determine the temperature history in the solid.

  b. Approximately plot the temperature profiles for the following BCs:

   i. Constant q'' at the surface.

   ii. Constant temperature at the surface, $T_s(0,t) > T_i$.

   iii. Constant fluid temperature T_f and h, $T_f > T_i$.

4.6. A semiinfinite carbon steel block is initially at a uniform temperature T_i and its surfaces is suddenly exposed to a constant irradiation flux q'', simultaneously to an ambient air of h, T_∞.

  a. Determine the temperature history in the solid and sketch the temperature profiles in the solid assuming $T_i = T_\infty$. Analytical method.

  b. If a semiinfinite pure copper block will be operated at the same condition, state that the time required for the surface of the block to reach T_s will be longer or shorter than that of the steel block. Why?

4.7. A liquid confined in a half-space $x > 0$ is initially at a temperature T_i which is higher than its freezing temperature T_m. For times $t > 0$, the surfaces at $x = 0$ are subjected to the following BCs. Plot the temperature history both in the liquid and the solid.

  a. Constant heat flux is q''; this q'' is removed away from the surface.

  b. Constant surface temperature T_0, $T_0 < T_m$.

  c. If natural convection takes place in the liquid region with a constant h, plot the temperature profiles and freezing distance δ with time in both case (a) and case (b). Explain the difference.

4.8. A solid simulated as a half-space, $x > 0$, is initially at a temperature T_i which is equal to its melting temperature T_m. For time $t > 0$, the surface at $x = 0$ is subjected to the following BCs. Plot the temperature history both in liquid and solid:

  a. Constant heat flux is q''; this q'' is applied to the surface.

  b. Constant surface temperature T_s, $T_s > T_m$.

  c. Convection to the surface with heating fluid at T_∞, h.

4.9. Refer to Equation 4.14, and solve the temperature distributions for the 3-D block with the following BCs:

$$
\begin{array}{lll}
(1) & x = \pm a, & T = T_s \\
    & y = \pm b, & T = T_s \\
    & z = \pm c, & T = T_s \\
\end{array}
$$

$$
\begin{array}{lll}
(2) & x = \pm a, & (-k\partial T(\pm a, t)/\partial x) = h[T(\pm a, t) - T_\infty] \\
    & y = \pm b, & (-k\partial T(\pm b, t)/\partial y) = h[T(\pm b, t) - T_\infty] \\
    & z = \pm c, & (-k\partial T(\pm c, t)/\partial z) = h[T(\pm c, t) - T_\infty] \\
\end{array}
$$

4.10. A semiinfinite solid is initially at uniform temperature T_s. The surface of the semiinfinite solid is suddenly exposed to a constant temperature T_w.

  a. Write the governing equations and the BCs.

b. Nondimensionalize the governing equations and BCs by appropriate choice of temperature variable, length scale, and timescale.

4.11. An infinite body of cold liquid initially at uniform temperature T_s is brought in contact with a heated horizontal wall of infinite length maintained at a constant temperature (T_w). It is expected that after infinite time the liquid temperature profile will be linear within a thermal boundary layer of thickness δ. Neglect gravity or body forces and liquid convection and assume that heat transfer in the liquid is by conduction only.

   a. Write the governing equations and the BCs.
   b. Nondimensionalize the governing equations and BCs by appropriate choice of temperature variable, length scale, and timescale.
   c. Solve the governing equations to obtain the transient temperature profile. Verify if the solution for the transient temperature profile satisfies the linear temperature profile when steady-state conditions are reached (at infinite time).

4.12. A plate is initially at temperature T_i when laid on an insulated surface and cooled by air flow at temperature T_∞, with the heat transfer coefficient h. The length of the plate is l, width w, and thickness b ($l \gg w$, $l \gg b$, $w \gg b$). The thermal diffusivity of the plate is α and the thermal conductivity of the plate material is k. The viscosity of air is μ and density is ρ. Estimate the time required for the bottom surface of the plate to cool to T_b, when

   a. The plate material is made of copper (the conduction resistance in the slab is negligible).
   b. The plate material is made of plastic (the convection resistance on the slab is negligible).

4.13. A long metal plane wall with a thickness of $2L$ is initially at temperature T_i and suddenly both sides are heated by a convection fluid flow at temperature T_∞. Outline the procedures and solve the temperature distributions in the plane wall and sketch the temperature profiles in the plane wall during the heating process for two different cases.

   a. Fluid flow is natural convection air.
   b. Fluid flow is forced convection water.

4.14. a. Consider a large wall, separating two fluids at $T_{\infty 1}$ and $T_{\infty 2}$ ($T_{\infty 2} < T_{\infty 1}$). To prevent heat transfer from the hot fluid at $T_{\infty 1}$ to the wall, a thin foil guard heater (of negligible thickness) on the surface of the wall exposed to the hot fluid at $T_{\infty 1}$ is used to raise the surface temperature to $T_{\infty 1}$. Sketch the instantaneous temperature distributions at several different times in the fluids near the wall and in the wall, before and after the surface temperature is raised with the thin foil guard heater, until a steady state is reached.
   b. Consider a large wall, separating two fluids at $T_{\infty 1}$ and $T_{\infty 2}$ ($T_{\infty 2} < T_{\infty 1}$). Instead of the thin foil guard heater in (a), heat is

generated uniformly in the wall to prevent heat transfer from the fluid at $T_{\infty 1}$ to the wall. Sketch the instantaneous temperature distributions in the fluids at several different times near the wall and in the wall, before and after heat is generated uniformly in the wall to prevent heat transfer from the hot fluid to the wall, until a steady state is reached.

c. Consider a large wall with a thickness of $2L$ its surfaces maintained at T_1 and T_2 ($T_2 \neq T_1$). Heat is generated uniformly in the wall. Beginning with the steady-state, 1-D, heat conduction equation and appropriate BCs ($T = T_1$ at $x = -L$ and $T = T_2$ at $x = +L$), derive an expression for the steady-state 1-D temperature distribution in the wall, $T(x)$. Using the expression, *show* that the rate of heat generation is equal to the sum of the rates of heat transfer from the two surfaces.

4.15. The plane wall has constant properties and no internal generation and is initially at a uniform temperature T_i. Suddenly, the surface $x = L$ is exposed to a heating process with a fluid at T_∞ having a convection coefficient h. At the same instant, the electrical heater is energized providing a constant heat flux q_0'' at $x = 0$.

a. On $T - x$ coordinates, sketch the temperature distributions for the following conditions: initial condition ($t < 0$), steady-state condition ($t \to \infty$), and for two intermediate times.

b. On $q_x'' - x$ coordinates, sketch the heat flux corresponding to the four temperature distributions of (a).

c. On $q_x'' - t$ coordinates, sketch the heat flux at the locations $x = 0$ and $x = L$. That is, show qualitatively how $q_x''(0,t)$ and $q_x''(L,t)$ vary with time.

d. Derive an expression for the steady-state temperature at the heater surface, $T(0, \infty)$, in terms of q_0'', T_∞, k, h, and L.

4.16. The plane wall has constant properties and a uniform internal generation of $\dot{q}(W/m^3)$ that activates only when the electric heater is energized. The wall is initially at a uniform temperature T_i. Suddenly, the surface $x = L$ is exposed to a cooling process with a fluid at T_∞ having a convection coefficient h. At the same instant, the electrical heater is energized providing a constant heat flux q_0'' at $x = 0$.

a. On $T - x$ coordinates, sketch the temperature distributions for the following conditions: initial condition ($t \leq 0$), steady-state condition ($t \to \infty$), and for two intermediate times.

b. On $q_x'' - x$ coordinate, sketch the heat flux corresponding to the four temperature distributions of (a).

c. On $q_x'' - t$ coordinates, sketch the heat flux at the locations $x = 0$ and $x = L$. That is, show qualitatively how $q_x''(0,t)$ and $q_x''(L,t)$ vary with time.

d. Derive an expression for the steady-state temperature at the heater surface, $T(0,\infty)$, in terms of q_0'', $\dot{q}$, T_∞, k, h, and L.

4.17. A 10 m-long 2 cm-diameter copper rod is immersed in a heating bath at a uniform temperature of 100°C. This rod is suddenly

exposed to an air stream at 20°C with a heat transfer coefficient of 200 W/m^2 K. Find the time required for the copper rod to cool to an average temperature of 25°C. Write all assumptions if necessary. Thermal conductivity, specific heat, and density of copper are 401 W/m K, 385 J/kg K, and 8933 kg/m^3, respectively.

4.18. Determine the temperature profile for 1-D transient heat conduction problem in a cylinder at constant surface temperature as shown in Figure 4.1b.

4.19. Determine the temperature profile for 1-D transient heat conduction problem in a sphere at constant surface temperature as shown in Figure 4.1c.

4.20. Determine the temperature profile for 1-D transient heat conduction problem in a vertical plate at constant surface temperature as shown in Figure 4.4b.

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5

Numerical Analysis in Heat Conduction

5.1 Finite-Difference Energy Balance Method for 2-D Steady-State Heat Conduction

The above-discussed principle of separation of variable is a powerful method to solve the 2-D heat conduction problem. However, the solutions become tedious for various nonhomogenous BCs. With the help of modern computers, the heat conduction problem can be easily solved by using the finite-difference energy balance method for complex BCs. For example, Figure 5.1 shows the typical numerical grid distribution for 2-D heat conduction with given surface temperatures as BCs. It requires detailed mathematical procedures if we choose the separation of variable method to solve this problem. Of course, the accuracy of the numerical solutions depends on the number of finite-difference grids used for energy balance calculations. In general, the accuracy improves with the increase of grid points. It should be noted that each grid point actually represents the temperature of a small area $\Delta x \Delta y$. So we obtain the discrete temperature distribution by using the finite-difference method. However, when Δx and Δy become very small (approaching zero), the temperature distribution predicted by the finite-difference method will be the same as those calculated using the separation of variable method.

In general, the grid size in the x-direction is not necessarily the same as that in the y-direction. We need to use smaller grid size (more grid points) in the high-temperature gradient direction. The energy balance can be performed for each grid point shown in Figure 5.1. The number of unknown temperatures is the same as the number of energy balance equations (the number of grid points). Therefore, the unknown temperatures can be solved. Note that we do not need to perform energy balance on the boundary points if the boundary temperatures are already given. But, we need to perform energy balance on the boundary points if the boundary is exposed to heat flux or convection in which their boundary temperatures are unknown and to be determined using the finite-difference method. The following outlines the finite-difference method to solve the 2-D heat conduction problem shown in Figure 5.2. We can begin the energy balance at the interior points and then extend to energy balance at the boundary points with various BCs.
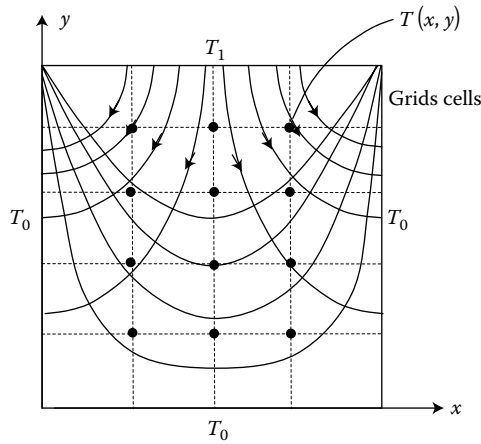
FIGURE 5.1
Finite difference method to solve 2-D heat conduction problem.

Energy balance at the interior nodes:

$$\sum_{i=1}^{4} q_{i \to 0} + \dot{q}(\Delta x \cdot \Delta y \cdot 1) = 0 \tag{5.1}$$

$$k\Delta y \cdot 1 \cdot \frac{T_1 - T_0}{\Delta x} + k\Delta y \cdot 1 \cdot \frac{T_2 - T_0}{\Delta x} + k\Delta x \cdot 1 \cdot \frac{T_3 - T_0}{\Delta y}$$

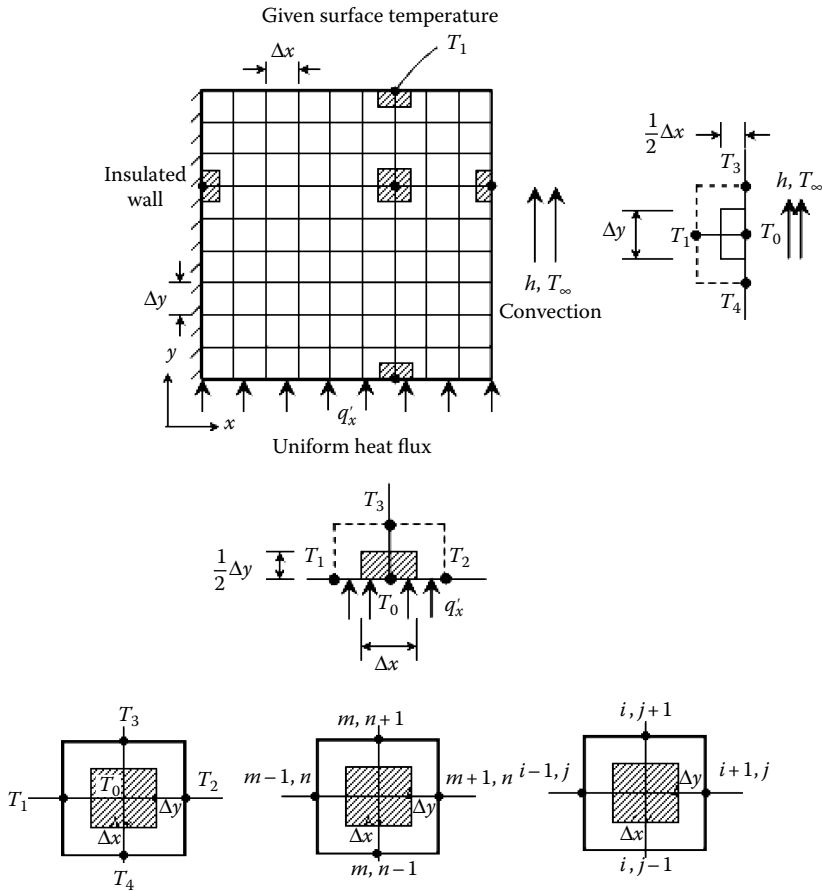$$+ k\Delta x \cdot 1 \cdot \frac{T_4 - T_0}{\Delta y} + \dot{q}(\Delta x \cdot \Delta y) = 0 \tag{5.2}$$

If $\Delta x = \Delta y$,

$$T_0 = \frac{1}{4}\left(T_1 + T_2 + T_3 + T_4 + \frac{\dot{q}\Delta x \Delta x}{k}\right) \tag{5.3}$$

Energy balance at boundary nodes (not needed if the surface temperature is given):

Convection boundary on the surface nodes:

$$k\Delta y \cdot 1 \cdot \frac{T_1 - T_0}{\Delta x} + k\frac{\Delta x}{2} \cdot 1 \cdot \frac{T_4 - T_0}{\Delta y} + k\frac{\Delta x}{2} \cdot 1 \cdot \frac{T_3 - T_0}{\Delta y}$$

$$+ h\Delta y(T_\infty - T_0) + \dot{q}\frac{\Delta x}{2} \cdot \Delta y = 0 \tag{5.4}$$

FIGURE 5.2
Finite difference method to solve 2-D heat conduction problem.

Uniform heat flux at the surface nodes:

$$k\frac{\Delta y}{2} \cdot 1 \cdot \frac{T_1 - T_0}{\Delta x} + k\Delta x \cdot 1 \cdot \frac{T_3 - T_0}{\Delta y} + k\frac{\Delta y}{2} \cdot 1 \cdot \frac{T_2 - T_0}{\Delta x}$$

$$+ q_s'' \Delta x \cdot 1 + \dot{q}\frac{\Delta y}{2} \cdot \Delta x = 0 \tag{5.5}$$

If insulation BCs applies to the surface nodes, then $q_s'' = 0$.

In general, T_0, T_1, T_2, T_3, and T_4 can be replaced by $T_{m,n}$, $T_{m-1,n}$, $T_{m+1,n}$, $T_{m,n-1}$, and $T_{m,n+1}$ or by $T_{i,j}$, $T_{i-1,j}$, $T_{i+1,j}$, $T_{i,j-1}$, and $T_{i,j+1}$, where $m = 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$ or $i = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots$, to obtain $T_{1,1}, T_{1,2}, T_{1,3}, \ldots, T_{2,1}, T_{2,2}, T_{2,3}, \ldots$, and $T_{3,1}, T_{3,2}, T_{3,3}, \ldots$.

Another approach is to make grid nodes directly from the heat conduction equation. The 2-D steady-state heat conduction equation with heat generation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{k} = 0$$

The finite-differential format of the steady-state 2-D heat conduction equation with heat generation can be written as

$$\frac{(T_{m-1,n} - T_{m,n})/\Delta x + (T_{m+1,n} - T_{m,n})/\Delta x}{\Delta x}$$

$$+ \frac{(T_{m,n-1} - T_{m,n})/\Delta y + (T_{m,n+1} - T_{m,n})/\Delta y}{\Delta y} + \frac{\dot{q}}{k} = 0 \qquad (5.6)$$

Let $\Delta x = \Delta y$, and one obtains

$$(T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}) + \frac{\dot{q}}{k}(\Delta x)^2 = 4T_{m,n} \qquad (5.7)$$

where $m = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$.

The above linear equation can be applied to any interior nodes. Theoretically, one would obtain $m \times n$ linear equations, and therefore, temperature $T(x, y) = T_{m,n}$ can be solved using the matrix method [1,2]. For example, let $T_{m,n} = T_1, T_2, T_3, \ldots, T_N$, and the above linear equations can be applied $T_1, T_1, T_3, \ldots, T_N$. Rearranging the equation, one obtains

$$a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1N}T_N = C_1$$

$$a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2N}T_N = C_2$$

$$\vdots \qquad\qquad\qquad (5.8)$$

$$a_{N1}T_1 + a_{N2}T_2 + a_{N3}T_3 + \cdots + a_{NN}T_N = C_N$$

Using the matrix notation, these equations can be expressed as

$$[A][T] = [C] \qquad (5.9)$$

where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \quad [T] = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad [C] = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

The solution may be expressed as

$$[T] = [A]^{-1}[C] \qquad (5.10)$$

where $[A]^{-1}$ is the inverse of $[A]$ that is defined as

$$[A]^{-1} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & & & \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}$$

Therefore, temperature can be determined by

$$\begin{aligned} T_1 &= b_{11}C_1 + b_{12}C_2 + \cdots + b_{1N}C_N \\ T_2 &= b_{21}C_1 + b_{22}C_2 + \cdots + b_{2N}C_N \\ &\vdots \\ T_N &= b_{N1}C_1 + b_{N2}C_2 + \cdots + b_{NN}C_N \end{aligned} \qquad (5.11)$$

Example 5.1

We use Figure 5.3 as an example to demonstrate how to solve the 2-D heat conduction problem by using the finite-difference method. Let $\Delta x = \Delta y$, $\dot{q} = 0$. Rearrange the temperatures from the energy balance on node 1, 2, 3, ...

$$T_1 = \frac{1}{4}(T_s + T_s + T_2 + T_3) \implies$$
$$-4T_1 + T_2 + T_3 + 0 + 0 + 0 + 0 + 0 = -2T_s$$
$$T_2 = \frac{1}{4}(T_1 + T_4 + T_1 + T_s)$$
$$2T_1 - 4T_2 + 0 + T_4 + 0 + 0 + 0 + 0 = -T_s$$
$$T_3 = \frac{1}{4}(T_s + T_5 + T_4 + T_1)$$
$$T_4 = \frac{1}{4}(T_3 + T_6 + T_3 + T_2)$$
$$T_5 = \frac{1}{4}(T_s + T_7 + T_6 + T_3)$$
$$T_6 = \frac{1}{4}(T_5 + T_8 + T_5 + T_4)$$
$$T_7 = \frac{1}{(4 + 2(h\Delta x/k))}\left(2T_5 + T_8 + T_s + 2\frac{h\Delta x}{k}T_\infty\right)$$
$$T_8 = \frac{1}{(2 + (h\Delta x/k))}\left(T_6 + T_7 + \frac{h\Delta x}{k}T_\infty\right)$$
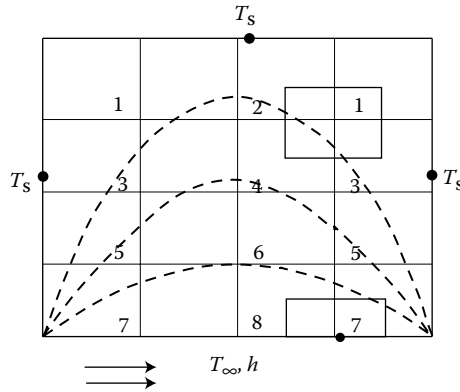
FIGURE 5.3
Example of using finite difference method to solve 2-D heat conduction problem.

   Place temperatures on the left-hand side of the equation and the constants on the right-hand side. We can form a coefficient matrix $[A]$, temperature matrix $[T]$, and column matrix $[C]$. The linear equations of the finite-difference energy balance on each grid point can be represented by the product of $[A][T] = [C]$. Therefore, the temperature distribution can be obtained if we know how to solve $[T]$ from $[A]$ and $[C]$. So the main job for this method is how to obtain $[A]$ and $[C]$. The solution for $[T]$ is

$$[T] = [A]^{-1}[C]$$

$$[A] = \begin{bmatrix}
-4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -4 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 2 & -4 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -4 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 2 & -4 & 0 & 1 \\
0 & 0 & 0 & 0 & 2 & 0 & -\left(4 + \dfrac{2h}{k}\Delta x\right) & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -\left(2 + \dfrac{h}{k}\Delta x\right)
\end{bmatrix}$$

$$[C] = \begin{bmatrix}
-T_s \\
-T_s \\
-T_s \\
0 \\
-T_s \\
0 \\
-T_s - \dfrac{2h}{k}\Delta x T_\infty \\
-\dfrac{h}{k}\Delta x T_\infty
\end{bmatrix} \qquad [T] = \begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
T_5 \\
T_6 \\
T_7 \\
T_8
\end{bmatrix}$$

Example 5.2

We use Figure 5.4 with $T(x, y) = T_{m,n}$, when $m = 1, 2, 3, 4, 5$, $n = 1, 2, 3, 4, 5$, respectively. We need to solve the temperature column matrix $[T]$. Let $\Delta x = \Delta y$, $\dot{q} = 0$, and write down the linear equations of the finite-difference energy balance method on each mode, and obtain a coefficient matrix $[A]$ and a column matrix $[C]$. Therefore, the temperature matrix can be solved by $[A][T] = [C]$, $[T] = [A]^{-1}[C]$, where all $[T]$, $[A]$, and $[C]$ are equal:

$$[T] = \begin{bmatrix} T_{1,1} & T_{2,1} & T_{3,1} & T_{4,1} & T_{5,1} \\ T_{1,2} & T_{2,2} & T_{3,2} & T_{4,2} & T_{5,2} \\ T_{1,3} & T_{2,3} & T_{3,3} & T_{4,3} & T_{5,3} \\ T_{1,4} & T_{2,4} & T_{3,4} & T_{4,4} & T_{4,5} \\ T_{1,5} & T_{2,5} & T_{3,5} & T_{4,5} & T_{5,5} \end{bmatrix} = \begin{bmatrix} T_{1,1} \\ T_{2,1} \\ \vdots \\ T_{1,1} \\ T_{2,2} \\ \vdots \\ T_{1,5} \\ T_{2,5} \\ \vdots \\ T_{5,5} \end{bmatrix}$$

$$[C] = \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} & C_{4,1} & C_{5,1} \\ C_{1,2} & C_{2,2} & C_{3,2} & C_{4,2} & C_{5,2} \\ C_{1,3} & C_{2,3} & C_{3,3} & C_{4,3} & C_{5,3} \\ C_{1,4} & C_{2,4} & C_{3,4} & C_{4,4} & C_{4,5} \\ C_{1,5} & C_{2,5} & C_{3,5} & C_{4,5} & C_{5,5} \end{bmatrix} = \begin{bmatrix} C_{1,1} \\ C_{2,1} \\ \vdots \\ C_{1,1} \\ C_{2,2} \\ \vdots \\ C_{1,5} \\ C_{2,5} \\ \vdots \\ C_{5,5} \end{bmatrix}$$

$$[A] = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{54} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}$$

The above-mentioned finite-difference energy balance method can be used for solving 2-D and 3-D heat conduction problems with Cartesian, cylindrical, and spherical coordinates with various BCs; for example, for the 3-D problem as shown in Figure 5.5.

T_s given surface temperature

$q_s'' = 0$
insulated
wall

h, T_∞
Convection

$n = 1, ..., 5$

y

x

$m = 1, ..., 5$

q_s'' uniform heat flux
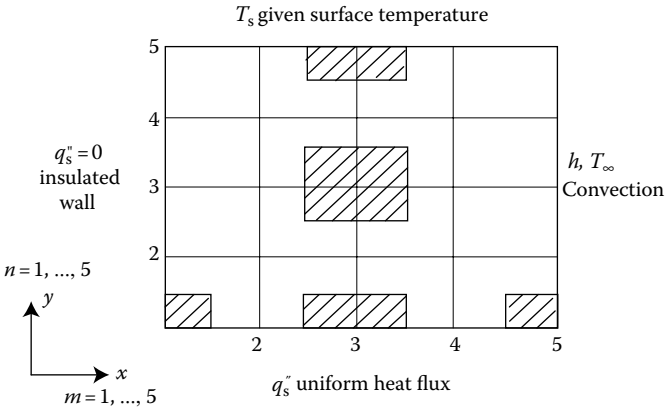
FIGURE 5.4
Example of using finite difference method to solve 2-D heat conduction problem with various boundary conditions.

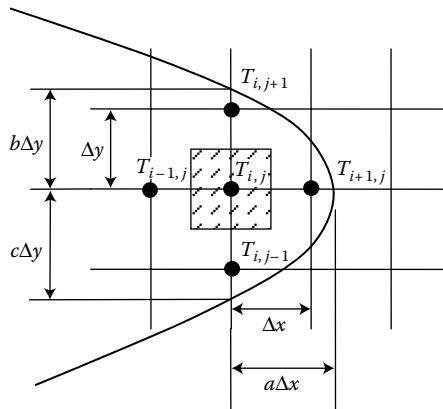Let $T(x, y, z) = T_{i,j,k}$, with $i = 1, 2, 3, \ldots i$, $j = 1, 2, 3, \ldots j$, and $k = 1, 2, 3, \ldots k$.
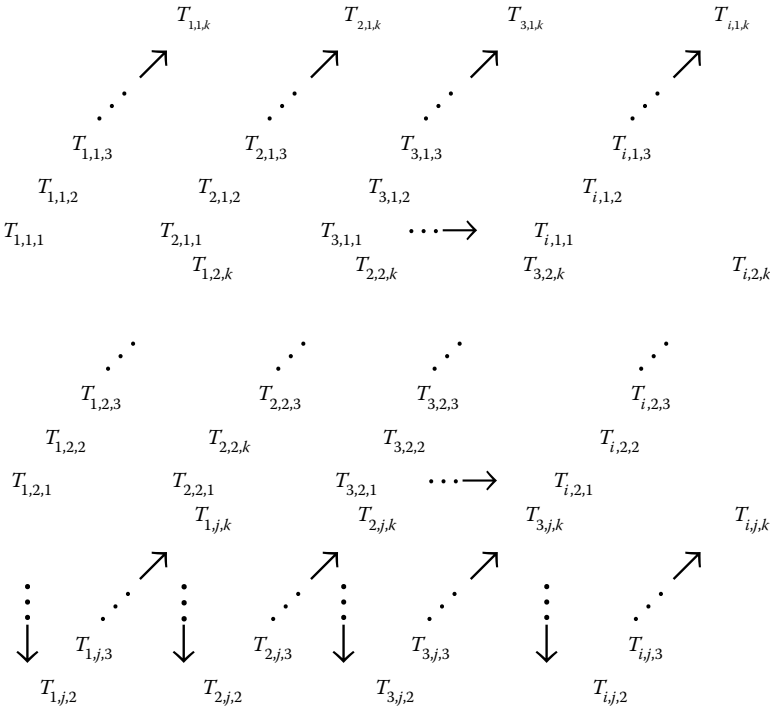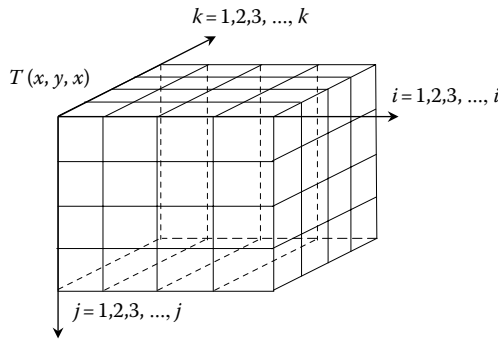
FIGURE 5.5
Finite difference method to solve 3-D heat conduction problem.

For the case of a curved boundary [3], the interior nodes can be determined as shown in Figure 5.6.

$$\text{Given} : T_{i+1,j}, T_{i-1,j}, T_{i,j+1}, T_{i,j-1}$$

$$k\Delta y \frac{T_{i-1,j} - T_{i,j}}{\Delta x} + k\Delta y \frac{T_{i+1,j} - T_{i,j}}{a\Delta x} \tag{5.12}$$

$$+k\Delta x \frac{T_{i,j+1} - T_{i,j}}{b\Delta y} + k\Delta x \frac{T_{i,j-1} - T_{i,j}}{c\Delta y} = 0$$

$$\text{Unknown} : T_{i,j}$$

FIGURE 5.6
Finite difference method to solve the interior nodes next to the curved boundary.

5.2 Finite-Difference Energy Balance Method for 1-D Transient Heat Conduction

The heat equation for 1-D transient heat conduction with heat generation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}.$$

This equation is a parabolic equation. As discussed in Chapter 4, this equation can be solved analytically by using the separation of variables method, similarity method, Laplace transform method, or integral method. In this chapter, we would like to solve 1-D and 2-D transient conduction problems with various BCs by using the finite-difference explicit method and the finite-difference implicit method [1].

5.2.1 Finite-Difference Explicit Method

This is finite difference in space with an explicit form (lower bond). At a given interior point, heat conductions from the neighborhood points are based on the previous time-step temperatures in order to increase that point temperature during the incremental time-step change. This method is limited by the instability problem, but easy to understand and calculate.

The finite-difference format of the above 1-D transient heat conduction equation can be written as

$$\frac{(T_1^P - T_2^P)/\Delta x + (T_3^P - T_2^P)/\Delta x}{\Delta x} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{T_2^{P+1} - T_2^P}{\Delta t} \qquad (5.13)$$

Example 5.3

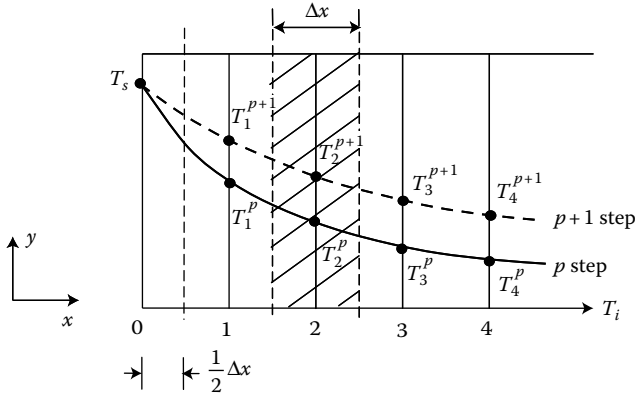Energy balance at the interior nodes, for example, node 2, as shown in Figure 5.7:

$$\sum q = \text{energy storage}$$

Lower bond:

$$k \cdot y \cdot 1 \frac{T_1^P - T_2^P}{\Delta x} + k \cdot y \cdot 1 \frac{T_3^P - T_2^P}{\Delta x} + \dot{q} \cdot \Delta x \cdot y \cdot 1 = \frac{\rho C_p \Delta x \cdot y \cdot 1 \cdot (T_2^{P+1} - T_2^P)}{\Delta t}. \qquad (5.14)$$

Let $\dot{q} = 0$, $\alpha = (k/\rho C_p)$, one obtains

$$T_1^P + T_3^P - 2T_2^P = \frac{1}{Fo}(T_2^{P+1} - T_2^P)$$

$$T_2^{P+1} = Fo(T_1^P + T_3^P) + (1 - 2Fo)T_2^P \qquad (5.15)$$

FIGURE 5.7
Finite difference energy balance method for one-dimensional transient heat conduction with given surface temperature boundary condition.

where time increment Δt with time step P, that is, $t = P\Delta t$, with $P = 0, 1, 2, \ldots$. *Fo* is a finite-difference form of the Fourier number $Fo = (\alpha\Delta t/\Delta x^2)$.

By hand calculation, temperature at $(P + 1)$ step is determined by the preceding time temperature at P step as sketched in Figure 5.7. For example, $P = 0$, $t = 0$, initial condition, $T_s^0 = T_1^0 = T_2^0 = T_3^0 = T_i$.

For stability, $(1 - 2Fo) \geq 0$, that is, $Fo \leq 1/2$, consider Δt as a very small value and Δx as a very large value.

Example 5.4

For the case of surface convection BC (node 0) as sketched in Figure 5.8:

$$h(T_\infty - T_0^p) + k\frac{T_1^p - T_0^p}{\Delta x} = \frac{T_0^{p+1} - T_0^p}{\Delta t}\rho C_p \left(\frac{\Delta x}{2}\right) \tag{5.16}$$

$$T_0^{p+1} = T_0^p + \frac{2h}{\rho C_p}\frac{\Delta t}{\Delta x}(T_\infty - T_0^p) + \frac{2\alpha\Delta t}{\Delta x^2}(T_1^p - T_0^p)$$

$$= 2Fo\left(T_1^p + Bi\, T_\infty\right) + (1 - 2\,Fo - 2\,Bi\,Fo)\,T_0^p \tag{5.17}$$

where

$$Fo = (\alpha\Delta t/\Delta x^2) = \text{Fourier number,}$$

$$Bi = \frac{h\Delta x}{k} = \text{Biot number.}$$

For stability criterion, $1 - 2Fo - 2Bi\,Fo \geq 0$, that is, $Fo\,(1 + Bi) \leq 1/2$.
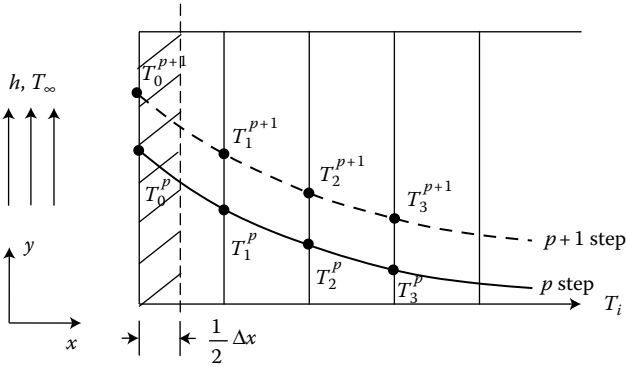
FIGURE 5.8
Finite difference energy balance method for one-dimensional transient heat conduction with convection boundary conditions.
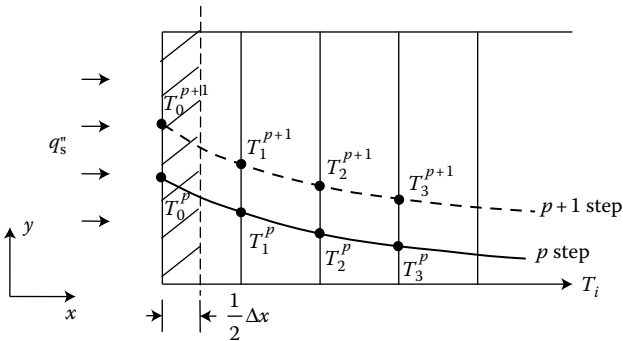
Example 5.5

For the case of surface heat flux BC as sketched in Figure 5.9:

$$q_s'' + k\frac{T_1^p - T_0^p}{\Delta x} = \frac{T_0^{p+1} - T_0^p}{\Delta t}\rho C_p\left(\frac{\Delta x}{2}\right) \tag{5.18}$$

$$T_0^{p+1} = \frac{\Delta t}{(\Delta x/2)\rho C_p}q_s'' + \frac{k\Delta t}{(\Delta x/2)\rho C_p}T_1^p + (1 - 2Fo)T_0^p \tag{5.19}$$

For stability, $Fo \leq 1/2$

FIGURE 5.9
Finite difference energy balance method for 1-D transient heat conduction with surface heat flux boundary condition.

5.2.2 Finite-Difference Implicit Method

This is finite difference in space with an implicit form (upper bond). At a given interior point, heat conductions from the neighborhood points are based on the new time-step temperatures in order to increase that point temperature during the incremental time-step change. This method has no instability problem, best accuracy, large Δt, and small Δx, but requires a computer to solve the matrix inverse problem.

Energy balance at the interior nodes (e.g., node 2, as shown in Figures 5.7 through 5.9):

Upper bond:

$$\sum q = \text{energy storage}$$

$$k\frac{T_1^{p+1} - T_2^{p+1}}{\Delta x} + k\frac{T_3^{p+1} - T_2^{p+1}}{\Delta x} + \dot{q} \cdot \Delta x \cdot 1 = \rho C_p \Delta x \frac{T_2^{p+1} - T_2^p}{\Delta t} \qquad (5.20)$$

Let $\dot{q} = 0$, $\alpha = (k/\rho C_p)$, $Fo = (\alpha \Delta t/\Delta x^2)$, one obtains

$$T_1^{p+1} - T_2^{p+1} + T_3^{p+1} - T_2^{p+1} = \frac{\Delta x^2}{\alpha \Delta t}(T_2^{p+1} - T_2^p)$$

$$(1 + 2\,Fo)T_2^{p+1} - Fo\,(T_1^{p+1} + T_2^{p+1}) = T_2^p \qquad (5.21)$$

There is no stability issue.

In general, $T_0, T_1, T_2, T_3, \ldots$ can be replaced by T_m or T_i, when $m = 1, 2, 3, \ldots$ or $i = 1, 2, 3, \ldots$.

5.3 2-D Transient Heat Conduction

The above-mentioned finite-difference energy balance method can be used for solving the 2-D transient heat conduction problem [1].

Let $T(x, y, t) = T(m, n, t)$ or $T(i, j, t)$, with $m = i = 1, 2, 3, \ldots$, $n = j = 1, 2, 3, \ldots$.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

x-direction net heat conduction + y-direction net heat conduction = temperature change of a small element $(\Delta x \, \Delta y \cdot 1)$.

Finite-difference explicit form (lower bond):

$$\frac{\left(T^p_{m+1,n} - T^p_{m,n}\right)\big/ \Delta x + \left(T^p_{m-1,n} - T^p_{m,n}\right)\big/ \Delta x}{\Delta x}$$

$$+ \frac{\left(T^p_{m,n+1} - T^p_{m,n}\right)\big/ \Delta y + \left(T^p_{m,n-1} - T^p_{m,n}\right)\big/ \Delta y}{\Delta y} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta t}$$

$$(5.22)$$

Let $\Delta x = \Delta y$, $\dot{q} = 0$, $Fo = \alpha \Delta t/\Delta x^2$, and temperature at $(P+1)$ step is determined by the preceding time temperature at P step as

$$T^{p+1}_{m,n} = Fo\left(T^p_{m+1,n} + T^p_{m-1,n} + T^p_{m,n+1} + T^p_{m,n-1}\right) + (1 - 4Fo)T^p_{m,n} \qquad (5.23)$$

For the stability criterion, $(1 - 4Fo) \geq 0$, that is, $Fo \leq 1/4$.
For the 1-D transient heat conduction problem, one obtains

$$T^{p+1}_m = Fo\,(T^p_{m+1} + T^p_{m-1}) + (1 - 2\,Fo)T^p_m$$

For stability, $(1 - 2Fo) \geq 0$, that is, $Fo \leq 1/2$
Finite-difference implicit form (upper bond):

$$\frac{\left(T^{p+1}_{m+1,n} - T^{p+1}_{m,n}\right)/\Delta x + \left(T^{p+1}_{m-1,n} - T^{p+1}_{m,n}\right)/\Delta x}{\Delta x}$$

$$+ \frac{\left(T^{p+1}_{m,n+1} - T^{p+1}_{m,n}\right)/\Delta y + \left(T^{p+1}_{m,n-1} - T^{p+1}_{m,n}\right)/\Delta y}{\Delta y} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{T^{p+1}_{m,n} - T^p_{m,n}}{\Delta t}$$

$$(5.24)$$

Let $\Delta x = \Delta y$, $\dot{q} = 0$, $Fo = \alpha \Delta t/\Delta x^2$, and temperature at $(P+1)$ step is determined by

$$T^{p+1}_{m+1,n} - T^{p+1}_{m,n} + T^{p+1}_{m-1,n} - T^{p+1}_{m,n} + T^{p+1}_{m,n+1}$$

$$- T^{p+1}_{m,n} + T^{p+1}_{m,n-1} - T^{p+1}_{m,n} = \frac{1}{Fo}\left(T^{p+1}_{m,n} - T^p_{m,n}\right)$$

$$(1 + 4\,Fo)T^{p+1}_{m,n} - Fo\left(T^{p+1}_{m+1,n} + T^{p+1}_{m-1,n} + T^{p+1}_{m,n+1} + T^{p+1}_{m,n+1}\right) = T^p_{m,n} \qquad (5.25)$$

For a 1-D transient heat conduction problem, one obtains

$$(1 + 2\,Fo)T^{p+1}_m - Fo(T^p_{m+1} + T^p_{m-1}) = T^p_m$$

It can be solved by a computer matrix, and there is no instability issue.
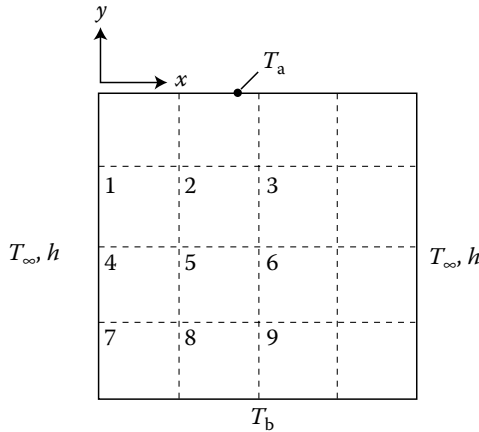
FIGURE 5.10
Finite difference method to solve a 2-D conduction problem.

In general, the finite-difference energy balance method (explicit or implicit method) can be used to solve 1-D, 2-D, and 3-D transient heat conduction problems for Cartesian, cylindrical, and spherical coordinates with various BCs.

Example 5.6

Figure 5.10 shows a long, square bar with opposite sides maintained at T_a and T_b, and the other two sides lose heat by convection to a fluid at T_∞. The conductivity of the bar material is k convective heat transfer coefficient is h. For the given mesh, use the finite-difference energy balance method to obtain a coefficient matrix $[A]$, temperature matrix $[T]$, and a column matrix $[C]$.

SOLUTION

Figure 5.10 shows the prescribed surface conditions and the nodes. Symmetry allows us to consider just nine nodes. For the interior, nodal temperatures are

$$T_2 = \frac{1}{4}(T_a + T_1 + T_3 + T_5)$$

$$T_3 = \frac{1}{4}(T_a + T_2 + T_2 + T_6)$$

$$T_5 = \frac{1}{4}(T_2 + T_4 + T_6 + T_8)$$

$$T_6 = \frac{1}{4}(T_3 + T_5 + T_5 + T_9)$$

$$T_8 = \frac{1}{4}\left(T_5 + T_7 + T_9 + T_b\right)$$

$$T_9 = \frac{1}{4}\left(T_6 + T_8 + T_8 + T_b\right)$$

For the boundary, nodal temperatures are

$$-\left(\frac{k\Delta y}{\Delta x} + \frac{k\Delta x}{\Delta y} + h\Delta y\right)T_1 + \frac{k\Delta y}{\Delta x}T_2 + \frac{k\Delta x}{2\Delta y}T_4 = -\frac{k\Delta x}{2\Delta y}T_a - h\Delta y T_\infty$$

$$\frac{k\Delta x}{2\Delta y}T_1 - \left(\frac{k\Delta y}{\Delta x} + \frac{k\Delta x}{\Delta y} + h\Delta y\right)T_4 + \frac{k\Delta y}{\Delta x}T_5 + \frac{k\Delta x}{2\Delta y}T_7 = -h\Delta y T_\infty$$

$$\frac{k\Delta x}{2\Delta y}T_4 - \left(\frac{k\Delta y}{\Delta x} + \frac{k\Delta x}{\Delta y} + h\Delta y\right)T_7 + \frac{k\Delta y}{\Delta x}T_8 = -\frac{k\Delta x}{2\Delta y}T_b - h\Delta y T_\infty$$

In matrix form $[A][T] = [C]$,

$$[A] =$$

$$
\begin{bmatrix}
-\left(\dfrac{k\Delta y}{\Delta x} + \dfrac{k\Delta x}{\Delta y} + h\Delta y\right) & \dfrac{k\Delta y}{\Delta x} & & & \dfrac{k\Delta x}{2\Delta y} \\
1 & -4 & 1 & & \\
& 2 & -4 & & \\
\dfrac{k\Delta x}{2\Delta y} & & & -\left(\dfrac{k\Delta y}{\Delta x} + \dfrac{k\Delta x}{\Delta y} + h\Delta y\right) \\
& & 1 & & 1 \\
& & 1 & & \\
& & & & \dfrac{k\Delta x}{2\Delta y}
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & & & & & & \\
& 1 & & & & & \\
\dfrac{k\Delta y}{\Delta x} & & & \dfrac{k\Delta x}{2\Delta y} & & & \\
-4 & 1 & & & & 1 & \\
2 & -4 & & & & & 1 \\
& & -\left(\dfrac{k\Delta y}{\Delta x} + \dfrac{k\Delta x}{\Delta y} + h\Delta y\right) & \dfrac{k\Delta y}{\Delta x} & & \\
1 & & 1 & & -4 & 1 \\
& 1 & & & & 2 & -4
\end{bmatrix}
$$

$$[T] = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} \quad [C] = \begin{bmatrix} -\dfrac{k\Delta x}{2\Delta y}T_a - h\Delta y T_\infty \\ -T_a \\ -T_a \\ -h\Delta y T_\infty \\ 0 \\ 0 \\ -\dfrac{k\Delta x}{2\Delta y}T_b - h\Delta y T_\infty \\ -T_b \\ -T_b \end{bmatrix}$$

Remarks

The finite-difference method is a very powerful numerical technique to solve many engineering application problems. As long as you know how to perform the basic energy balance at the interior nodes as well as at the boundary nodes, this method essentially can solve all kinds of heat conduction problems with complex thermal BCs. In the undergraduate-level heat transfer, students are normally required to perform simple energy balance at any specified node inside a 2-D steady-state solid material and on the boundary.

   In the intermediate-level heat transfer, we are more focused on how to perform simple energy balance as well as how to discretize the heat conduction equation in order to solve the 1-D and 2-D steady-state heat conduction problems with various BCs by using the matrix inverse method. We also put in effort to solve the 1-D and 2-D transient heat conduction problems with various BCs by using the finite-difference implicit method and explicit method. In general, the same technique can be used to solve heat conduction problems with cylindrical and spherical coordinates.

PROBLEMS

5.1. Refer to Figure 5.4, show the matrices $[A]$, $[T]$, and $[C]$, with the following grid distributions:

(1)	$m = 1,2,3,4,5$	(2)	$m = 1,2,3,4$	(3)	$m = 1,2,3$
	$n = 1,2,3,4,5$		$n = 1,2,3,4$		$n = 1,2,3$

5.2. Derive finite-difference energy balance equations, and show the matrices $[A]$, $[T]$, and $[C]$, for a 1-D hollow cylinder with the following BCs:

(1) $r = r_1, T = T_1$	(2) $r = r_1, -k(\partial T/\partial r)$
	$\quad = h_1(T_{\infty 1} - T_1)$
$\quad r = r_N, T = T_N$	$r = r_N, -k(\partial T/\partial r)$
	$\quad = h_N(T_N - T_{\infty N})$
(3) $r = r_1, -k(\partial T/\partial r) = h_1(T_{\infty 1} - T_1)$	
$\quad r = r_N, -k(\partial T/\partial r) = q_s''$	

5.3. A semi-infinite stainless-steel block is initially at $T_i = 20°C$. The surface has an emissivity $\varepsilon = 1.0$ and is placed in a large enclosure of T_{sur} of $20°C$. Suddenly, the surface is exposed to a hot-air flow of $T_\infty = 600°C$ and $h = 100$ w/m²K.

At an instant time of t s, temperatures T_1 and T_2 have been calculated as shown below. Predict the surface temperature after $(t + 100)$ s. Use the forward-difference energy balance method. Given

$\alpha = 4 \times 10^{-6} m^2/s$	$k = 15.07$ w/m K	$\rho = 7900$ kg/m³
$C = 477$ J/kg K	$\sigma = 5.67 \times 10^{-8}$ w/m² K⁴	$\varepsilon = 0.1$

At time t s: $T_1 = 400°C$ and $T_2 = 380°C$
Find: $T_1 = ?$ after $(t = \Delta t)$ s where $\Delta t = 100$ s.

5.4. Given a very long and wide fin with a height of $2L$. The base of the fin is maintained at a uniform temperature of T_b. The top and bottom surfaces of the fin are exposed to a fluid whose temperature is T_∞ $(T_\infty < T_b)$. The convective heat transfer coefficient between the fin surfaces and the fluid is h.

a. Sketch the steady 2-D temperature distribution in the fin.

b. If you were to determine the steady 2-D temperature distribution in the fin using a finite-difference numerical method, you would solve a set of algebraic nodal equations simultaneously for the temperatures at a 2-D array of nodes. Derive the equation for a typical node on one of the surfaces of the fin. Please do not simplify the equation.

c. Using the method of separation of variables, *derive* an expression for the steady local temperature in the fin, in terms of the thermal conductivity of the fin, k, the convective heat transfer coefficient, h, the half-height of the fin, L, and the base and fluid temperatures, T_b and T_∞.

Note that

$$\int_0^W [\cos^2(aw)] \, dw = (1/4a)[2aW + \sin(2aW)]$$

$$\text{and} \int_0^W [\cos(aw) \cdot \cos(bw)] \, dw = 0, \quad \text{when } a \neq b.$$

5.5. A 3-mm-diameter rod that is 120 mm in length is supported by two electrodes within a large vacuum enclosure. Initially, the rod is in

equilibrium with the electrodes and its surroundings. Suddenly, an electrical current is passed through the rod.

  a. Using a first law analysis, what is the energy balance on the rod?

  b. Using $\left(A_c = (\pi D^2/4), P = \pi D\right)$, derive the explicit finite-difference expression for node (n). Recall that heat generation follows the $I^2 R_e$ law and that R_e is defined as $R_e = \rho_e \Delta x/A_c$. Express your answer using the Fourier number in the explicit finite-difference form.

  c. What are the stability criteria at node (n)?

5.6. Refer to Figure 5.4, use the finite-difference explicit method to derive the energy balance during the transient for the following grid distributions:

(1)	$m = 1,2,3,4,5$	(2)	$m = 1,2,3,4$	(3)	$m = 1,2,3$
	$n = 1,2,3,4,5$		$n = 1,2,3,4$		$n = 1,2,3$

5.7 Refer to Figure 5.4, use the finite-difference implicit method to derive the energy balance during the transient for the following grid distributions:

(1)	$m = 1,2,3,4,5$	(2)	$m = 1,2,3,4$	(3)	$m = 1,2,3$
	$n = 1,2,3,4,5$		$n = 1,2,3,4$		$n = 1,2,3$

5.8. Use the finite-difference explicit method, and derive finite-difference energy balance equations for a 1-D hollow cylinder during the transient with the following BCs:

(1) $r = r_1, T = T_1$ (2) $r = r_1, -k(\partial T/\partial r) = h_1(T_{\infty 1} - T_1)$
    $r = r_N, T = T_N$ $r = r_N, -k(\partial T/\partial r) = h_N(T_N - T_{\infty N})$

(3) $r = r_1, -k(\partial T/\partial r)$
    $\quad = h_1(T_{\infty 1} - T_1)$
    $r = r_N, -k(\partial T/\partial r) = q_s''$

5.9. Use the finite-difference implicit method, and derive finite-difference energy balance equations for a 1-D hollow cylinder during the transient with the following BCs:

(1) $r = r_1, T = T_1$ (2) $r = r_1, -k(\partial T/\partial r) = h_1(T_{\infty 1} - T_1)$
    $r = r_N, T = T_N$ $r = r_N, -k(\partial T/\partial r) = h_N(T_N - T_{\infty N})$

(3) $r = r_1, -k(\partial T/\partial r)$
    $\quad = h_1(T_{\infty 1} - T_1)$
    $r = r_N, -k(\partial T/\partial r) = q_s''$

5.10. Derive Equations 5.23 and 5.25 for 3-D transient heat conduction problems.

References

1. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.
2. A. Mills, *Heat Transfer*, Richard D. Irwin, Inc., Boston, MA, 1992.
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6

Heat Convection Equations

6.1 Boundary-Layer Concepts

For flow moving over a solid body, a hydrodynamic (or velocity) boundary layer is formed near the solid surface. Hydrodynamic (or velocity) boundary layer is the region where the fluid velocity changes from its free-stream value to zero at the solid surface. For example, considering the flow over a flat plate as shown in Figure 6.1, due to viscosity of the fluid, velocity gradually decreases from its free-stream maximum value to zero at the flat plate (assuming that the fluid particle on the surface is not moving). The fluid particle wants to move faster as its free-stream value, but viscous force tries to resist it from moving over the solid surface. Therefore, a hydrodynamic boundary-layer thickness (where velocity is about 99% of the free-stream value) is developed over a solid surface due to fluid viscosity.

The velocity profile and the associated hydrodynamic boundary-layer thickness over a flat plate will be solved in the later sections. In general, hydrodynamic boundary-layer thickness grows with the square root of distance from the leading edge of the flat plate. Figure 6.1 shows a typical velocity profile over a flat plate in the laminar flow ($Re < 300 \times 10^3$) and turbulent flow ($Re > 300 \times 10^3$) boundary-layer region, respectively. Once the velocity profile over a flat plate, $u(y)$, at a given distance x is determined, the hydrodynamic boundary-layer thickness, the shear stress on the surface, and the friction factor (or friction coefficient) can be obtained as follows:

For external flow,

$$Re = \frac{\text{Inertia force}}{\text{Viscous force}} = \frac{\rho U_\infty x}{\mu} = \frac{U_\infty x}{\nu}$$

The flow is a laminar flow if $Re < 300 \times 10^3$, and a turbulent flow if $Re > 300 \times 10^3$.

Flow boundary-layer thickness $\delta(x) \sim \sqrt{x}$
Dynamic viscosity μ,
Kinematic viscosity $\nu = \mu/\rho$
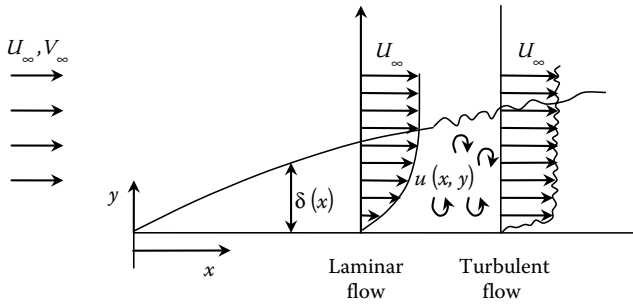
FIGURE 6.1
Hydrodynamic and thermal boundary layer over a flat plate (if $Pr = 1$).

Shear stress

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = C_f \cdot \frac{1}{2}(\rho U_\infty^2 - 0)$$

Friction coefficient:

$$C_f = \frac{\tau_w}{(1/2)(\rho U_\infty^2 - 0)} = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{\mu (\partial u/\partial y)|_{y=0}}{(1/2)\rho U_\infty^2} \sim \frac{\mu}{\rho} \frac{(U_\infty/\delta)}{U_\infty^2} \sim \nu \frac{1}{\delta U_\infty}$$

C_f is a function of Reynolds number, that is,

$$C_f = aRe^b; \quad C_f = aRe^b \sim \frac{1}{Re};$$

Reynolds number is defined as the fluid inertia force against viscous force (i.e., the fluid particle tries to move but viscosity tries to resist it from moving) and is a combination of velocity, viscosity, and length (distance measured from the leading edge of the plate). When Reynolds number is approximately less than 300×10^3, the fluid particle moves like laminar, layer to layer from free-stream velocity to zero velocity on the solid surface, and creates shear stress over the solid surface. When the Reynolds number is greater than 300×10^3, the fluid particle tends to become unstable (random motion) and gradually transitions into the turbulent flow boundary layer. In the laminar boundary layer, the velocity profile gradually changes from free-stream value to zero on the surface as a parabolic shape. But, in the turbulent boundary layer, the velocity profile remains fairly uniform as the free-stream value till near the surface and then suddenly changes to zero on the surface. This is due to turbulent mixing (the particle moves up and down, back and forth) so that free-stream velocity is able to move closer to the surface. From the application point of view, shear stress (viscosity × velocity gradient at the surface, i.e., $\tau_w = \mu(\partial u/\partial y)|_{y=0}$) decreases with decreasing velocity gradient and viscosity.
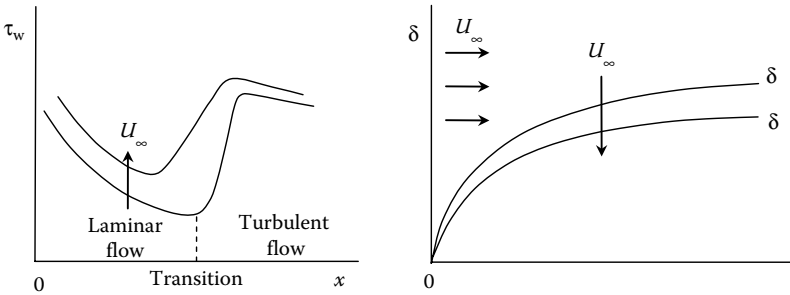
FIGURE 6.2
Hydrodynamic boundary layer, friction factor, and shear stress profile.

Since velocity gradient decreases (because boundary-layer thickness grows) with increasing distance due to viscosity, shear stress (related to pressure loss) and friction factor decrease with increasing distance from the leading edge of the flat plate. However, when the flow transitions into the turbulent boundary layer, pressure loss is much greater than that in the laminar flow portion. This is because a major portion of pressure loss is required in order to maintain turbulence random motion in the turbulent boundary layer.

It is noted that boundary-layer thickness decreases with increasing square root of free-stream velocity, and friction factor decreases with increasing free-stream velocity; however, shear stress increases with increasing free-stream velocity (thinner boundary layer and larger velocity gradient) as sketched in Figure 6.2. Similarly, friction factor decreases with increasing Reynolds number, but, shear stress increases with increasing Reynolds number.

For hot flow moving over a cold solid body, a thermal (temperature) boundary layer is formed around the solid surface. The thermal (or temperature) boundary layer is the region where the fluid temperature changes from its free-stream value to that at the solid surface. Heat transfer can take place either from hot flow to the cold surface or from the heated surface to cold flow as shown in Figure 6.3. For example, considering the hot flow over a cold flat
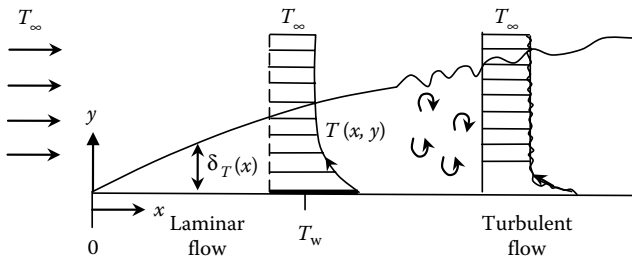
FIGURE 6.3
Thermal boundary layer over a heated flat plate.

plate as shown in Figure 6.1, due to conductivity of fluid and velocity distribution, the temperature gradually decreases from its free-stream maximum value to that at the flat plate. The hot fluid particle conducts heat from the free-stream into the cold surface through the velocity boundary layer. Therefore, a thermal boundary-layer thickness is developed over a solid surface due to fluid flow.

The temperature profile and associated thermal boundary-layer thickness over a flat plate is solved in Chapter 7. In an ideal case (assume $Pr = 1$), the thermal boundary layer is identical to hydrodynamic boundary layer as shown in Figure 6.1 or 6.3. In this ideal case, the temperature profile is the same as the velocity profile through the entire boundary layer over the flat plate. Once we determine the temperature profile over a flat plate, $T(y)$ at a given distance x, the thermal boundary-layer thickness, the heat flux on the surface, and the heat transfer coefficient (or Nusselt number) can be obtained as follows:

Thermal boundary-layer thickness $\delta_T(x) \sim \sqrt{x}$

If $Pr = 1$, then $\delta(x) = \delta_T(x)$.

At the body surface, the heat flux is

$$q_w'' = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} = -k_f \left.\frac{\partial T}{\partial y}\right|_{y=0} \equiv h(T_w - T_\infty)$$

The heat transfer coefficient h with the unit of $W/m^2\,k$ can be expressed as

$$h = \frac{-k_f\,(\partial T/\partial y)|_{y=0}}{T_w - T_\infty} \sim \frac{-k_f\,((T_\infty - T_w)/\delta_T)}{T_w - T_\infty} \sim \frac{k_f}{\delta_T} \sim \frac{k_f}{\delta} \sim k_f U_\infty$$

In the laminar boundary layer, the temperature profile gradually changes from the free-stream value to the surface as a parabolic shape, but, in the turbulent boundary layer, the temperature profile remains fairly uniform from the free-stream to near the surface and then suddenly changes to the surface value. This is due to turbulent mixing (particle moves up and down, back and forth); the hot (or cold) free-stream particle is able to move next to the cold (or heated) surface due to random motion. From application point of view, the heat flux (fluid conductivity × temperature gradient at the surface) and heat transfer coefficient (heat flux/temperature difference between the free-stream and the surface) decrease with decreasing temperature gradient and fluid conductivity. Since temperature gradient decreases (because thermal boundary-layer thickness increases) with increasing distance due to fluid thermal conductivity, the heat flux (related to heat transfer rate) and heat transfer coefficient decrease with increasing distance from the leading edge of the flat plate. However, when flow transitions into the turbulent boundary layer, the heat flux (and heat transfer coefficient) is much greater than the laminar flow portion. This is because a major portion of heat transfer is due to turbulent random motion in the turbulent boundary layer.

It is noted that heat flux is proportional to the heat transfer coefficient and temperature difference between the free-stream and the surface. The heat transfer coefficient increases with increasing free-stream velocity (thinner hydrodynamic and thermal boundary-layer thickness) and fluid thermal conductivity as sketched in Figure 6.4. This implies that the heat transfer coefficient, heat flux, and Nusselt number (the dimensionless heat transfer coefficient) increase with Reynolds number. Another import parameter in heat transfer study is the role of Prandtl number (*Pr*). Prandtl number is a ratio of kinematic viscosity to thermal diffusivity, or a ratio of velocity to temperature boundary-layer thickness as sketched in Figure 6.4. For example, thermal boundary-layer thickness is identical to hydrodynamic boundary-layer thickness if $Pr = 1$ as discussed above. However, in real life, different fluids have different Prandtl numbers. In Chapters 7, 8, and 10, we will see that Nusselt number is proportional to Reynolds number and Prandtl number for both laminar and turbulent flow problems (with different power and constants).

Nusselt number:

$$Nu = \frac{hx}{k} = aRe^b Pr^n$$

Prandtl number:

$$Pr = \frac{\nu}{\alpha} = \frac{\mu/\rho}{k/\rho C_p} = \frac{\mu C_p}{k} \sim \frac{\delta}{\delta_T}$$

Prandtl number is a property of fluid; it shows the ratio of momentum transfer versus heat transfer.

Air or gas $Pr = 0.7$
Water $Pr = 2 \sim 20$
Oil $Pr = 100 \sim 1000$
Liquid metal $Pr = 0.01 \sim 0.001$

When temperature increases, the viscosity and Prandtl number for oil decrease.
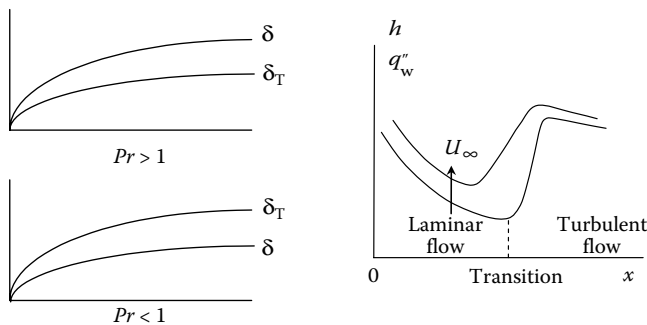
FIGURE 6.4
Thermal boundary layer, heat transfer coefficient, and heat flux profile.

6.2   General Heat Convection Equations

For flow moving over a heated or cooled solid body, the general 3-D pressure profiles $P(x,y,z,t)$, velocity profiles $u(x,y,z,t)$, $v(x,y,z,t)$, and $w(x,y,z,t)$, and temperature profiles $T(x,y,z,t)$ can be obtained by solving the following continuity, momentum, and energy equations inside the hydrodynamic and thermal boundary layers over the heated or cooled solid surface [1–3].

Conservation of mass (continuity equation):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \tag{6.1}$$

where

$$V = iu + jv + kw$$

and

$$\nabla = i\left(\frac{\partial}{\partial x}\right) + j\left(\frac{\partial}{\partial y}\right) + k\left(\frac{\partial}{\partial z}\right)$$

are the velocity vector and the del operator for unit vectors, i, j, and k in the x-, y-, and z-direction, respectively.

Conservation of momentum:

$$\rho\frac{DV}{Dt} = -\nabla P + \mu\nabla^2 V + \rho g \tag{6.2}$$

Conservation of energy:

$$\rho\frac{Dh}{Dt} = \frac{DP}{Dt} + \nabla \cdot k\nabla T + \mu\Phi + \dot{q} \tag{6.3}$$

where $h = e + (1/2)V \cdot V$, and e is the specific internal energy. Φ is often called the dissipation function with the form

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \tag{6.4}$$

$\dot{q}$ is the heat generation in unit volume.

6.3 2-D Heat Convection Equations

Many real-life applications for flow moving over a solid body can be modeled as a 2-D boundary-layer flow and heat transfer problems. We need to know the 2-D velocity profiles $u(x, y, t)$ and $v(x, y, t)$ in order to calculate the wall shear stress (related to pressure loss) and the friction factor along the surface for a given fluid at given flow conditions, as well as the 2-D temperature profiles $T(x, y, t)$ in order to calculate the wall heat flux (related to heat transfer rate) and the heat transfer coefficient along the surface for a given fluid at given flow and thermal BCs. Since flow moving due to pressure difference between upstream and downstream flow and viscous boundary layer effect over the solid surface, it is necessary to perform conservation of mass (continuity equation) and momentum (momentum equation) through the boundary layer in order to solve for velocity distributions over the surface. Similarly, since heat transfer due to temperature difference between the free-stream and the solid surface and flow moving carrying energy, it is necessary to perform conservation of energy (energy equation) through the thermal boundary layer in order to solve for temperature distributions over the heated (or cooled) surface. Consider a small 2-D differential fluid element $(dxdy)$ at any point of the boundary layer, the following shows a step-by-step derivation of 2-D conservation equations for mass, momentum, and energy through hydrodynamic and thermal boundary layers over a solid surface [1–3].

Conservation of mass:

Perform mass balance shown in Figure 6.5:

$$-\frac{\partial}{\partial x}(\rho u\, dy)\, dx - \frac{\partial}{\partial y}(\rho v\, dx)\, dy = \frac{\partial}{\partial t}(\rho\, dx\, dy) \tag{6.5}$$
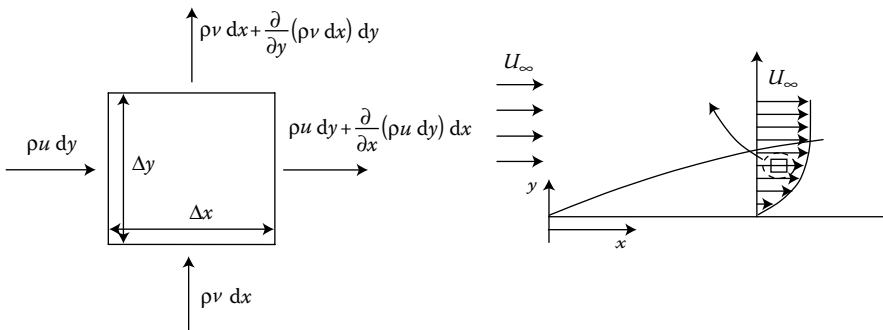
FIGURE 6.5
Conservation of mass.

If the flow is steady-state,

$$-\frac{\partial}{\partial x}(\rho u\, dy)\, dx - \frac{\partial}{\partial y}(\rho v\, dx)\, dy = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{6.6}$$

For incompressible flow, $\rho = \text{const}$, the continuity equation can be simplified as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.7}$$

Conservation of momentum:
From Newton's second law, net force exerting on a body equals the momentum change.

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

where a_x is the acceleration in the x–direction, and a_y is the acceleration in the y-direction.

The force exerting on a control volume and the momentum change are shown in Figure 6.6.

$$\underbrace{\frac{\partial}{\partial x}\sigma_x}_{\substack{\text{normal}\\\text{stress}}} + \underbrace{\frac{\partial}{\partial y}\tau_{xy}}_{\substack{\text{shear}\\\text{stress}}} - \underbrace{\frac{\partial P_x}{\partial x}}_{\substack{\text{pressure}\\\text{gradient}}} = \underbrace{\frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial y}\left(\rho u v\right)}_{\text{convective term}} + \underbrace{\frac{\partial}{\partial t}\left(\rho u\right)}_{\substack{\text{unsteady}\\\text{term}}} \tag{6.8}$$

By Navier–Stokes for Newtonian incompressible fluid:

$$\sigma_x = 2\mu\frac{\partial u}{\partial x} \tag{6.9}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6.10}$$

Substituting Equations 6.9 and 6.10 into Equation 6.8, we obtain

$$\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] - \frac{\partial P_x}{\partial x} = \frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial y}\left(\rho u v\right) + \frac{\partial}{\partial t}\left(\rho u\right)$$
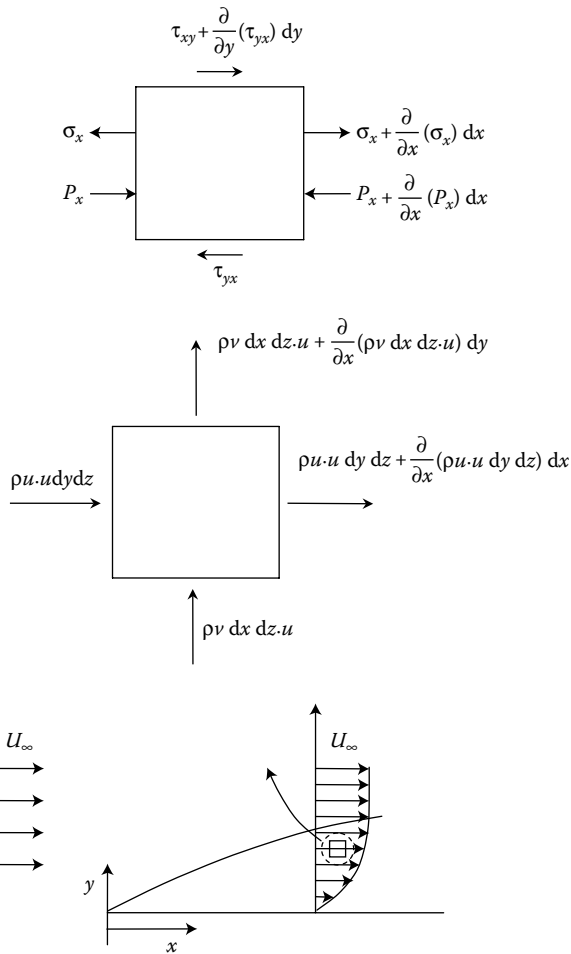
FIGURE 6.6
Conservation of momentum.

For a steady-state, constant-property flow, the left-hand side can be expressed as

$$2\mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 v}{\partial x \partial y} = \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 v}{\partial x \partial y}$$

$$= \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2} + \mu \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

$$\sim \mu \frac{\partial^2 u}{\partial y^2} \quad \text{if } \mu \frac{\partial^2 u}{\partial x^2} \ll \mu \frac{\partial^2 u}{\partial y^2}$$

The right-hand side can be expanded as

$$\frac{\partial}{\partial x}\left(\rho u^2\right) + \frac{\partial}{\partial y}\left(\rho uv\right) = 2\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho u\frac{\partial v}{\partial y}$$

$$= \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^{\!0}$$

Substitute the mass conservation equation into and we will obtain the *x*-direction momentum equation

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{convection}} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x}}_{\text{pressure}} + \underbrace{v\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial y^2}}_{\text{stress}}. \tag{6.11}$$

Similarly, we will obtain the *y*-direction momentum equation from Equation 6.11 by changing *u* and *v*, *x* and *y*:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + v\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^2 v}{\partial y^2} \tag{6.12}$$

Conservation of energy:
Unsteady state:
Perform energy balance shown in Figure 6.7,

$$-\frac{\partial q_x}{\partial x}\,dx - \frac{\partial q_y}{\partial y}\,dy - \frac{\partial}{\partial x}\left(\rho u\,dy \cdot C_p \cdot T\right)dx - \frac{\partial}{\partial y}\left(\rho v\,dx \cdot C_p \cdot T\right)dy$$

$$= \frac{\partial\left(\rho\,dx\,dy \cdot C_p \cdot T\right)}{\partial t} \tag{6.13}$$

$$\underbrace{\frac{\partial\left(\rho C_p T\right)}{\partial t}}_{\substack{\text{unsteady steady}}} + \underbrace{\frac{\partial}{\partial x}\left(\rho u C_p T\right) + \frac{\partial}{\partial y}\left(\rho v C_p T\right)}_{\text{convection}}$$

$$= \underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}_{\text{heat diffusion}} + \underbrace{\frac{\dot{q}}{k}}_{\substack{\text{heat}\\\text{generation}}} + \underbrace{\mu\Phi}_{\substack{\text{heat dissipation}\\\text{source due to fricion}}} \tag{6.14}$$

For steady-state constant properties,

$$\frac{\partial\left(uT\right)}{\partial x} + \frac{\partial\left(vT\right)}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{\mu}{\rho C_p}\Phi. \tag{6.15}$$

$$\mu\Phi = \mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]\right\} \sim \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{6.16}$$
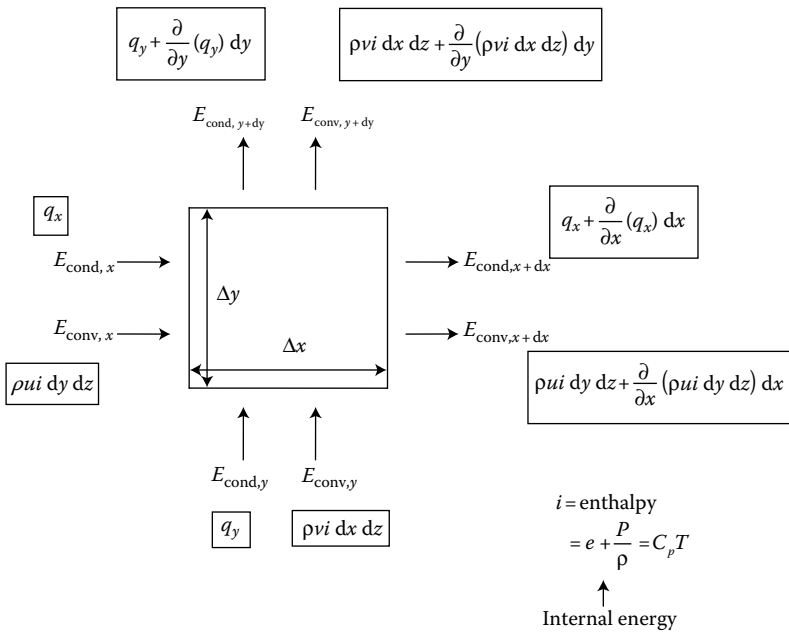
FIGURE 6.7
Conservation of energy.

6.4 Boundary-Layer Approximations

The above-derived boundary equations are not easy to solve analytically. Boundary-layer approximations, as shown in Figure 6.8, can be employed to simplify the boundary equations as follows: velocity in the x-direction is greater than that in the y-direction, streamwise velocity change in the y-direction is greater than that in the x-direction; temperature change in the
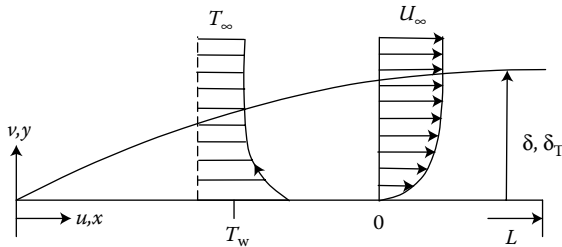
FIGURE 6.8
Boundary-layer approximations.

y-direction is greater than that in the x-direction.

$$u \gg v$$

$$\frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

$$\frac{\partial T}{\partial y} \gg \frac{\partial T}{\partial x}$$

Therefore, the continuity equation remains the same. Assume steady-state constant properties; the momentum equation in the x- and y-directions can be simplified as

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\frac{\partial^2 u}{\partial y^2} \tag{6.17}$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial y} = 0 \tag{6.18}$$

For an incompressible flow, that is, $M < 0.2$, $\Phi \sim 0$, and so the energy equation becomes

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{6.19}$$

Outside of the boundary layer, it is potential flow (μ effect $\to 0$, $v \to 0$, $\partial u/\partial y \to 0$); Equation 6.17 reduces to

$$U_\infty\frac{\partial U_\infty}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \tag{6.20}$$

Note that Equation 6.18 implies that there is no pressure change in the y-direction within the boundary layer. Equation 6.20 implies that pressure change in the x-direction within the boundary layer can be predetermined from velocity and its velocity change in the x-direction outside of the boundary layer. Therefore, Equation 6.20 can be substituted into Equation 6.17 to solve for the velocity profiles inside the boundary layer.

6.4.1   Boundary-Layer Similarity/Dimensional Analysis

Here we want to generalize the application of the above-derived boundary-layer approximation equations. Most often, one wants to apply the boundary-layer equations from a small-scale test model to a large-scale application or from a large-scale test model to a small-scale application. This is called boundary-layer similarity or dimensional analysis. The following is

a common way of converting dimensional parameters into nondimensional parameters [2].

$$\text{Let } x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{U_\infty},$$

$$T^* = \frac{T - T_w}{T_\infty - T_w}, \quad P^* = \frac{P}{\rho U_\infty^2}, \tag{6.21}$$

Then, the conservation equations can be written as

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \frac{v}{U_\infty L} \cdot \frac{\partial^2 u^*}{\partial y^{*2}}$$

$$\frac{\partial P^*}{\partial y^*} = 0$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\alpha}{U_\infty L} \cdot \frac{\partial^2 T^*}{\partial y^{*2}}$$

And the coefficient is

$$\frac{\alpha}{U_\infty L} = \frac{1}{(U_\infty L / v) \cdot (v / \alpha)} = \frac{1}{Re \cdot Pr} \tag{6.22}$$

The above similarity functional solutions can be written as

$$u^* = f_1 \left(x^*, y^*, Re_L, \frac{dP^*}{dx^*} \right)$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \mu \frac{U_\infty}{L} \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0}$$

where

$$\left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} = f_2 \left(x^*, Re_L, \frac{dP^*}{dx^*} \right)$$

$$C_f = \frac{\tau_w}{(1/2)\rho V_\infty^2} = \frac{m(U_\infty/L)}{(1/2)\rho U_\infty^2} f_2 \left(x^*, Re_L, \frac{dP^*}{dx^*} \right)$$

$$C_f = \frac{2}{Re_L} f_2 \left(x^*, Re_L, \frac{dP^*}{dx^*} \right) \tag{6.23}$$

Special case: when flow over a flat plate $dP^*/dx^* = 0$, the average friction factor can be determined from Reynolds number as

$$\bar{C}_f = \frac{2}{Re_L} f_2\, (Re_L) = aRe_L^m \tag{6.24}$$

Similarly, the temperature and heat transfer coefficient can be obtained as

$$T^* = f_3\left(x^*,\, y^*,\, Re_L,\, Pr,\, \frac{dP^*}{dx^*}\right)$$

$$h = \frac{-k\,(\partial T/\partial y)\big|_{y=0}}{T_w - T_\infty} \sim \frac{k}{L}\frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} = \frac{k}{L} f_4\left(x^*,\, Re_L,\, Pr,\, \frac{dP^*}{dx^*}\right)$$

$$Nu_L \equiv \frac{hL}{k} = f_4\left(x^*,\, Re_L,\, Pr,\, \frac{dP^*}{dx^*}\right) \tag{6.25}$$

For flow over a flat plate, $dP^*/dx^* = 0$, the average Nusselt number can be determined from Reynolds number and Prandtl number as

$$\overline{Nu}_L \equiv \frac{\overline{h}L}{k} = f_5\,(Re_L,\, Pr) = aRe_L^m Pr^n \tag{6.26}$$

The above analysis concludes that, for flow over a flat plate, the local friction factor (at a given location x) is a function of Reynolds number only, and the local heat transfer coefficient or Nusselt number (at a given location x) is a function of Reynolds number as well as Prandtl number.

6.4.2   Reynolds Analogy

Assuming that $Pr = 1$ (approximation for air, $Pr = 0.7$), the above friction factor and the Nusselt number can be reduced to the following:

$$C_{fx} = \frac{2}{Re_L} f_2\left(x^*,\, Re_L\right) \tag{6.27}$$

$$Nu_x = f_4\left(x^*,\, Re_L,\, Pr\right) \tag{6.28}$$

If $f_2 = f_4$, $Pr = 1$,

$$C_f \frac{Re_L}{2} = f_2 = f_4 = Nu$$

$$\frac{1}{2}C_f = \frac{Nu}{Re \cdot Pr} = St = \frac{(hL/k)}{(\rho VL/\mu)\cdot(\mu C_p/k)} = \frac{h}{\rho C_p V} \tag{6.29}$$

Reynolds analogy:

$$\frac{1}{2}C_f = St \tag{6.30}$$

Experimentally, we obtained

$$\frac{1}{2}C_f \, Pr^{-2/3} = St \tag{6.31}$$

where $0.6 \leq Pr \leq 60$.

The importance of Reynolds analogy is that one can estimate the heat transfer coefficient (or the Stantan number (St)) from a given (or a predetermined) friction factor, or one can calculate the friction factor from a given (or predetermined) heat transfer coefficient (or the St) for a typical 2-D boundary layer flow and the heat transfer problem. The original Reynolds analogy is shown in Equation 6.30. However, Equation 6.31 still can be called Reynold's analogy including Prandtl number effect.

Remarks

This chapter provides the basic concept of boundary-layer flow and heat transfer; it focuses on how to derive 2-D boundary-layer conservations for mass, momentum, and energy; boundary-layer approximations; non-dimensional analysis; and Reynolds analogy. Students have come across these equations in their undergraduate-level heat transfer. However, in the intermediate-level heat transfer, students are expected to fully understand how to obtain these equations.

PROBLEMS

6.1. For hot-gas flow (velocity V_∞, temperature T_∞) over a cooled convex surface (surface temperature T_s), answer the following questions:

   a. Sketch the "thermal boundary-layer thickness" distribution on the entire convex surface and explain the results.

   b. Sketch the possible local heat transfer coefficient distribution on the convex surface and explain the results.

   c. Define the similarity parameters (dimensionless parameters) that are important to determine the local heat transfer coefficient on the convex surface.

   d. Write down the relationship among those similarity parameters and give explanations.

   e. Write down how to determine the local heat flux from the convex surface.

6.2. For cold-gas flow (velocity V_∞, temperature T_∞) over a heated convex surface (surface temperature T_s), answer the following questions:

    a. Sketch the "thermal boundary-layer thickness" distribution on the entire convex surface and explain the results.

    b. Sketch the possible local heat transfer coefficient distribution on the convex surface and explain the results.

    c. Define the similarity parameters (dimensionless parameters) that are important to determine the local heat transfer coefficient on the convex surface.

    d. Write down the relationship among those similarity parameters and give explanations.

    e. Write down how to determine the local heat flux from the convex surface.

6.3. Derive Equations 6.11, 6.12, 6.15, and 6.16.

6.4. Derive Equations 6.23 and 6.25.

6.5. Derive Equation 6.30.

References

1. W. Rohsenow and H. Choi, *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.
2. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.
3. H. Schlichting, *Boundary-Layer Theory*, Sixth Edition, McGraw-Hill Book Company, New York, NY, 1968.

7

External Forced Convection

7.1   Laminar Flow and Heat Transfer over a Flat Surface: Similarity Solution

External forced convection is that flow moves over the external surface of a solid body and forms hydrodynamic and thermal boundary layers around the surface. There are two well-known methods to solve external boundary-layer flow and heat transfer problems. One is the similarity method to obtain the exact solution. The other is the integral method to obtain the approximate solution. This section begins with the similarity method [1–6]. Figure 7.1 shows stream lines for flow over a flat plate. The velocity along each stream line looks quite similar to each other. Define stream function Ψ and derive two nonlinear PDEs to one nonlinear PDE, then use the similarity concept to derive the nonlinear PDE to the nonlinear ODE.

$$u = \frac{\partial \Psi}{\partial y} \tag{7.1}$$

$$v = -\frac{\partial \Psi}{\partial x} \tag{7.2}$$

The continuity equation is automatically satisfied.

$$\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \Psi}{\partial x}\right) = 0 \tag{7.3}$$

The x-momentum equation becomes

$$\frac{\partial \Psi}{\partial y}\left(\frac{\partial^2 \Psi}{\partial x \partial y}\right) - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = \upsilon\left(\frac{\partial^3 \Psi}{\partial y^3}\right) \tag{7.4}$$

$$\Psi(x, y) \Rightarrow \Psi(\eta)$$
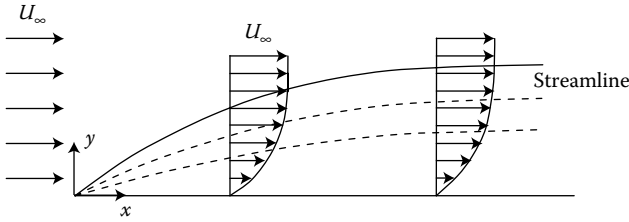
FIGURE 7.1
Stream lines for flow over a flat plate.

Applying the similarity concept from Figure 7.1,

$$y \sim \sqrt{x} \Rightarrow \eta = \frac{c_2 y}{\sqrt{x}} \tag{7.5}$$

$$\Psi \sim \sqrt{x} \Rightarrow f = c_1 \frac{\Psi}{\sqrt{x}} = f(\eta) \tag{7.6}$$

η is the similarity variable.
f is the similarity function.

$$\left.\begin{array}{c} u(x,y) \\ v(x,y) \end{array}\right\} \Rightarrow \Psi(x,y) \Rightarrow \Psi(\eta) \Rightarrow f(\eta)$$

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\sqrt{x}}{c_1}f'\frac{c_2}{\sqrt{x}} = \frac{c_2}{c_1}f'$$

$$\frac{\partial \Psi}{\partial x} = \left(\frac{\partial f}{\partial x}\frac{\sqrt{x}}{C_1} + \frac{f}{C_1}\frac{1}{2}\frac{1}{\sqrt{x}}\right)$$

$$= \left(\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}\frac{\sqrt{x}}{C_1} + \frac{f}{C_1}\frac{1}{2}\frac{1}{\sqrt{x}}\right)$$

$$= f'\eta\left(-\frac{1}{2}\frac{1}{\sqrt{x}}\frac{1}{C_1}\right) + \frac{f}{C_1}\frac{1}{2}\frac{1}{\sqrt{x}}$$

$$= \frac{1}{2}\frac{1}{C_1\sqrt{x}}\left(f - f'\eta\right)$$

where

$$f' = \frac{\partial f}{\partial \eta}, \quad \frac{\partial \eta}{\partial y} = \frac{C_2}{\sqrt{x}}, \quad \frac{\partial \eta}{\partial x} = -\frac{1}{2}C_2 y x^{-(3/2)} = -\frac{\eta}{2x},$$

Similarly,

$$\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{c_2}{c_1}f'\right) = \frac{c_2}{c_1}\frac{\partial f'}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{c_2^2}{c_1}\frac{f''}{\sqrt{x}}$$

$$\frac{\partial^3 \Psi}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \Psi}{\partial y^2}\right) = \frac{\partial}{\partial y}\left(\frac{C_2^2}{C_1}\frac{f''}{\sqrt{x}}\right) = \frac{C_2^2}{C_1}\frac{\partial}{\partial \eta}\left(\frac{f''}{\sqrt{x}}\right)\cdot\frac{\partial \eta}{\partial y}$$

$$= \frac{C_2^2}{C_1}\frac{1}{\sqrt{x}}f'''\frac{C_2}{\sqrt{x}} = \frac{C_2^3}{C_1}\frac{f'''}{x}$$

$$\frac{\partial^2 \Psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{C_2}{C_1}f'\right) = \frac{\partial}{\partial \eta}\left(\frac{C_2}{C_1}f'\right)\frac{\partial \eta}{\partial x} = \frac{C_2}{C_1}f''\left(-\frac{\eta}{2x}\right)$$

Inserting them into the above stream function momentum equation, we obtain

$$ff'' + 2\upsilon c_1 c_2 f''' = 0 \tag{7.7}$$

This is the similarity momentum equation.
Assume

$$\upsilon c_1 c_2 = 1 \tag{7.8}$$

at

$$y = \infty, \quad u = U_\infty = \frac{\partial \Psi}{\partial y}\bigg|_{y=\infty} = f'\frac{C_2}{C_1}$$

Let

$$\frac{C_2}{C_1} = U_\infty. \tag{7.9}$$

From Equations 7.8 and 7.9, we obtain

$$c_2 = \sqrt{\frac{U_\infty}{\upsilon}}, \quad c_1 = \frac{1}{\sqrt{\upsilon U_\infty}}$$

Therefore,

$$\eta = C_2\frac{y}{\sqrt{x}} = \frac{y}{\sqrt{x}}\sqrt{\frac{U_\infty}{\upsilon}} = \frac{y}{x}\sqrt{Re_x} \tag{7.10}$$

$$f = C_1\frac{\Psi}{\sqrt{x}} = \frac{\Psi}{\sqrt{\upsilon U_\infty x}} \tag{7.11}$$

where f and η are similarity functions and similarity variables, respectively. Finally, we obtain the following:

$$u = \frac{\partial \Psi}{\partial y} = \frac{C_2}{C_1}f' = U_\infty f' \tag{7.12}$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{1}{2}\frac{1}{C_1\sqrt{x}}(f - f'\eta) = \frac{1}{2}\sqrt{\frac{\upsilon U_\infty}{x}}(f'\eta - f) \tag{7.13}$$

$$f' = \frac{df}{d\eta} = \frac{u}{U_\infty} = \text{velocity profile} \tag{7.14}$$

$$f'' = \frac{d^2f}{d\eta^2} = \frac{d(u/U_\infty)}{d\eta} = \text{velocity gradient} \tag{7.15}$$

Boundary conditions:
at

$$y = 0, u = v = 0, \Rightarrow \eta = 0, f' = \frac{u}{U_\infty} = 0; v = 0, \Rightarrow f = 0 \Rightarrow f(0) = 0 \tag{7.16}$$

at $y = \infty, u = U_\infty, \Rightarrow f'(\infty) = 1$

$$f''' = -\frac{1}{2}ff'' \tag{7.17}$$

Equation 7.17 may be solved numerically by expressing $f(\eta)$ in a power series (1908 Blasius Series Expansion) with the above-mentioned BCs as

$$f = f_i = \frac{\alpha\eta^2}{2!} - \frac{1}{2}\frac{\alpha^2\eta^5}{5!} + \frac{11}{4}\frac{\alpha^3\eta^8}{8!} - \frac{375}{8}\frac{\alpha^4\eta^{11}}{11!} + \cdots \text{for small } \eta \tag{7.18}$$

$$f = f_0 = \eta - \beta - \gamma \int\limits_\eta^\infty \left\{ \exp\left[-\left(\frac{\eta - \beta}{2}\right)^2\right] d\eta \right\} d\eta \ldots \text{for large } \eta \tag{7.19}$$

where $\alpha = 0.332$, $\beta = 1.73$, and $\gamma = 0.231$. Then u and v can be determined.

Equation 7.17 can also be solved by numerical integration as

$$\int \frac{df''}{f''} = -\int \frac{1}{2}f \, d\eta$$

$$\ln f'' = -\int\limits_0^\eta \frac{1}{2}f \, d\eta + C$$

$$f'' = e^{-\int_0^\eta 1/2(f \, d\eta)} \cdot C_1$$

$$f' = \int df' = \int\limits_0^\eta \left(e^{-\int_0^\eta 1/2(f \, d\eta)}\right) d\eta \cdot C_1 + C_2$$

where $C_2 = 0$ at $\eta = 0, f' = 0$; $C_1 = (1/\int_0^\infty e^{-\int_0^\eta 1/2(f \, d\eta)} d\eta)$ at $\eta = \infty, f' = 1$.

$$f = c_1 \int\limits_0^\eta \int\limits_0^\eta e^{-\int 1/2(f \, d\eta)} d\eta \, d\eta + c_3$$

where $C_3 = 0$ at $\eta = 0, f = 0$.

Therefore,

$$f = \frac{\int_0^\eta \int_0^\eta e^{-\int_0^\eta 1/2(f\,d\eta)}d\eta\,d\eta}{\int_0^\infty e^{-\int_0^\eta 1/2(f\,d\eta)}d\eta} \tag{7.20}$$

For example, use of the trapezoidal rule for the numerical integrations:
Choose $\eta_{max} = 5 = \Delta\eta \cdot N$, when $N = 50$ (the number of integration steps), $\Delta\eta = 0.1$

Let $\eta_{i+1} = \eta_i + \Delta\eta$ for $i = 0, 1, 2, \ldots$, with $\eta_o = 0$

Initial guess $f_i = \eta_i$

Calculate $\int_0^{\eta_i} f\,d\eta$, $e^{-\int_0^{\eta_i} f\,d\eta}$, $\int_0^{\eta_i} e^{-\int_0^{\eta_i} f\,d\eta}d\eta$, and $\int_0^{\eta_i} \int_0^{\eta_i} e^{-\int_0^{\eta_i} f\,d\eta}d\eta\,d\eta$

Calculate $C_1 = 1/\left(\int_0^{\eta_{max}} e^{-\int_0^{\eta_i} f\,d\eta}d\eta\right)$, f_i, f_i', f_i''

Check convergence $\left|1 - f_i^{new}/f_i^{old}\right| < \varepsilon$?, for $i = 0, 1, 2, \ldots, N$. If not, go back again.

From the tabulated data shown in Table 7.1 [2] or Figure 7.2, for given $Re_x = \rho U_\infty x/\mu$, the velocity profile $u(x, y)$ at any location (x, y) and the shear stress f'' at the wall ($y = 0$, $\eta = 0$) can be determined.

TABLE 7.1

Flat Plate Laminar Boundary Layer Functions [2]

$\eta = y\sqrt{\frac{U_\infty}{vx}}$	f	$f' = \frac{u}{U_\infty}$	f''
0	0	0	0.332
0.4	0.027	0.133	0.331
0.8	0.106	0.265	0.327
1.2	0.238	0.394	0.317
1.6	0.420	0.517	0.297
2.0	0.650	0.630	0.267
2.4	0.922	0.729	0.228
2.8	1.231	0.812	0.184
3.2	1.569	0.876	0.139
3.6	1.930	0.923	0.098
4.0	2.306	0.956	0.064
4.4	2.692	0.976	0.039
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
5.6	3.880	0.997	0.005
6.0	4.280	0.999	0.002
6.4	4.679	1.000	0.001
6.8	5.079	1.000	0.000
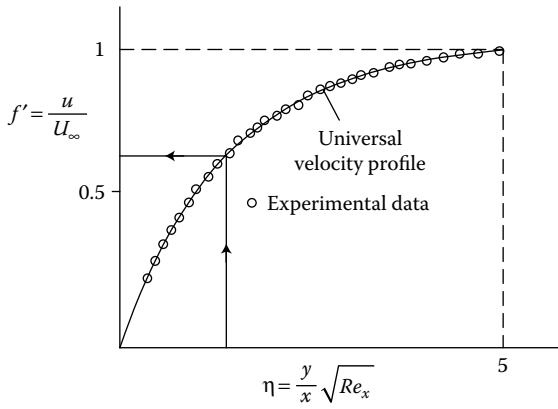
FIGURE 7.2
Graphical sketch of velocity profile from similarity.

From η, obtain or derive f' and $u(x, y)$.

The similarity function for temperature is shown in Figure 7.3.

Let

$$\theta = \frac{T - T_{\mathrm{w}}}{T_\infty - T_{\mathrm{w}}} \left(\cong \frac{u - 0}{U_\infty - 0} = f' \right) \tag{7.21}$$

The energy equation becomes

$$\frac{\partial \Psi}{\partial y} \frac{\partial \theta}{\partial x} - \frac{\partial \Psi}{\partial x} \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \tag{7.22}$$
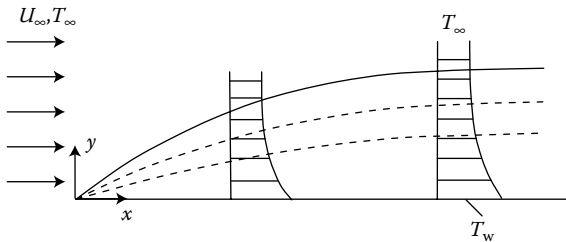
FIGURE 7.3
Concept of thermal boundary layer.

Performing

$$\frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial \eta} \frac{\partial \eta}{\partial y} = \theta' \frac{C_2}{\sqrt{x}}$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial \theta}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{C_2}{\sqrt{x}} \theta' \right) = \frac{\partial}{\partial \eta} \left(\frac{C_2}{\sqrt{x}} \theta' \right) \cdot \frac{\partial \eta}{\partial y} = \frac{C_2^2}{x} \theta''$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial \theta}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \theta' \cdot \frac{-\eta}{2x}$$

Inserting this into the energy equation we obtain

$$\theta'' + \frac{1}{2} Pr f \theta' = 0 \tag{7.23}$$

Boundary conditions:

$$\theta(0) = 0$$
$$\theta(\infty) = 1 \tag{7.24}$$

Equation 7.23 can be solved as

$$\int \frac{\theta''}{\theta'} \, d\eta = - \int \frac{Pr}{2} f \, d\eta$$

$$\ln \theta' = - \int_0^\eta \frac{Pr}{2} f \, d\eta + C$$

$$\theta' = e^{-\int_0^\eta \frac{Pr}{2} f \, d\eta} \cdot C_1$$

$$\theta = C_1 \int_0^\eta e^{-\int_0^\eta \frac{Pr}{2} f \, d\eta} \, d\eta + C_2$$

at $\eta = 0, \theta(0) = 0, C_2 = 0$
at $\eta = 0, \theta(\infty) = 1,$

$$C_1 = \frac{1}{\int_0^\eta e^{-\int_0^\eta Pr/2(f \, d\eta)} d\eta}$$

Therefore,

$$\theta = \frac{\int_0^\eta e^{-\int_0^\eta Pr/2(f \, d\eta)} d\eta}{\int_0^\infty e^{-\int_0^\eta Pr/2(f \, d\eta)} d\eta} \tag{7.25}$$

$$\theta = \theta'(0) \int_0^\eta \exp \left[-\int_0^\eta \frac{Pr}{2} f \, d\eta \right] d\eta \tag{7.26}$$

where

$$\theta'(0) = \frac{1}{\int_0^\infty \exp\left[-\int_0^\eta Pr/2(f\ d\eta)\right] d\eta} \tag{7.27}$$

If $Pr = 1$, $f' = \theta$, the thermal boundary layer is the same as the hydro-dynamic boundary layer (Equation 7.20 = Equation 7.25).

For a given Pr, θ and θ' can be determined if f'' has been solved previously. For example,

$$f = -\frac{f'''}{(1/2)f''}$$

$\int_0^\eta f\ d\eta = -2 \ln f''$

$$f'' = e^{-\int_0^\eta 1/2(f\ d\eta)}$$

$$\theta = \frac{\int_0^\eta \left(f''\right)^{Pr} d\eta}{\int_0^\infty \left(f''\right)^{Pr} d\eta}$$

From the tabulated data or Figure 7.4, the temperature profile $T(x,y)$ at any location (x, y) and the heat flux at the wall ($y = 0$, $\eta = 0$) can be determined.

From η, we obtain θ and $T(x,y)$ for the given Prandtl number.

7.1.1 Summary of the Similarity Solution for Laminar Boundary-Layer Flow and Heat Transfer over a Flat Surface

The following outlines that the equations can be used to calculate boundary-layer thickness, shear stress, and the friction factor for a given Reynolds number; as well as heat flux, the heat transfer coefficient, and Nusselt number for a given Reynolds number and Prandtl number.
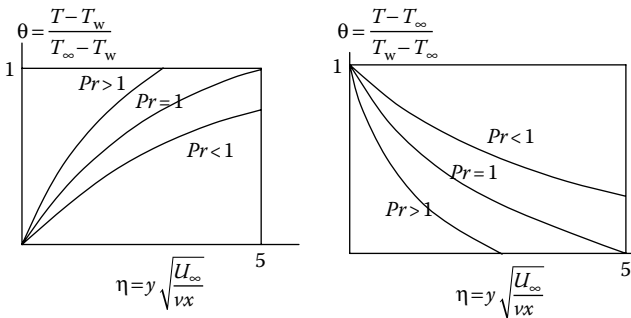
FIGURE 7.4
Graphical sketch of temperature profile from similarity solutions.

Let $u/U_\infty = 0.99$ at $\eta = 5$, that is, $\eta = (y/x)\sqrt{Re_x} = 5$, where $y = \delta$, and the edge of the boundary layer, $\delta = y = 5x/\sqrt{Re_x}$

$$\frac{\delta}{x} = 5.0/\sqrt{Re_x} \tag{7.28}$$

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \frac{\partial}{\partial y}\left(\frac{\partial \Psi}{\partial y}\right)\bigg|_{y=0} = \mu \frac{C_2^2}{C_1}\frac{f''(\eta)}{\sqrt{x}}\bigg|_{\eta=0}$$

$$= \mu U_\infty \sqrt{\frac{U_\infty}{\upsilon}}\frac{f''(\eta)}{\sqrt{x}}\bigg|_{\eta=0} = \mu U_\infty f''(0)\sqrt{\frac{U_\infty}{\upsilon x}}$$

$f'' = 0.332$ from the Table 7.1 or Figure 7.2

$$C_{fx} = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{2f''(0)}{\sqrt{\rho U_\infty x/\mu}} = \frac{0.664}{\sqrt{Re_x}} \sim \frac{1}{\sqrt{x}} \tag{7.29}$$

$$\bar{C}_{fx} = \frac{1.328}{Re_L}$$

when

$$Re_L = \frac{\rho U_\infty L}{\mu}$$

$$q'' = -k \frac{\partial T}{\partial y}\bigg|_{y=0} = -k \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial y}\bigg|_{y=0} = k(T_w - T_\infty)\frac{\partial \theta}{\partial \eta}\sqrt{\frac{U_\infty}{\upsilon x}}\bigg|_{\eta=0}$$

$$= -k(T_\infty - T_w)\theta'(0)\sqrt{\frac{U_\infty}{\upsilon x}}$$

$\theta'(0) = 0.332\,Pr^{1/3}$ from the Table 7.1 (for $Pr = 1$) or Figure 7.4

$$h = \frac{q''}{T_w - T_\infty} = \frac{-k\,(\partial T/\partial y)|_{y=0}}{T_w - T_\infty} = k\sqrt{\frac{U_\infty}{\upsilon x}}\cdot\theta'(0) \sim \frac{1}{\sqrt{x}} \tag{7.30}$$

$$Nu_x = \frac{hx}{k} = \theta'(0)\sqrt{Re_x} = 0.332\sqrt{Re_x}Pr^{1/3}$$

Remarks

There are many engineering applications involving external laminar flow heat transfer such as electronic components cooling and plate-type heat exchangers design. In the undergraduate-level heat transfer, there are many heat transfer relations between Nusselt numbers and Reynolds and Prandtl

numbers. Students are expected to calculate heat transfer coefficients from these relations by giving Reynolds and Prandtl numbers.

For the similarity method, students are expected to know how to derive the similarity momentum and energy equations with proper velocity and thermal BCs. Students are also expected to know how to sketch and predict velocity profiles inside the boundary layer, for a given Reynolds number, from the velocity similarity solution by using tables or figures; how to sketch and predict temperature profiles inside the thermal boundary layer, for a given Reynolds number and Prandtl number, from the temperature similarity solution by using tables or figures. Here we focus on flow over a flat plate (zero-pressure gradient flow) with constant surface temperature BC, and do not include the one at constant surface heat flux BC.

In advanced heat convection, the similarity solution can be extended to include various constant pressure gradient flows (such as flow acceleration or deceleration) with variable surface temperature BCs. These can be solved using the fourth-order Runge–Kutta method in order to obtain the velocity and temperature profiles across the forced convection boundary-layer flow.

7.2 Laminar Flow and Heat Transfer over a Flat Surface: Integral Method

The other powerful method to solve boundary-layer flow and the heat transfer problem is using the integral approximate solution [1–6]. Instead of performing mass, momentum, and energy balance through a differential fluid element inside the boundary layer as the similarity method, the integral method performs conservation of mass, momentum, and energy across the boundary-layer thickness at a given differential x-direction. It is noted that, for fluid with a Prandtl number different from unity, such as gases, water, and oils, the hydrodynamic boundary-layer thickness is different from the thermal boundary layer.

7.2.1 Momentum Integral Equation by Von Karman

Employing a control volume that is infinitesimal in the x-direction but finite in the y-direction across the boundary-layer thickness, as shown in Figure 7.5, we apply the mass and momentum conservation to the control volume.

From mass conservation,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \qquad v(\delta) = v|_0 - \int_0^\delta \frac{\partial u}{\partial x}\, dy = -\int_0^\delta \frac{\partial u}{\partial x}\, dy$$

$$\rho v\, dx = \frac{d}{dx} \int (\rho u\, dy)\, dx \qquad\qquad (7.31)$$
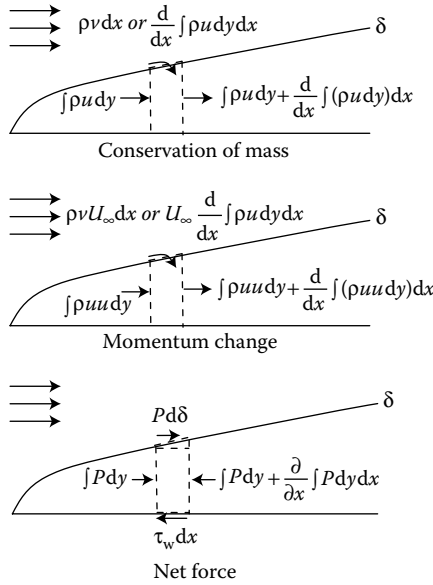
FIGURE 7.5
Integral method.

From momentum conservation,

$$
\underbrace{-\tau_w \, dx + P\frac{d\delta}{dx}\, dx - \frac{d}{dx}\left(\int_0^\delta P \, dy\right) dx}_{\text{net force}} = \underbrace{\frac{d}{dx}\int(\rho uu \, dy)\, dx - U_\infty \frac{d}{dx}\int(\rho u \, dy)\, dx}_{\text{momentum change}}
$$

(7.32)

where

$$
-\frac{d}{dx}\int P \, dy = -\frac{d}{dx}P\int dy = -P\frac{d\delta}{dx} - \delta\frac{dP}{dx}
$$

$dP/dx = -\rho U_\infty (dU_\infty/dx)$ (from the momentum differential equation, at outside of the boundary layer, $u = U_\infty$, $v = 0$, $\partial^2 U_\infty/\partial y^2 = 0$).
Therefore,

$$
-\tau_w + P\frac{d\delta}{dx} - P\frac{d\delta}{dx} - \delta\left(-\rho U_\infty \frac{dU_\infty}{dx}\right) = \frac{d}{dx}\int \rho uu \, dy - U_\infty \frac{d}{dx}\int \rho u \, dy
$$

$$
-\tau_w = \frac{d}{dx}\int \rho uu \, dy - \rho U_\infty \frac{dU_\infty}{dx}\int dy - U_\infty \frac{d}{dx}\int \rho u \, dy
$$

$$= \frac{d}{dx} \int \rho uu \, dy - \frac{dU_\infty}{dx} \int \rho U_\infty \, dy - U_\infty \frac{d}{dx} \int \rho u \, dy$$

$$+ \frac{dU_\infty}{dx} \int \rho u \, dy - \frac{dU_\infty}{dx} \int \rho u \, dy$$

$$= \frac{d}{dx} \int \rho uu \, dy + \frac{dU_\infty}{dx} \int \rho \, (u - U_\infty) \, dy - \frac{d}{dx} \int \rho u U_\infty \, dy$$

$$= \frac{d}{dx} \int \rho u \, (u - U_\infty) \, dy + \frac{dU_\infty}{dx} \int \rho \, (u - U_\infty) \, dy$$

$$\tau_w = \frac{d}{dx} \int \rho u \, (U_\infty - u) \, dy + \frac{dU_\infty}{dx} \int \rho \, (U_\infty - u) \, dy \qquad (7.33)$$

For a flat plate flow, $dU_\infty/dx = 0$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{d}{dx} \int \rho u (U_\infty - u) \, dy \qquad (7.34)$$

7.2.2   Energy Integral Equation by Pohlhausen

From energy conservation, as shown in Figure 7.6,

$$q_s'' dx = \frac{d}{dx} \left(\int_0^{\delta_T} \rho C_p u T \, dy \right) dx - C_p T_\infty \frac{d}{dx} \left(\int_0^{\delta} \rho u \, dy \right) dx \qquad (7.35)$$

$$= \frac{d}{dx} \int_0^{\delta_T} \rho C_p u (T - T_\infty) \, dy \, dx$$

$$q_s'' = -k \left. \frac{\partial T}{\partial y} \right|_{y=0} = \frac{d}{dx} \int_0^{\delta_T} \rho C_p u (T - T_\infty) \, dy \qquad (7.36)$$
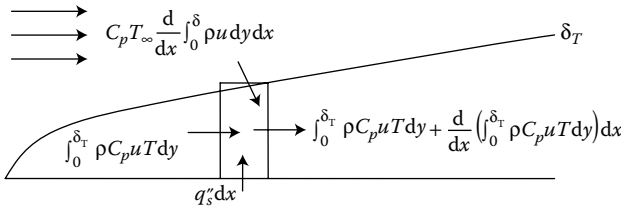
FIGURE 7.6
Conservation of energy.

7.2.3 Outline of the Integral Approximate Method

For a hydrodynamic boundary layer:
Step 1: assume the velocity profile, such as

$$u = a + by$$

$$u = a + by + cy^2$$

$$u = a + by + cy^2 + dy^3$$

$$u = a + by + cy^2 + dy^3 + ey^4$$

Step 2: from the velocity BCs to determine the coefficients a, b, c, d, and e
Step 3: put the velocity profile into momentum integral to solve $\delta(x)$
Step 4: put $\delta(x)$ back to the velocity profile
Step 5: the friction factor

$$C_{fx} = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{\mu \, (\partial u/\partial y)|_0}{(1/2)\rho U_\infty^2} \quad \text{can be determined}$$

Similarly, for a thermal boundary layer:
Step 1: assume a temperature profile in one of the following forms:

$$T = a + by$$

$$T = a + by + cy^2$$

$$T = a + by + cy^2 + dy^3$$

$$T = a + by + cy^2 + dy^3 + ey^4$$

Step 2: from the temperature BCs to determine the coefficients a, b, c, d, and e
Step 3: put the temperature profile into energy integral to solve $\delta_T(x)$
Step 4: put $\delta_T(x)$ back to the temperature profile and

$$h = \frac{q_w''}{T_w - T_\infty} = \frac{-k(\partial T/\partial y)_0}{T_w - T_\infty} \quad \text{can be determined.}$$

Examples

7.1 Assume third-order velocity and temperature profiles for the boundary layer
flow to satisfy the BCs:

$$u = a + by + cy^2 + dy^3$$

$$T = a + by + cy^2 + dy^3$$

$$y = 0 \quad u = 0 \qquad \partial^2 u/\partial y^2 = 0 \quad T = T_w \quad \partial^2 T/\partial y^2 = 0$$
$$y = \delta \quad u = U_\infty \quad \partial u/\partial y = 0 \qquad T = T_\infty \quad \partial T/\partial y = 0$$

$$\frac{u - 0}{U_\infty - 0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 = \frac{u}{U_\infty}$$

Put the above velocity profile into momentum integral to solve for $\delta(x)$

$$\mu U_\infty \frac{3}{2}\frac{1}{\delta} = \frac{d}{dx}\int_0^\delta \rho U_\infty^2 \left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[1 - \frac{3}{2}\left(\frac{y}{\delta}\right) + \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right] dy$$

$$= \frac{d}{dx}\left(\frac{39}{280}\delta\rho U_\infty^2\right)$$

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\upsilon}{U_\infty}$$

$$\frac{1}{2}\delta^2 = \frac{140}{13}\frac{\upsilon}{U_\infty}x + C$$

at $x = 0$, $\delta = 0$, $C = 0$
Therefore, $\delta(x) = 4.64\sqrt{(\upsilon x/U_\infty)}$
Put $\delta(x)$ back to velocity profile to obtain the final velocity profile. And the friction factor can be calculated as

$$C_{fx} = \frac{\tau_w}{(1/2)\rho U_\infty^2} = \frac{\mu(\partial u/\partial y)_0}{(1/2)\rho U_\infty^2} = \frac{\mu(3/2)(1/\delta)U_\infty}{(1/2)\rho U_\infty^2} = \frac{0.646}{\sqrt{Re_x}} \qquad (7.37)$$

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3$$

or

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - \frac{3}{2}\left(\frac{y}{\delta_T}\right) + \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3$$

Put the above temperature profile into the energy integral to solve for $\delta_T(x)$.
If $Pr = 1$, $\delta = \delta_T$, $u = T - T_w$

$$\frac{\delta_T}{x} = \frac{4.64}{\sqrt{Re_x}}$$

The coefficient 4.64 is from momentum integral. It equals 5.0 from the similarity solution.

From the energy integral, we obtain

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=0} = k\left(T_{\rm w} - T_{\infty}\right)\frac{3}{2}\frac{1}{\delta_{\rm T}} = \frac{d}{dx}\int_{0}^{\delta_{\rm T}}\rho C_{\rm p}\left[\frac{3}{2}\left(\frac{y}{\delta}\right)\right.$$

$$\left.-\frac{1}{2}\left(\frac{y}{\delta}\right)^{3}\right]U_{\infty}\left(T_{\rm w} - T_{\infty}\right)\left[1 - \frac{3}{2}\left(\frac{y}{\delta_{\rm T}}\right) + \frac{1}{2}\left(\frac{y}{\delta_{\rm T}}\right)^{3}\right]dy$$

Then,

$$k\frac{3}{2}\frac{1}{\delta_{\rm T}} = \rho C_{p}U_{\infty}\left\{3\delta\left[\frac{\delta_{\rm T}}{\delta}\frac{1}{10} - \left(\frac{\delta_{\rm T}}{\delta}\right)^{3}\frac{1}{70}\right]\frac{d}{dx}\left(\frac{\delta_{\rm T}}{\delta}\right)\right.$$

$$\left.+3\left[\left(\frac{\delta_{\rm T}}{\delta}\right)^{2}\frac{1}{20} - \left(\frac{\delta_{\rm T}}{\delta}\right)^{4}\frac{1}{280}\right]\frac{d\delta}{dx}\right\}$$

Let

$$r = \frac{\delta_{\rm T}}{\delta} < 1, \qquad \left(\frac{\delta_{\rm T}}{\delta}\right)^{3} \sim \left(\frac{\delta_{\rm T}}{\delta}\right)^{4} \sim 0$$

We obtain

$$2r^{2}\delta^{2}\frac{dr}{dx} + r^{3}\delta\frac{d\delta}{dx} = 10\frac{\alpha}{U_{\infty}}$$

From above

$$\delta^{2} = \frac{280}{13}\frac{\upsilon x}{U_{\infty}}, \quad \delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\upsilon}{U_{\infty}}$$

We obtain

$$2r^{2}\frac{280}{13}\frac{\upsilon x}{U_{\infty}}\frac{dr}{dx} + r^{3}\frac{140}{13}\frac{\upsilon}{U_{\infty}} = 10\frac{\alpha}{U_{\infty}}$$

Then,

$$4r^{2}x\frac{dr}{dx} + r^{3} = \frac{13}{14Pr}$$

Let $s = r^{3}$

$$r^{2}\frac{dr}{dx} = \frac{1}{3}\frac{d}{dx}\left(r^{3}\right) = \frac{1}{3}\frac{ds}{dx}$$

$$s + \frac{4}{3}x\frac{ds}{dx} = \frac{13}{14Pr}$$

$$s = r^3 = C_1 x^{-3/4} + \frac{13}{14\,Pr}$$

at $x = 0$, $C_1 = 0$, $r^3 = \dfrac{13}{14Pr}$

$$r = \frac{\delta_T}{\delta} = \left(\frac{13}{14Pr}\right)^{1/3} \cong 0.975\,(Pr)^{-1/3} \sim Pr^{-1/3}$$

$$\delta_T = \delta\,Pr^{-1/3} = \frac{4.64}{\sqrt{Re_x}} \cdot Pr^{-1/3}$$

$$h = \frac{-k(\partial T/\partial y)_{y=0}}{T_w - T_\infty} = \frac{3}{2}\frac{k}{\delta_T}$$

$$= \frac{3}{2}\frac{k}{\delta Pr^{-1/3}} = \frac{3}{2}\frac{k}{x(4.64/\sqrt{Re_x})Pr^{-1/3}}$$

For this typical case,

$$Nu_x = \frac{hx}{k} = 0.323\,Re_x^{1/2}\,Pr^{1/3} \tag{7.38}$$

From Figure 7.7,
at $x = x_o$, $\delta_T = 0$, $r = 0$

$$0 = C_1\frac{1}{x_o^{3/4}} + \frac{13}{14Pr}, \quad C_1 = -\frac{13}{14Pr}x_o^{3/4}$$

Therefore,

$$r = \frac{\delta_T}{\delta} = 0.975Pr^{-1/3}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3}$$
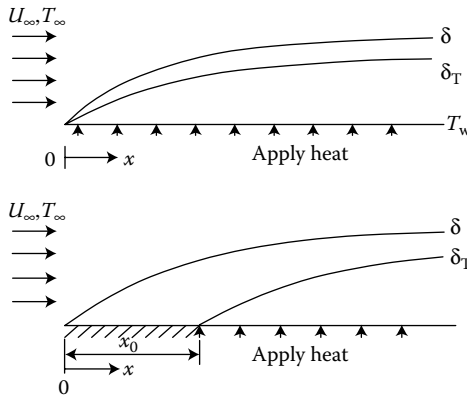
FIGURE 7.7
Integral approximation method.

For a typical case, $x_o = 0$, $\delta_T = 0$,

$$\frac{\delta_T}{\delta} = 0.975 Pr^{-1/3} \simeq Pr^{-1/3}$$

For the nonheating leading length problem, $x_o > 0$,

$$Nu_x = \frac{hx}{k} = \frac{0.323 \, Re_x^{1/2} Pr^{1/3}}{\sqrt[3]{1 - (x_0/x)^{3/4}}} \qquad (7.39)$$

If $x_0 = 0$, go back to the typical case.

Remarks

For the integral method, students are expected to know how to sketch and derive momentum and energy integral equations from mass, force, and heat balance across the boundary layer. Students are also expected to know how to solve velocity boundary-layer thickness and the friction factor from the derived momentum integral equation by assuming any velocity profile to satisfy velocity BCs across the boundary layer; as well as how to solve thermal boundary thickness and the heat transfer coefficient from the derived energy integral equation by assuming any temperature profile to satisfy thermal BCs (given surface temperature or surface heat flux) across the thermal boundary layer. Note that one will get a slightly different velocity boundary-layer thickness (and friction factor) by using different velocity profiles across the boundary layer, and a slightly different thermal boundary-layer thickness (and a heat transfer coefficient) by using different temperature profiles across the thermal boundary layer. This is the nature of the integral method. Another note is that velocity and thermal boundary-layer thickness is the same if Prandtl number unity is assumed.

PROBLEMS

7.1. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W. Assume that there exist no body force and constant thermal and fluid properties. The similarity momentum and energy equations are listed here for reference: $f''' + (1/2)ff'' = 0$ and $\theta'' + (1/2)Pr_f f \theta' = 0$.

a. From the Blasius solution of the above similarity equations, sketch the relations between the similarity functions (f', θ) and the similarity variable (η) for both water and air (i.e., sketch f' versus η for both water and air on the same plot; and θ versus η for both water and air on the same plot). Explain why they

have differences, if any. You do not need to solve the above equations.

b. For a given problem (i.e., U_∞, T_∞, ρ, μ, T_w, P_r are given), explain briefly how to determine the local velocity $u(y)$ and temperature $T(y)$ at a specified location x, for both water and air, from the sketches in (a)?

7.2. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_w or at a uniform wall heat flux q''_w, respectively. Assume that there exist no body force and constant thermal and fluid properties.

a. Based on the approximate integral method, assuming a uniform velocity profile inside the boundary layer, that is, $u = U_\infty$, and assuming a linear temperature profile inside the thermal boundary layer as $T = a + b \times y$, determine the local thermal boundary-layer growth along the flat plate (i.e., δ_T versus x) at a uniform wall temperature T_w condition. (*Note*: a and b are unknown constants that need to be determined.)

b. Based on (a), determine the local Nusselt number distribution along the flat plate (i.e., Nu versus x) at a uniform wall temperature.

7.3. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_w. Assume that there exist no body force and constant thermal and fluid properties. The similarity momentum and energy equations are listed here for reference: $f''' + (1/2)ff'' = 0$ and $\theta'' + (1/2)P_r f\theta' = 0$.

a. Sketch both the velocity and the thermal boundary-layer thickness distribution for both water and liquid metal, respectively, flowing over the flat plate? (i.e., sketch δ versus x and δ_T versus x on the same plot for water, and δ versus x and δ_T versus x on the same plot for the liquid metal). Explain why they have differences, if any.

b. Define the similarity variable (η), the similarity function for temperature (θ), and the derivative of the similarity function for velocity (f')? Write the BCs which can be used for this problem?

c. From the Blasius solution of the above similarity equations, sketch the relations between the similarity functions (f', θ) and the similarity variable (η) for both water and liquid metal (i.e., sketch f' versus η for both water and liquid metal on the same plot; and θ versus η for both water and liquid metal on the same plot). Explain why they have differences, if any. You do not need to solve the above equations.

d. For a given problem (i.e., U_∞, T_∞, ρ, μ, T_w, P_r are given), explain briefly how to determine the local velocity $u(y)$ and temperature $T(y)$ at a specified location x, for both water and liquid metal, from the sketches in (c)?

e. Explain briefly how to determine the local heat transfer coefficient from the sketches in (c)? Sketch the local heat transfer coefficient over the flat plate for both water and liquid metal (i.e., sketch h versus x for both water and liquid metal on the same plot), if both are at the same free-stream velocity (U_∞)? Explain why they have differences, if any.

7.4. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W or at a uniform wall heat flux q''_W, respectively. Assume that there exist no body force and constant thermal and fluid properties.

a. Based on the integral method, sketch and write down the momentum balance as well as the energy balance across the boundary layers? (If you cannot remember, derive it).

b. Assuming a uniform velocity profile inside the boundary layer, that is, $u = U_\infty$, and assuming a linear temperature profile inside the thermal boundary layer as $T = a + b \times y$, determine the local thermal boundary-layer growth along the flat plate (i.e., δ_T versus x) at a uniform wall temperature T_W condition. (*Note*: a and b are unknown constants that need to be determined.)

c. Based on (b), determine the local Nusselt number distribution along the flat plate (i.e., Nu versus x) at a uniform wall temperature.

d. Assuming a uniform velocity profile inside the boundary layer, that is, $u = U_\infty$, and assuming a linear temperature profile inside the thermal boundary layer as $T = c + d \times y$, determine the local thermal boundary-layer growth along the flat plate (i.e., δ_T versus x) at a uniform wall heat flux q''_W condition. (*Note*: c and d are unknown constants that need to be determined.)

e. Based on (d), determine the local Nusselt number distribution along the flat plate (i.e., Nu versus x) at a uniform wall heat flux. Explain and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than that in (c)?

7.5. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W. Assume that there exist no body force and constant thermal and fluid properties. The similarity momentum and energy equations are listed here for reference: $f''' + (1/2)ff'' = 0$ and $\theta'' + (1/2)Pr f\theta' = 0$.

a. Sketch both the velocity and the thermal boundary-layer thickness distributions, respectively for both water and air flowing over the flat plate? (i.e., sketch δ versus x and δ_T versus x on the same plot for water and for air, respectively). Explain why they have differences, if any.

b. Define the similarity variable (η), the similarity function for temperature (θ), and the derivative of the similarity function for velocity (f')? Write the BCs that can be used for this problem?

c. From the Blasius solution of the above similarity equations, sketch the relations between the similarity functions (f', θ) and the similarity variable (η) for both water and air. (i.e., sketch f' versus η for both water and air on the same plot; and θ versus η for both water and air on the same plot). Explain why they have differences, if any. You do not have to solve the above equations.

d. For a given problem (i.e., U_∞, T_∞, ρ, μ, T_w, P_r are given), explain briefly how to determine the local velocity $u(y)$ and temperature $T(y)$ at a specified location x, for both water and air, from the sketches in (c)?

e. Explain briefly how to determine the local heat transfer coefficient from the sketches in (c)? Answer whether water or air will provide a higher convective heat transfer coefficient from the surface, if both are at the same free-stream velocity (U_∞)? Why?

7.6. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall heat flux q''_w. Assume that there exist no body force and constant thermal and fluid properties.

a. Based on the integral method, sketch and write down the momentum balance as well as the energy balance across the boundary layers? (If you cannot remember, derive it.)

b. Assuming a uniform velocity profile inside the boundary layer, that is, $u = U_\infty$, and assuming a linear temperature profile inside the thermal boundary layer as $T = a + b \times y$, determine the local thermal boundary-layer growth along the flat plate (i.e., δ_t versus x).

c. Based on (b), determine the local Nusselt number distribution along the flat plat (i.e., Nu versus x).

d. Consider a parabolic velocity and temperature profile inside the thermal boundary layer as $u = a + b \times y + c \times y^2$, $T = a + b \times y + c \times y^2$, and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than those of a uniform velocity profile and a linear temperature profile as indicated in (b)? Explain why.

e. Consider a uniform suction through the wall (i.e., $v = -v_0$), and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than that without boundary-layer suction? Explain why.

7.7. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W. Assume that there exist no body force and constant thermal and fluid properties. The similarity momentum and

energy equations are listed here for reference: $f''' + (1/2)ff'' = 0$ and $\theta'' + (1/2)\,Pr f\theta' = 0$.

a. Define the similarity variable (η), the similarity function for temperature (θ), and the derivative of the similarity function for velocity (f')? Write the BCs that can be used for this problem?

b. From the Blasius solution of the above similarity equations, sketch the relations between the similarity functions (f', θ) and the similarity variable (η) for water, air, and liquid metal (i.e., sketch f' versus η for water, air, and liquid metal on the same plot; and θ versus η for water, air, and liquid metal on the same plot). Explain why they have differences, if any. Explain briefly how to determine the local velocity (u) and temperature (T) from the sketches? You do not need to solve the above equations.

c. Explain briefly how to determine the local heat transfer coefficient from the sketches in (b)? Answer whether water or liquid metal will provide a higher convective heat transfer coefficient from the surface, if both are at the same free-stream velocity (U_∞)? Explain why?

7.8. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W. Assume that there exist no body force and constant thermal and fluid properties.

a. Based on the integral method, sketch and write down the momentum balance as well as the energy balance across the boundary layers? (If you cannot remember, derive it.)

b. Assuming a uniform velocity profile inside the boundary layer that is, $u = U_\infty$, and assuming a linear temperature profile inside the thermal boundary layer as $T = a + b \times y$, determine the local thermal boundary-layer growth and the local Nusselt number distribution along the flat plate (i.e., δ_t versus X and Nu versus X).

c. Consider a uniform blowing through the wall (i.e., $v = v_0$), and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than that without boundary-layer blowing? Explain why.

7.9. Consider a steady, incompressible, low-speed 2-D laminar boundary-layer flow (at U_∞, T_∞) over a flat plate at a uniform wall temperature T_W. Assume that there exist no body force and constant thermal and fluid properties.

a. Based on the integral method, write down the final form of momentum integral equation. (If you cannot remember, please derive it.)

b. Based on the integral method, can you remember to write down the final form of the energy integral equation? (If you cannot remember, please derive it.)

c. Assuming a uniform velocity profile inside the boundary layer that is, $u = U_\infty$, and assuming a linear temperature profile

inside the thermal boundary layer as $T = a + by$, determine the local Nusselt number distribution along the flat plate (i.e., Nu versus X).

d. Consider a uniform suction through the wall (i.e., $v = -v_0$), and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than those without boundary-layer suction? Explain why.

7.10. Consider a laminar air flow (at U_∞, T_∞) over a flat plate at a uniform wall heat flux q_W''.

   a. Based on the integral method, can you remember to write down the momentum and energy integral equations? (if you cannot remember, please derive them).

   b. Assuming a linear velocity profile inside the boundary layer as $u = a + by$, derive the velocity boundary-layer thickness distribution along the flat plate (i.e., δ versus X).

   c. Assuming a linear temperature profile inside the thermal boundary layer as $T = c + dy$, derive the thermal boundary-layer thickness distribution along the flat plate (i.e., δ_T versus X).

   d. Based on (b) and (c), determine the local Nusselt number distribution along the flat plate (i.e., Nu versus X).

   e. Consider a uniform wall temperature (T_W) as a thermal BC at the wall, and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than those of uniform wall heat flux as a thermal BC? Explain why.

   f. Consider a uniform suction through the wall (i.e., $v = -v_0$), and comment on whether the local Nusselt number distribution along the flat plate will be higher, the same, or lower than those without boundary-layer suction? Explain why.

7.11. The similarity method for laminar flow over a flat plate:
   U_∞ is the free-stream velocity, T_∞ is the free-stream temperature, and T_W is the flat plate wall temperature.

   a. Write down the similarity variable, differential equations, and BCs for velocity and temperature, respectively. Then, determine velocity (u) and temperature (if $P_r = 1$) at

$$(x, y) = (2\,\text{cm}, 1/3\delta),$$

$$= (4\,\text{cm}, 1/3\delta),$$

$$= (6\,\text{cm}, 1/3\delta).$$

   b. At any given x, if U_∞ increases, the friction factor will be increased or decreased. Why? How about shear stress? At any given U_∞, if x increases, the heat transfer coefficient will be increased or decreased. Why? How about heat transfer rate?

7.12. The similarity method for laminar flow over a flat plate:
   U_∞ – free-stream velocity, T_∞ – free-stream temperature, and T_W – flat plate wall temperature.

a. Write down the similarity variable, differential equations, and BCs for velocity and temperature, respectively. Then determine velocity (u) and temperature (if $P_r = 1$) at

$$(x, y) = (1\,\text{cm}, 1/2\delta) \quad \text{and} \quad (x, y) = (1\,\text{cm}, 1/4\delta)$$

$$= (3\,\text{cm}, 1/2\delta) \qquad\qquad\qquad = (3\,\text{cm}, 1/4\delta)$$

$$= (9\,\text{cm}, 1/2\delta) \qquad\qquad\qquad = (9\,\text{cm}, 1/4\delta)$$

b. At any given x, if U_∞ increases, the friction factor will be increased or decreased. Why? How about shear stress? At any given U_∞: if x increases, the heat transfer coefficient will be increased or decreased. Why? How about heat transfer rate?

7.13. Consider the development of velocity and thermal boundary layers on a porous flat plate where air passes into the flat plate at a velocity V_0.

a. Assume that no pressure gradient exists in either the x- or the y-direction and that all fluid properties are constant. Derive the differential equation that relates boundary-layer thickness δ to distance x. A linear profile may be assumed.

b. The exact solution of the boundary-layer equations with the BCs of part (a) shows that δ approaches a constant value for large x, and that for large x, V_x and V_y are given by

$$V_x = V_\infty \left[1 - \exp\left(\frac{V_0 y}{v} \right) \right]$$

$$V_y = -V_0$$

Suppose now that at some large $x = \ell$ (i.e., where δ has become constant and where V_x and V_y are given above), a step change in wall temperature occurs. Using the integral technique, derive a differential equation relating the thermal boundary-layer thickness δ_T to x ($x > \ell$). Again assume that fluid properties are constant. A linear temperature profile may be assumed. Integrals and derivatives need not be evaluated.

7.14. Air at 1 atm and at a temperature of 30°C flows over a 0.3-m-long flat plate at 100°C with a free-stream velocity of 3 m/s. At the position $x = 0.05$ m, determine the values of the boundary-layer thickness, displacement thickness, moments thickness, wall stress, and heat transfer coefficient. Determine the values of the velocity parallel and normal to the plate surface, the values of the shear stress, and the values of temperature in the fluid at the positions ($x = 0.05$ m, $y = 0.002$ m); ($x = 0.05$ m, $y = 0.004$ m).

7.15. Using the integral method and assuming that velocity and temperature vary as

$$u = a + by + cy^2 + dy^3$$

$$T = a + by + cy^2 + dy^3$$

determine the "x" variations of the heat transfer coefficient and wall temperature for the case of laminar flow over a flat plate with a uniform wall heat flux $(q/A)_W$. Compare the result of the heat transfer coefficient to the solution obtained in the textbook for the case of uniform wall temperature.

7.16. Laminar air flow at 1 atm pressure over a flat plate at a uniform T_W. Assume that $U_\infty = 3$ m/s, $T_\infty = 20°C$, and $T_W = 100°C$, and determine local u and T at $x = 3$ cm and $y = 0.2$ cm by using the similarity solution. Also determine the heat transfer coefficient at the same $x = 3$ cm location. Will the heat transfer coefficient be increased or decreased with increasing x for the same U_∞? Why? Will the Nusselt number be increased or decreased with increasing x for the same U_∞? Why?

7.17. Use similarity solutions: air at 300 K and 1 atm flows along a flat plate at 5 m/s. At a location 0.2 m from the leading edge, plot the u and v velocity profiles using the exact solution to the Blasius equation Also, determine the boundary-layer thickness, if it is defined as the location where $u = 0.99u_\infty$. Plot the temperature profile and determine the thermal boundary thickness if the plate temperature is 500 K.

7.18. Using integral method solutions: air at 300 K and 1 atm pressure flows along a flat plate at 5 m/s. For $x < 10$ cm, $T_s = 300$ K, whereas for 10 cm $< x < 20$ cm, $T_s = 500$ K.

   Calculate the heat loss from the plate and compare the result with the heat loss if the plate were isothermal at 500 K. Assume that there exists a laminar boundary layer. Compare and discuss the two cases.

7.19. Consider a boundary-layer flow over a flat plate.
   a. Using the integral method derive the continuity equation in the boundary layer for flow over a flat plate.
   b. Using the integral method derive the energy conservation equation in the boundary layer for flow over a flat plate.
   c. Using appropriate BCs and boundary-layer theory, show that for an inviscid fluid,

$$Nu = 0.564 \, Pe^{1/2}$$

   where Nu is the Nusselt number and Pe is the Peclet number. (Hint: Use a plug flow model for velocity.)

7.20. Consider a fluid approaching the leading edge of a flat plate with uniform velocity and temperature profiles U_∞ and T_∞. The flat plate is frictionless and is held at a constant heat flux of q_s''. The temperature profile at a distance x from the leading edge is given by

$$T = a_0 + a_1 y + a_2 y^2 + a_3 y^3$$

   a. Using appropriate BCs evaluate a_0, a_1, a_2, and a_3.
   b. Qualitatively sketch velocity and temperature profiles at a distance x_1 and x_2, respectively, from the leading edge.

    c. Determine the local heat transfer coefficient (h_x) and the Nusselt number Nu_x. Express the answer in terms of the thermal boundary-layer thickness δ_t.

7.21. Consider a 2-D laminar air flow over a friction less plate. The flow approaches the leading edge of the plate with uniform velocity U_∞ and temperature T_∞. The plate is subjected to a constant wall temperature, $T_s(> T_\infty)$.

    a. Qualitatively sketch hydrodynamic (δ) and thermal boundary layer (δ_t) growth as a function of distance x from the leading edge.

    b. Qualitatively sketch velocity and temperature profiles at a distance x from the leading edge.

    c. A third-order polynomial of the form

$$\frac{T - T_s}{T_\infty - T_s} = a_0 + a_1(y/\delta_t) + a_2(y/\delta_t)^2 + a_3(y/\delta_t)^3$$

is used to describe the temperature profile. Determine the constants a_0, a_1, a_2, and a_3 using appropriate BCs.

    d. Express local Nusselt numbers (Nu_x) in terms of local thermal boundary-layer thickness δ_t.

    e. Set up an integral energy balance equation. Do not attempt to solve the equation.

7.22. Consider a 2-D, steady, incompressible laminar flow over a flat plate. The flow approaches the leading edge with free-stream velocity of U_∞ and temperature T_∞. The flat plate is frictionless and it is kept at a uniform temperature of $T_s(> T_\infty)$.

    a. State clearly all the boundary-layer assumptions.

    b. If the temperature distribution at any axial distance x is approximated by a linear profile $(T - T_s)/(T_\infty - T_s) = y/\delta_t$, derive an expression for the local Nusselt number distribution.

7.23. Consider a steady laminar viscous fluid with a free-stream velocity V_∞ and temperature T_∞ flows over a flat plate at a uniform wall temperature T_w. Assume that the thermal fluids properties are constant.

    a. If the fluid has a Prandtl number of one (i.e., $Pr = 1.0$), determine the local heat transfer coefficient along the plate. You may use the method of integral approximation with the assumptions of the linear velocity and temperature profiles across the boundary layers, that is, $u = a + by$ and $T = c + dy$, where a, b, c, and d are constants.

    b. If the fluid's Prandtl number is not equivalent to one (i.e., $Pr > 1$ or $Pr < 1$), outline the methods (no need to solve) in order to determine the surface heat transfer. You may use the same assumptions as in part (a). Does the heat transfer coefficient increase or decrease with the fluid Prandtl number? Explain your answers.

7.24. A flat horizontal plate has a dimension of $10\,\text{cm} \times 10\,\text{cm}$. The plate is maintained at a constant surface temperature of $300°\text{K}$ with a water jacket.

a. If the plate is placed in a hot air stream with a pressure of 1 atm, temperature of 400°K, and velocity of 10 m/s, sketch the local heat transfer coefficient, h_x, along the plate. Also, determine the surface heat flux at the trailing edge of the plate.

Given: air properties at 350°K, $k = 30 \times 10^{-3}$ w/mk, $v = 20.92 \times 10^{-6}$ m²/s, $Pr = 0.7$:

$$Nu_x = a\, Re_x^m Pr^n,$$

where $a = 0.332$, $m = 1/2$, $n = 1/3$ for laminar flow

$a = 0.0296$, $m = 4/5$, $n = 1/3$ for turbulent flow

7.25. Constant properties laminar viscous fluids ($\rho, C_p, K, \mu = $ constant) with a free-stream velocity U_∞ and temperature T_∞ move over a flat plate at a uniform wall heat flux ($q_w'' = $ constant).

a. Determine the local Nusselt number ($Nu_x = h_x \cdot x/k$) along the plate by using the integral approximation method with the velocity and temperature profiles across the boundary layers as $u = a + by$ and $T = c + dy$, respectively. Assume $Pr = 1$ for this problem.

b. On the same plot, sketch h_x, Nu_x versus x for the uniform wall heat flux ($q_w'' = $ constant) and uniform wall temperature ($T_w = $ constant) BCs, respectively. For the same flow velocity, which wall BC (q_w'' or T_w) will provide a higher heat transfer coefficient or Nusselt number? Explain why.

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8

Internal Forced Convection

8.1 Velocity and Temperature Profiles in a Circular Tube or between Parallel Plates

Internal forced convection is that flow moves through the internal surface of a passage and forms an internal boundary layer on the surface. For example, fluid flowing through a circular tube or between two parallel plates is a most common application. Figure 8.1 shows the hydrodynamic boundary-layer development (due to viscosity) for flow entering a circular tube (or between two parallel plates). The boundary layer starts from the tube (or plate) entrance and grows along the tube (or plate) length. The velocity profile keeps changing in the entrance region of the tube (or plate). The flow becomes a "hydrodynamic fully developed flow" when the boundary thickness is the same as the tube radius (or half-spacing between the two plates). The velocity profile no longer changes after a fully developed flow. For a laminar flow, the entrance length to tube diameter ratio is about 5% of Reynolds number (based on the tube diameter). This implies that the entrance length increases with increasing Reynolds number (because a thinner boundary layer requires longer distance for the boundary layer to merge). Figure 8.1 also shows that shear stress decreases from the entrance along the tube and becomes a constant value when the flow reaches the fully developed condition, and shear stress increases with Reynolds number (because of a thinner boundary layer from the entrance and the longer entrance length). For a turbulent flow, the entrance length is harder to determine; the entrance length is around 10–20 tube diameter. It is hard to distinguish whether the turbulent flow is fully developed or not from 10 to 20 tube diameter downstream [1–4].

Figure 8.2 shows the thermal boundary-layer development (due to thermal conductivity and velocity) for flow entering a circular tube (or between two parallel plates). The thermal boundary layer starts from the tube (or plate) entrance and grows along the tube (or plate) length. The temperature profile keeps changing from the entrance due to adding heat along the tube (or plate) wall. The flow becomes "thermally fully developed flow" when the thermal boundary thickness is the same as the tube radius (or half-spacing between the two plates). The dimensionless temperature profile no longer changes after being thermally fully developed (but the temperature still keeps increasing).
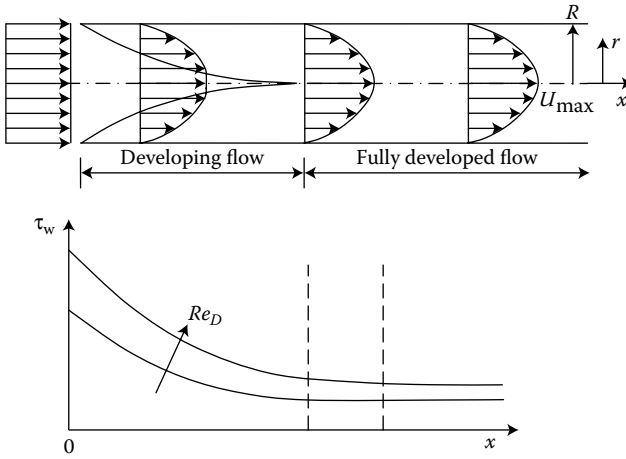
FIGURE 8.1
Velocity profile and shear stress distribution in a circular tube or between two parallel plates.

$$\left(\frac{X_{f,h}}{D}\right)_{Lam} = 0.05 Re_D$$

$$\left(\frac{X_{f,h}}{D}\right)_{Turb} \cong 20+$$

$$\left(\frac{X_{f,t}}{D}\right)_{Lam} = 0.05 Re_D \, Pr$$
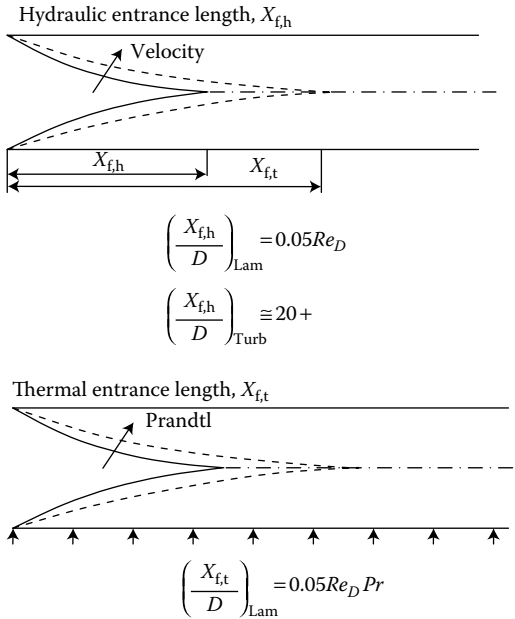
FIGURE 8.2
Hydraulic entrance length and thermal entrance length in a circular tube or between two parallel plates.

For the laminar flow, the thermal entrance length to tube diameter ratio is about 5% of Reynolds number (based on the tube diameter) times Prandtl number. This implies that the thermal entrance length increases with increasing Reynolds number (because a thinner boundary layer requires longer distance for the boundary layer to merge) and Prandtl number (because lower thermal conductivity requires longer distance to merge) [1–4].

Figure 8.2 also shows that the heat transfer coefficient decreases from the entrance along the tube and becomes a constant value when thermal boundary layer reaches the fully developed condition, and the heat transfer coefficient increases with Reynolds number (because of a thinner boundary layer from the entrance and the longer entrance length). It is noted that the thermal entrance length is identical to the hydrodynamic entrance length if $Pr = 1$. For the turbulent flow, the thermal entrance length is harder to determine; just like the hydrodynamic entrance length, the thermal entrance length is around 10–20 tube diameter. It is hard to distinguish whether the turbulent flow is thermally fully developed or not from 10 to 20 tube diameter downstream.

8.2 Fully Developed Laminar Flow and Heat Transfer in a Circular Tube or between Parallel Plates

For fluid flow in a circular tube, the Reynolds number is defined as

$$Re_D = \frac{\rho \overline{V} D}{\mu} = \frac{\overline{V} D}{\nu} = \frac{4\dot{m}}{\pi D \mu} \tag{8.1}$$

Laminar flow is observed if $Re_D \leq 2300$.

At a certain distance from the entrance, the velocity profile $u(r)$ remains unchanging along the tube (if the fluid properties remain constant). Correspondingly, there will be no velocity component in the radial direction. Also, the axial pressure gradient required to sustain the flow against the viscous forces will be constant along the tube (no momentum change). The flow is hydrodynamically fully developed. The differential governing equations for the flow inside a circular tube are [1–4]

$$\frac{\partial u}{\partial x} + \frac{1}{r} \frac{\partial (vr)}{\partial r} = 0 \tag{8.2}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + v \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \tag{8.3}$$

$$\frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} = \alpha \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{8.4}$$

If the flow is hydrodynamically fully developed, then

$$v = 0 \tag{8.5}$$

$$\frac{\partial u}{\partial x} = 0 \tag{8.6}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{8.7}$$

and $u = u(r)$ only.

Solve momentum equation

$$\frac{1}{\rho}\frac{dP}{dx} = v\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$

$$\int \frac{dP}{dx}r\, dr = \int \mu d\left(r\frac{du}{dr}\right)$$

$$\frac{dP}{dx}\frac{1}{2}r^2 = \mu r\frac{du}{dr} + C_1$$

at $r = 0$, $du/dr = 0$, $C_1 = 0$

$$\int \frac{dP}{dx}\frac{1}{2}r\, dr = \int \mu\, du$$

$$\frac{dP}{dx}\frac{1}{4}r^2 = \mu u + C_2$$

at $r = R$, $u = 0$, $C_2 = (1/4)R^2(dP/dx)$

$$\frac{1}{4}r^2\frac{dP}{dx} = \mu u + \frac{1}{4}R^2\frac{dP}{dx}$$

$$u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 - R^2\right)$$

and $r = 0$, $u = u_{max} = -(1/4\mu)R^2(dP/dx)$.

We obtain $u = -u_{max}(r^2 - R^2)/R^2$

$$\frac{u}{U_{max}} = 1 - \left(\frac{r}{R}\right)^2 \tag{8.8}$$

The bulk mean velocity is defined as

$$\overline{V} = \frac{1}{\pi R^2} \int_0^R u 2\pi r \, dr \tag{8.9}$$

$$= \frac{1}{\pi R^2} \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] u_{max} 2\pi r \, dr$$

$$= \frac{2u_{max}}{R^2} \left[\frac{1}{2}r^2 - \frac{1}{4R^2}r^4\right]\Big|_0^R$$

$$\overline{V} = \frac{1}{2}U_{max} \tag{8.10}$$

8.2.1 Fully Developed Flow in a Tube: Friction Factor

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu U_{max} \left(-\frac{2r}{R^2}\right)\Big|_{r=R} \tag{8.11}$$

where $u/U_{max} = 1 - (r/R)^2$

$$\tau_w = \mu U_{max} \frac{-2}{R}$$

$$f = \frac{\tau_w}{(1/2)\rho \overline{U}^2} = \frac{\mu U_{max}(-2/R)}{(1/2)\rho \left((1/2)U_{max}\right)^2} = \frac{16\mu}{\rho \overline{U}D} = \frac{16}{Re_D} \sim \frac{1}{Re_D} \tag{8.12}$$

where $Re_D = \rho U D/\mu$.

Figure 8.3 shows the force balance in the fully developed flow region:

$$\Delta P A_c + \tau_w \pi D \Delta x = 0$$

$$\Delta P = -\frac{\tau_w \pi D \Delta x}{(1/4)\pi D^2}$$

$$\frac{\Delta P}{\Delta x} = \frac{-4\tau_w}{D} = -\frac{4}{D} f \frac{1}{2}\rho \overline{U}^2 \tag{8.13}$$

$$= -\frac{4}{D}\frac{1}{2}\rho \overline{U}^2 \frac{16}{Re_D} = -\frac{32\mu \overline{U}}{D^2} = -\frac{32\mu}{D^2}\frac{\rho}{\rho}\frac{D}{D}\frac{\mu}{\mu}\overline{U} = -\frac{32\mu^2}{D^3 \rho}\frac{\rho D \overline{U}}{\mu}$$

$$= -\frac{32\mu^2}{D^3 \rho} Re_D \sim Re_D \tag{8.14}$$

And pumping power can be obtained as $P \cong \Delta P$ (volume flow). Figure 8.4 shows that the friction factor decreases and the pressure drop increases with Reynolds number.
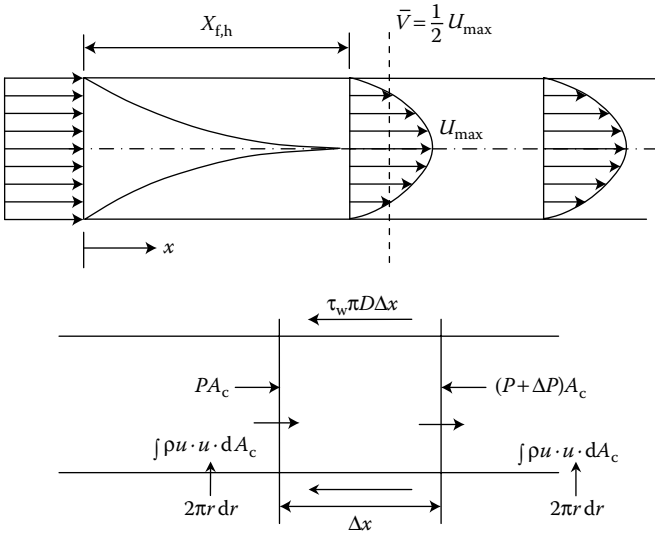
FIGURE 8.3
Force balance in fully developed flow region.

Laminar flow heat transfer depends on thermal BCs. However, turbulent flow heat transfer is fairly independent of thermal BCs (particularly for Prandtl number around one such as air). Typical thermal BCs are case 1, uniform heat flux and case 2, uniform wall temperature.

8.2.2   Case 1: Uniform Wall Heat Flux

Figure 8.5 shows the laminar flow in a circular tube with a uniform surface heat flux condition and the thermal boundary layer, temperature, and the heat transfer coefficient (Nusselt number) along the tube. The following outlines, step by step, how to obtain the results shown in Figure 8.5.
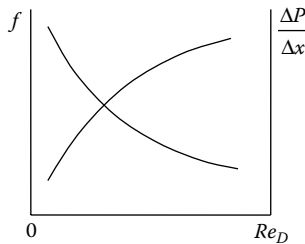
FIGURE 8.4
Friction factor and pressure drop versus Reynolds number in fully developed flow region.
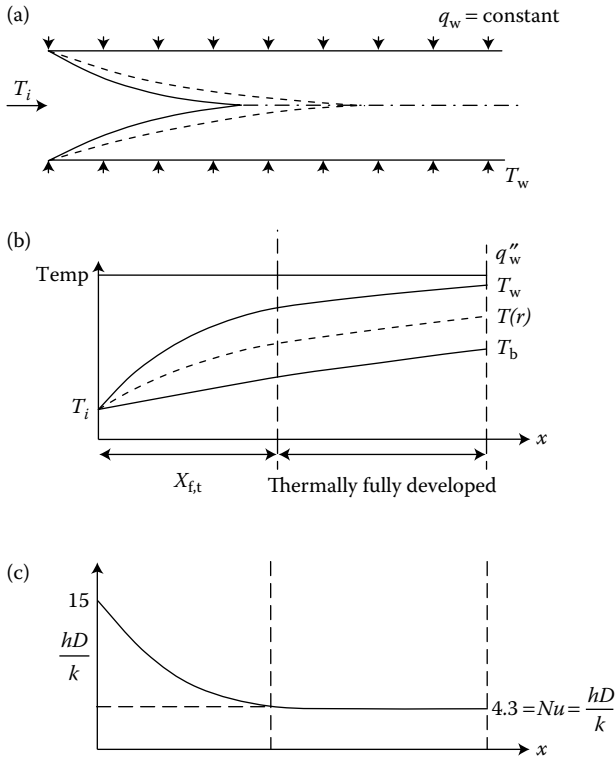
FIGURE 8.5
Laminar flow in a circular tube with uniform surface heat flux condition. (a) Thermal boundary layer; (b) temperature; and (c) heat transfer coefficient.

$$q''_w = \frac{q_w}{A_s} = h(T_w - T_b) \tag{8.15}$$

$$\underbrace{q''_w}_{\text{const.}} = \frac{q_w}{A_s} = h\underbrace{(T_w - T_b)}_{\text{const.}} \tag{8.16}$$

Bulk mean temperature

$$T_b = \frac{\int_0^R \rho u T \cdot 2\pi r \, dr}{\int_0^R \rho u \cdot 2\pi r \, dr} \tag{8.17}$$

From energy balance,

$$q''_w = \frac{q_w}{A_s} = \dot{m}C_p(T_{b,o} - T_{b,i})/A_s \tag{8.18}$$

$$q''_w = \frac{\dot{m}C_p}{\pi D} \frac{dT_b}{dx} \quad \Rightarrow \quad \frac{dT_b}{dx} = \text{const} \tag{8.19}$$

For thermally fully developed flow,

$$\frac{\partial}{\partial x}\left(\frac{T-T_w}{T_b-T_w}\right)=0 \tag{8.20}$$

$$\frac{T-T_w}{T_b-T_w}=f(r)\neq f(x)$$

$$\frac{\partial T}{\partial x}-\frac{\partial T_w}{\partial x}-\underbrace{\frac{T-T_w}{T_b-T_w}\left(\frac{\partial T_b}{\partial x}-\frac{\partial T_w}{\partial x}\right)}_{=0}=0$$

$$\frac{\partial T}{\partial x}=\frac{\partial T_w}{\partial x}=\frac{\partial T_b}{\partial x}=\text{constant} \tag{8.21}$$

From energy equation 8.4,

$$\rho C_p u\frac{\partial T}{\partial x}=k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=T(r)\text{ only} \tag{8.22}$$

$$\int r\frac{\rho C_p}{k}\underbrace{2\overline{V}\left[1-\left(\frac{r}{R}\right)^2\right]}_{u}\underbrace{\frac{q_w''}{\dot{m}C_p/\pi D}}_{\frac{\partial T}{\partial x}}\,dr=\int d\left(r\frac{dT}{dr}\right)$$

And thermal BCs are

$$r=0,\quad T=T_c\quad\text{or}\quad\frac{\partial T}{\partial r}=0$$

$$r=R,\quad T=T_w$$

$$T-T_c=\frac{\rho C_p}{k}\frac{dT}{dx}U_{\max}\left(\frac{r^2}{4}-\frac{r^4}{16R^2}\right)$$

$$T_w-T_c=\frac{\rho C_p}{k}\frac{\partial T}{\partial x}U_{\max}\left(\frac{R^2}{4}-\frac{R^4}{16R^2}\right)=\frac{\rho C_p}{k}\frac{\partial T}{\partial x}U_{\max}\frac{3}{16}R^2$$

where

$$T_b=\frac{\int_0^R\rho uT\cdot 2\pi r\,dr}{\int_0^R\rho u\cdot 2\pi r\,dr}=T_c+\frac{7}{96}\frac{\rho C_p}{k}\frac{dT}{dx}U_{\max}R^2$$

$$T_w-T_b=\frac{\rho C_p}{k}\frac{dT}{dx}U_{\max}\left(\frac{3}{16}R^2-\frac{7}{96}R^2\right) \tag{8.23}$$

Therefore, h can be determined by substituting Equations 8.19 and 8.23:

$$h = \frac{q''_w}{T_w - T_b} = \frac{(\dot{m}C_p/\pi D)(dT_b/dx)}{\rho(C_p/k)(dT/dx)u_{\max}\left((3/16)R^2 - (7/96)R^2\right)}$$

$$= \frac{(\rho\pi R^2 \bar{V}C_p/\pi 2R)(dT/dx)}{(\rho C_p/k)(dT/dx)u_{\max}(11/96)R^2} = \frac{(\rho C_p \pi R^2 (1/2)u_{\max}/\pi 2R)(dT/dx)}{(\rho C_p/k)(dT/dx)u_{\max}(11/96)R^2}$$

$$= \frac{k}{(44/96)R} = \frac{96}{22}\frac{k}{D}$$

$$Nu_D \equiv \frac{hD}{k} = \frac{96}{22} = 4.314 \tag{8.24}$$

8.2.3 Case 2: Uniform Wall Temperature

For the case of uniform wall temperature ($T_w = \text{constant}$),

$$\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x}\frac{T - T_w}{T_b - T_w} \tag{8.25}$$

The energy equation becomes

$$\rho C_p 2\bar{V}\left[1 - \left(\frac{r}{R}\right)^2\right]\frac{dT_b}{dx}\frac{T - T_w}{T_b - T_w} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{8.26}$$

Assuming a temperature profile, a final temperature distribution may be obtained by using an iterative procedure. The resulting Nusselt number is

$$Nu_D = \frac{hD}{k} = 3.66 \tag{8.27}$$

Examples

8.1. Internal flow, fully developed laminar forced convection: Consider a low-speed, constant-property, fully developed laminar flow between two parallel plates at $y = \pm H$, as shown in Figure 8.6. The plates are electrically heated to give a uniform wall heat flux. Determine the velocity profile, the friction factor, and the Nusselt number.
   Assumptions:
     Low speed $\Rightarrow \Phi = 0$
   Constant properties:
     Fully developed $\Rightarrow du/dx = 0$
     Thermally fully developed $\Rightarrow dT/dx = \text{const}$
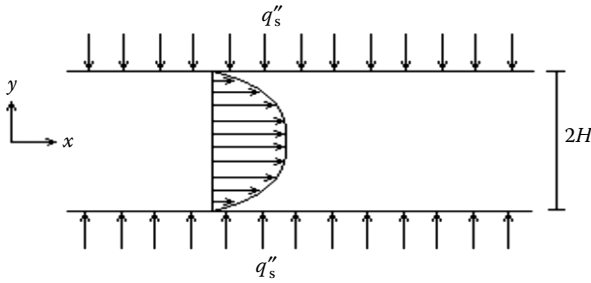
FIGURE 8.6
Two parallel plates at a uniform wall heat flux.

a. Velocity profile
   Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Momentum equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$v = 0$

$$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = v\frac{\partial^2 u}{\partial y^2} \quad \Rightarrow \quad \text{Governing equation} \qquad (8.28)$$

with solution

$$u = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 + c_1 y + c_2 \qquad (8.29)$$

where $v = \mu/\rho$
Boundary conditions:
   At $y = 0$, $\partial u/\partial y = 0$ (maximum velocity)
   At $y = H$, $u = 0$
   $c_1 = 0$

$$c_2 = -\frac{1}{2\mu}\frac{\partial P}{\partial x}H^2$$

Substituting this into Equation 8.29 and rearranging

$$u = -\frac{H^2}{2\mu}\frac{\partial P}{\partial x}\left[1 - \frac{y^2}{H^2}\right] \tag{8.30}$$

$$u_{max} \text{ at } y = 0$$

$$u_{max} = -\frac{H^2}{2\mu}\frac{\partial P}{\partial x}$$

$$u = u_{max}\left[1 - \frac{y^2}{H^2}\right] \tag{8.31a}$$

$$u_m = \frac{\int \rho u dA}{\rho A} = \frac{2\int_0^H u_{max}\left[1 - \frac{y^2}{H^2}\right]dy}{2H}$$

$$u_m = \frac{u_{max}\left[y - (y^3/3H^2)\right]_0^H}{H} = \frac{u_{max}\left((2/3)H\right)}{H}$$

$$u_m = \frac{2}{3}u_{max}$$

Thus,

$$u = \frac{3}{2}u_m\left[1 - \frac{y^2}{H^2}\right] \tag{8.31b}$$

b. Friction factor:

$$f \equiv \frac{-(\partial P/\partial x)\,D_h}{\rho u_m^2/2}$$

$$\Delta P A_c + \tau_w(2w)\Delta x = 0$$

$$\frac{\Delta P}{\Delta X} = \frac{-2\tau_w w}{A_c} = \frac{-2\tau_w w}{2wH} = -\frac{\tau_w}{H}$$

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=H} = \mu\left(\frac{3}{2}\right)u_m\left(\frac{2y}{H^2}\right)_0^H = 3\mu\frac{u_m}{H}$$

Therefore,

$$\frac{\Delta P}{\Delta x} = -\frac{3\mu u_m}{H^2}$$

$$D_H = \frac{4A_c}{P} = \frac{4(2wH)}{2w} = 4H$$

Thus,

$$f = \frac{(3\mu u_m/H^2)(4H)}{\rho u_m^2/2} = \frac{24\mu u_m H}{\rho u_m^2 H^2}$$

$$f = \frac{24\mu}{u_m H \rho}$$

But,

$$Re_{Dh} = \frac{\rho u_m D_H}{\mu} = \frac{4 u_m H \rho}{\mu}$$

Finally,

$$f = \frac{96}{Re_{Dh}} \tag{8.32}$$

c. Energy equation:

$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \dot{q} + \Phi$$

$\Phi = 0$ (because of low-speed flow)
$\dot{q} = 0$ (no internal heat generation)

$$\frac{dT}{dx} = \text{const} \implies \frac{d^2 T}{dx^2} = 0$$

The governing equation becomes

$$\frac{\partial^2 T}{\partial y^2} = \frac{u}{\alpha} \frac{\partial T}{\partial x} \tag{8.33}$$

With solution (after substitution of Equation 8.31b)

$$T = \frac{3}{2} \frac{u_m}{\alpha} \left[\frac{y^2}{2} - \frac{y^4}{12H^2} \right] \frac{dT}{dx} + c_1 y + c_2$$

Boundary conditions:
$y = 0, \quad dT/dy = 0 \rightarrow c_1 = 0$
$y = b, \quad T = T_s$

$$c_2 = T_s - \frac{5}{8} \frac{u_m}{\alpha} H^2 \frac{dT}{dx}$$

$$T = \frac{3}{2} \frac{u_m}{\alpha} H^2 \left[\frac{y^2}{2H^2} - \frac{y^4}{12H^4} - \frac{5}{12} \right] \frac{dT}{dx} + T_s \tag{8.34}$$

$$T_m \equiv \frac{\int \rho c_v u T \, dA}{\rho c_v u_m A}$$

$$T_m = \frac{2 \int_0^H \rho c_v \left\{ \frac{3}{2} u_m \left[1 - \frac{y^2}{H^2} \right] \right\} \left\{ \frac{3}{2} \frac{u_m}{\alpha} H^2 \left[\frac{y^2}{2H^2} - \frac{y^4}{12H^4} - \frac{5}{12} \right] \frac{dT}{dx} + T_s \right\} dy}{\rho c_v u_m (2H)}$$

$$T_m = \frac{9}{4} \frac{u_m H^2}{\alpha} \left\{ \left[\frac{1}{6} - \frac{1}{60} - \frac{5}{12} - \frac{1}{10} + \frac{1}{84} + \frac{5}{36} \right] \frac{dT}{dx} \right\} + \frac{3}{2} \left\{ T_s - \frac{T_s}{3} \right\}$$

$$T_m = -\frac{17}{35} \frac{u_m H^2}{\alpha} \frac{dT}{dx} + T_s \qquad (8.35)$$

$$q_s'' = h(T_s - T_m)$$

$$q_s'' = - k \frac{\partial T}{\partial y} \bigg|_{y=H}$$

$$h = \frac{k \left[(3/2)(u_m/\alpha) \cdot (2/3)H \right] (dT/dx)}{T_s + (17/35)(u_m/\alpha)H^2 (dT/dx) - T_s}$$

$$h = \frac{35}{17} \frac{k}{H}$$

$$Nu_D = \frac{h(4H)}{k} = \frac{35}{17} \frac{k}{H} \left(\frac{4H}{k} \right)$$

$$Nu_D = \frac{140}{17} = 8.235$$

Remarks

There are many engineering applications such as electronic equipments, mini-scale channels, and compact heat exchangers that required laminar flow heat transfer analysis and design. In the undergraduate-level heat transfer, students are expected to know many heat transfer relations between Nusselt numbers and Reynolds and Prandtl numbers for developing and fully developed flows inside circular tubes at various surface thermal BCs. Students are expected to calculate heat transfer coefficients from these relations by giving Reynolds and Prandtl numbers.

In the intermediate-level heat transfer, this chapter focuses on how to solve fully developed heat transfer problems for flow between two parallel plates or inside circular tubes at uniform surface heat flux BCs. Students are expected to know how to analytically determine the velocity profile, the friction factor, the temperature profile, and the Nusselt number for these cases. Here we do not include how to analytically determine the heat transfer coefficient at uniform surface temperature BCs.

In advanced heat convection, students will learn how to analytically predict heat transfer in both developing flow and thermal entrance regions; with

various thermal BCs such as variable surface heat flux as well as variable surface temperature BCs; for flow in rectangular channels with various aspect ratios and flow in annulus with various thermal BCs. They require more complex mathematics and are beyond the intermediate-level heat transfer.

PROBLEMS

8.1. Consider a steady constant-property laminar flow between two parallel plates at $y = \pm \ell$. The plates are electrically heated to give a uniform wall heat flux. The differential equations for momentum and energy are listed here for reference:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

a. Assume a low-speed, slug flow velocity profile (i.e., a uniform velocity profile) between two parallel plates, and also assume a thermally, fully developed condition, and write down the simplified equations for momentum and energy and the associated BCs that can be used for this problem.

b. Under the assumption in (a), determine the Nusselt number on the plate.

c. Consider a fully developed velocity profile (i.e., a parabolic velocity profile) between two parallel plates and a thermally, fully developed condition, and comment on whether the Nusselt number on the plate will be higher, the same, or lower than those of symmetry linear velocity profile (uniform velocity profile). Explain why.

8.2. Consider a steady constant-property laminar flow between two parallel plates at $y = \pm \ell$. The plates are electrically heated to give a uniform wall heat flux. The differential equations for momentum and energy are listed here for reference:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

a. Assume a low-speed, symmetry linear velocity profile (i.e., $u = a + by$ with maximum velocity at $y = 0$, zero velocity at $y = \pm\ell$) between two parallel plates, and also assume a thermally, fully developed condition, and write down the simplified equations for momentum and energy and the associated BCs that can be used for this problem.

b. Under the assumption in (a), determine the Nusselt number on the plate.

c. Consider a fully developed velocity profile (i.e., a parabolic velocity profile) between two parallel plates and a thermally, fully developed condition, and comment on whether the Nusselt number on the plate will be higher, the same, or lower than those of the symmetry linear velocity profile ($u = a + by$)? Explain why.

8.3. Internal flow, fully developed laminar forced convection: Consider a low-speed, constant-property, fully developed laminar flow between two parallel plates, with one plate insulated and the other uniformly heated. Determine the Nusselt number.

8.4. Consider a concentric circular-tube annulus with a radius ratio of $r_i/r_o = 0.6$. Let the inner tube wall be heated at a constant rate and the outer tube wall remains insulated. Let the fluid be a low-Prandtl number fluid, and assume a slug flow inside of the annulus. Develop an expression for the Nusselt number at the inner surface by means of the following steps:

a. Discuss how temperature and velocity profiles develop in the system by considering Pr and the type of flow present. Indicate the system conditions. (Is the flow and or temperature developed?)

b. Let T_m be the mass average fluid temperature, and T_{s,r_i} the surface temperature at the inner surface with radius r_i. Form the thermal BCs; what can be said about $(T_m - T_{s,r_i})$ and $\partial T/\partial x$?

c. Draw a diagram, indicating BCs for both temperature and velocity and indicate the assumptions used to simplify the energy equation.

d. Solve for T using the simplified velocity profile, BCs, and the energy equation.

e. Find the heat transfer coefficient and the Nusselt number by calculating T_m, the mass average fluid temperature, and heat flux at the inner surface.

Remember that the energy equation in cylindrical coordinates is given by

$$\rho c \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_x \frac{\partial T}{\partial x} \right)$$

$$= \frac{1}{r} \frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \dot{q},$$

where x represents the axial direction of the cylinder.

8.5. Consider a low-speed, constant-property fluid, fully developed laminar flow between two parallel plates located at $y = \pm b$. The plates are electrically heated to give a uniform heat flux.

a. Determine the velocity profile and define the bulk velocity (u_b) in terms of the pressure gradient driving the flow.

b. Using the hydraulic diameter $D_h = 4A_c/P$, show that the friction factor is given by $f = 96/Re_{D_h}$.

c. Obtain an expression for the heat convection coefficient and the Nusselt number.

(*Hint*: Consider the fully developed condition to approximate $\partial T/\partial x$; and define $Y(y) = T - T_b$, where $T_b = 2\int_0^b uT\,dy/2bu_b$.)

8.6. Consider a steady constant-property laminar flow between two parallel plates at $y \pm \ell$. The plates are electrically heated to give a uniform wall heat flux. The differential equations for momentum and energy are listed here for reference:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

a. Assume a low-speed, linear velocity profile (i.e., $u = u_m$ $(1 - y/\ell)$ with maximum velocity at $y = 0$, and zero velocity at $y \pm \ell$) between two parallel plates, and also assume a thermally, fully developed condition, and write down the simplified equations for momentum and energy and the associated BCs that can be used for this problem.

b. Under the assumption in (a), determine the Nusselt number on the plate.

c. Consider a fully developed velocity profile (i.e., a parabolic velocity profile) between two parallel plates and a thermally, fully developed condition, and comment on whether the Nusselt number on the plate will be higher, the same, or lower than those of symmetry linear velocity profile in (a)? Explain why.

8.7. Consider an incompressible laminar 2-D flow in a parallel plate channel as shown below. The top plate is pulled at a constant velocity U_T. The top and bottom plates are maintained at constant heat flux q_s. Flow is both hydrodynamically and thermally fully developed. Assume that the pressure gradient is zero in a parallel plate channel.

a. Obtain differential equations governing the velocity $U(Y)$ and temperature $T(Y)$ fields.

b. Use appropriate BCs to evaluate $U(Y)$. Obtain $T(Y)$. Do not attempt to evaluate constants of integration for the temperature field.

8.8. Find the Nusselt number for the following problems.

a. Fully developed Couette flow (i.e., assume that velocity and temperature profiles do not change along the channel) with the lower plane wall at uniform wall temperature T_0 and the upper plane wall at T_1. If the velocity profile is a linear profile ($U = 0$ at the lower plane wall, $U = V$ at the upper plane wall), find the temperature profile from the energy equation.

b. Fully developed Poiseuille flow (i.e., assume that velocity and temperature profiles do not change along the channel) with

the lower plane wall at uniform temperature T_0 and the upper plane wall at T_1. If the velocity profile is a parabolic profile, find the temperature profile from the energy equation.

8.9. A 2-D channel flow is subjected to a uniform heat flux on one wall and insulated on the other wall. Assume that the viscous dissipation is negligible and the properties are constant. Determine the following.

a. The governing momentum and energy equations with the appropriate BCs for the developing region. Do not solve the equations; however, show the details of simplifying the governing equations. Sketch the temperature and velocity profiles with respect to y at two x positions.

b. Repeat (a) for the fully developed region.

c. Sketch the temperature profile in the fully developed region if both walls are insulated. Consider two cases: (1) viscous dissipation is negligible and (2) viscous dissipation is not negligible.

8.10. Consider liquid metal flow in a parallel-plate channel at a uniform wall heat flux condition.

a. Using the momentum and energy differential equations and making the appropriate assumptions, derive the surface Nusselt number if flow is laminar and the temperature profile is in a fully developed condition.

b. Using the momentum and energy differential equations and making the appropriate assumptions, outline the methods (no need to solve) in order to determine the surface Nusselt number if the flow is laminar and the temperature profile is in a developing condition. Describe that the surface Nusselt numbers for (b) will be higher or lower than those for (a). Explain why.

c. Nu, T_w, and T_b versus x from the entrance to the fully developed region.

8.11. Consider a steady constant-property laminar flow between two parallel plates at $y = \pm\ell$. The plates are electrically heated to give a uniform wall heat flux. The differential equations for momentum and energy are listed here for reference:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \frac{v}{c_p}\left(\frac{\partial u}{\partial y}\right)^2$$

a. What is the physical meaning of the last term shown in the above differential energy equation? Explain under what conditions the last term should be included in order to solve the temperature distribution between two parallel plates.

b. Assume a low-speed, slug-flow velocity profile (i.e., a uniform velocity profile) between two parallel plates, and also

　　　　assume a thermally, fully developed condition, and write
　　　　down the simplified equations for momentum and energy and
　　　　the associated BCs that can be used for this problem.

　　c. Under the assumption in (b), determine the Nusselt number
　　　　on the plate.

8.12. Water at 43°C enters a 5-cm-ID pipe at a rate of 6 kg/s. If the pipe is
　　　　9 m long and maintained at 71°, calculate the exit water tempera-
　　　　ture and the total heat transfer. Assume that the Nusselt number of
　　　　the flow in the pipe is 3.657. Also, the thermal properties of water
　　　　are $\rho = 1000 \, \text{kg/m}^3$, $c_p = 4.18 \, \text{kJ/(kg K)}$, and $k = 0.6 \, \text{W/(m K)}$.

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9

Natural Convection

9.1 Laminar Natural Convection on a Vertical Wall: Similarity Solution

Natural convection can occur when the solid surface temperature is different from the surrounding fluid. For example, natural convection can take place between a heated (or cooled) vertical (or horizontal) plate or tube and the surrounding fluid. Figure 9.1 shows the velocity profile, temperature profile, and heat transfer from a hot vertical wall to a cold fluid due to natural convection. The hot vertical wall conducts heat to the fluid particle (fluid layer) next to the wall and the heated fluid particle (fluid layer) conducts heat to the next cooler fluid particle, and so on. Therefore, the fluid particle near the hot wall is lighter than that is away from the hot wall and natural circulation takes place (near the wall, the hot fluid moving up and away from the wall, cold fluid moving down) due to gravity. This buoyancy-driven natural convection flow is primarily due to density gradient (temperature gradient) from the hot vertical wall and cold surrounding fluid. The key parameter/driving force to determine natural convection is Grashof number, a ratio of buoyancy force to viscous force (buoyancy force tries to move the fluid up but viscous force tries to resist it from moving). Another parameter is Prandtl number, a fluid property showing the ratio of kinematic viscosity to thermal diffusivity. The product of Grashof number with Prandtl number is called Rayleigh number, another way of measuring the natural convection. The following shows the definition of Grashof number, Rayleigh number, and 2-D laminar natural convection boundary-layer equations from a heated vertical wall [1–4].

$$Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{v^2} \tag{9.1}$$

$$Ra_x = Gr_x \cdot Pr \tag{9.2}$$

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.3}$$
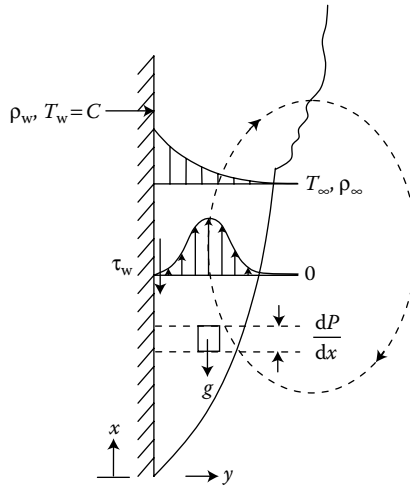
FIGURE 9.1
Natural convection boundary layer from a heated vertical wall.

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + v\frac{\partial^2 u}{\partial y^2} \tag{9.4}$$

Energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{9.5}$$

The following shows how to derive the above natural convection momentum equation from the original momentum equation:

$$\text{X-momentum} \quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} - g + v\frac{\partial^2 u}{\partial y^2}$$

From outside of the boundary layer $0 = -(1/\rho_\infty)(\partial P/\partial x) - g$

$$\because \beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P \cong -\frac{1}{\rho}\frac{\rho_\infty - \rho}{T_\infty - T}$$

$$\therefore \rho_\infty - \rho = \rho\beta(T - T_\infty)$$

$$\therefore \frac{\partial P}{\partial x} = -\rho_\infty g \Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial x} - g$$

$$= -\frac{1}{\rho}(-\rho_\infty g) - g = \frac{g}{\rho}(\rho_\infty - \rho) = g\beta(T_\infty - T)$$

From the above momentum equation, one can see that natural convection is due to temperature difference between the surface and fluid and the gravity force. This implies that there is no natural convection if there exists no temperature gradient or no gravity force. The larger delta T and gravity (means larger Grashof number) will cause larger natural circulation and results in thinner boundary-layer thickness and higher friction (shear) and higher heat transfer coefficient. The Grashof number in natural convection plays a similar role as Reynolds number does in forced convection; the larger Grashof number causes higher heat transfer in natural convection as the greater Reynolds number has higher heat transfer in forced convection. Prandtl number plays the same role in both natural and forced convection, basically the fluid property.

Just like in forced convection, both similarity and integral methods can be used to solve natural convection boundary-layer equations. The following only outline the similarity method from Ostrach in 1953.

Similarity variable:

$$\eta = y\sqrt[4]{\frac{g\beta(T_w - T_\infty)}{4\nu^2 x}} = \frac{y}{x}\left(\frac{Gr_x}{4}\right)^{1/4} \tag{9.6}$$

where

$$Gr_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2}$$

Similarity functions for velocity and temperature:

$$f(\eta) = \frac{\Psi(x, \eta)}{4\nu\,(Gr_x/4)^{1/4}} \tag{9.7}$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{9.8}$$

Put them into the above momentum equation and energy equation, respectively:

$$u = \frac{\partial \Psi}{\partial y} = \cdots$$

$$v = -\frac{\partial \Psi}{\partial x} = \cdots$$

$$\frac{\partial T}{\partial x} = \cdots$$

$$\frac{\partial T}{\partial y} = \cdots$$

The resultant similarity momentum and energy equations are

$$f''' + 3ff'' - 2f'^2 + \theta = 0 \tag{9.9}$$

$$\theta'' + 3Prf\theta' = 0 \tag{9.10}$$

The related BCs are

$$f(0) = f'(0) = 0, \quad f'(\infty) = 0 \tag{9.11}$$

$$\theta(0) = 1, \quad \theta(\infty) = 0 \tag{9.12}$$

These can be solved by the fourth-order Runge–Kutta method in order to obtain the velocity and temperature profiles across the natural convection boundary layer. Figure 9.2 shows typical dimensionless velocity and
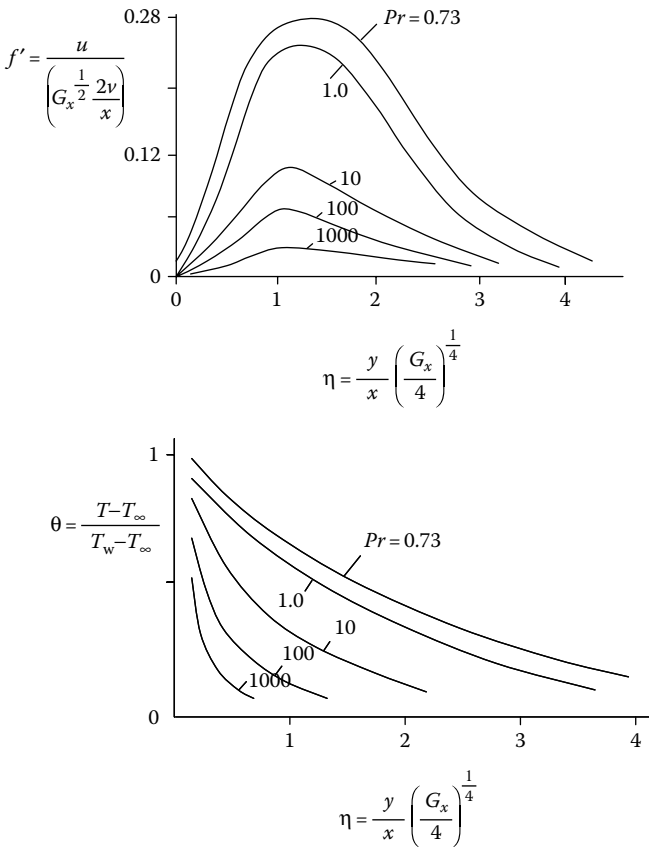
FIGURE 9.2
Dimensionless velocity and temperature profiles from heated vertical wall.

temperature profiles from the heated vertical wall, for different Prandtl fluids.

$$f' = \frac{u}{2\sqrt{gx}}\sqrt{\frac{T_\infty}{T_w - T_\infty}}$$

$$= \frac{ux}{2v}G_x^{-1/2}$$

$$= \frac{u}{\left(G_x^{1/2}(2v/x)\right)} \tag{9.13}$$

From the dimensionless velocity and temperature profiles, the associated heat flux and the heat transfer coefficient (or Nusselt number) can be determined.

$$q''_w = -k\left.\frac{\partial T}{\partial y}\right|_0 = -\frac{k}{x}(T_w - T_\infty)\left(\frac{Gr_x}{4}\right)^{1/4}\left.\frac{d\theta}{d\eta}\right|_{y=0} \tag{9.14}$$

where

$$\left.\frac{d\theta}{d\eta}\right|_{y=0} = \theta'(0) = f(Pr)$$

$$\therefore h = \frac{q''_w}{T_w - T_\infty} = \frac{-k(\partial T/\partial y)_0}{T_w - T_\infty}$$

$$Nu = \frac{hx}{k} = -\left(\frac{Gr_x}{4}\right)^{1/4}\left.\frac{d\theta}{d\eta}\right|_{y=0} \tag{9.15}$$

Numerical results: for laminar natural convection:

Pr	0.01	0.733	1	2	10	100	1000
$-\left.\dfrac{d\theta}{d\eta}\right|_0$	0.081	0.508	0.567	0.716	1.169	2.191	3.966

For air, $Pr = 0.733$,

$$Nu_x = \frac{hx}{k} = 0.359 Gr_x^{1/4} \tag{9.16}$$

$$\overline{Nu_x} = \frac{\overline{h_x}L}{k} = \frac{4}{3}Nu_L = 0.478 Gr_L^{1/4} \tag{9.17}$$

where

$$\overline{h_x} = \frac{1}{L}\int_0^L h_x dx = -\frac{4}{3}\frac{k}{L}\left(\frac{Gr_L}{4}\right)^{1/4}\frac{d\theta}{d\eta}\bigg|_0 = \frac{4}{3}h_L$$

$$Gr_L = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} \tag{9.18}$$

$$Nu_x = \frac{3}{4}\left[\frac{2Pr}{5\left(1 + 2Pr^{1/2} + 2Pr\right)}\right]^{1/4}(Gr_x Pr)^{1/4} \tag{9.19}$$

$$Nu_x = 0.6(Gr_x Pr^2)^{1/4}, \quad \text{if } Pr \to 0 \tag{9.20}$$

$$Nu_x = 0.503(Gr_x Pr)^{1/4}, \quad \text{if } Pr \to \infty \tag{9.21}$$

In general,

$$Nu_x = a(Gr_x Pr)^b = a\,Ra_x^b$$

where Rayleigh number,

$$Gr_x\,Pr = Ra_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu\alpha}$$

Compared to forced convection

$$Nu_x = a\,Re_x^m\,Pr^n$$

Note: The following is a simple guideline whether the problem can be solved by forced convection, natural convection, or mixed (combined forced and natural) convection.

If $Gr_x/Re_x^2 < 1$, the problem can be treated as forced convection.

If $Gr_x/Re_x^2 \cong 1$, the problem can be treated as mixed convection.

If $Gr_x/Re_x^2 > 1$, the problem can be treated as natural convection.

9.2 Laminar Natural Convection on a Vertical Wall: Integral Method

Integral approximate solution by Pohlhausen 1921: We apply the momentum and energy balance to the control volume across the boundary layer as shown in Figure 9.3.
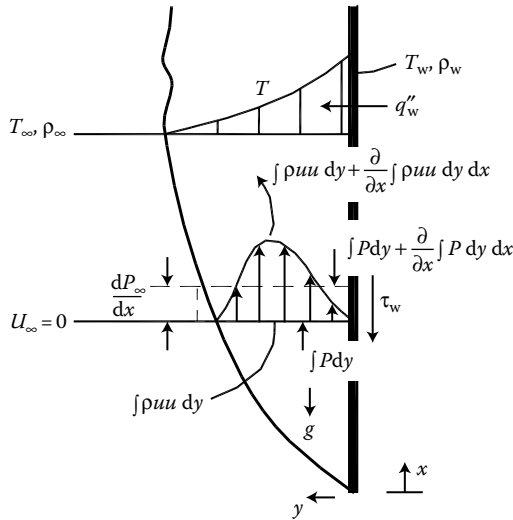
FIGURE 9.3
Integral method.

Net force = Momentum change

$$-\tau_w - \int \rho g \, dy - \int \frac{\partial P}{\partial x} dy = \frac{\partial}{\partial x} \int \rho u u \, dy$$

where $\partial P/\partial x = \partial P_\infty/\partial x = -\rho_\infty g$ (outside of the boundary layer).
  Also $\rho - \rho_\infty = -\rho_\infty \beta \, (T - T_\infty)$ from $\beta \equiv -1/\rho \, (\partial \rho/\partial T)_P$

$$\frac{\tau_w}{\rho_\infty} = \frac{\mu}{\rho_\infty} \frac{\partial u}{\partial y}\bigg|_0 = g\beta \int_0^\delta (T - T_\infty) \, dy - \frac{d}{dx} \int_0^\delta u^2 \, dy \qquad (9.22)$$

$$q'' = -k \frac{\partial T}{\partial y}\bigg|_0 = \frac{d}{dx} \int_0^{\delta_T} \rho c_p u \, (T - T_\infty) \, dy \qquad (9.23)$$

Boundary conditions: $u \, (x, 0) = 0 \quad T \, (x, 0) = T_w$

$$u \, (x, \delta) = 0, \quad T \, (x, \delta_T) = T_\infty$$
$$\frac{\partial u \, (x, \delta)}{\partial y} = 0, \quad \frac{\partial T \, (x, \delta_T)}{\partial y} = 0$$

Assuming velocity and temperature profiles to satisfy boundary-layer conditions,

$$u = u\,(x, y) = u_1 \eta\,(1 - \eta)^2 \tag{9.24}$$

$$\frac{T - T_\infty}{T_w - T_\infty} = (1 - \eta)^2 = f\,(x, y) \tag{9.25}$$

where $\eta = y/\delta(x)$; $u_1 = c_1 x^m$; $\delta\,(x) = c_2 x^n$;

$$m = \frac{1}{2}; \quad n = \frac{1}{4}$$

Put this into momentum and energy integral equations:

$$u_1\,(x) = \left(\frac{80}{3}\right)^{1/2} \left(\frac{G_x}{(20/21) + Pr}\right)^{1/2} \frac{\nu}{x}$$

$$\frac{\delta\,(x)}{x} = \left[\frac{240\,(1 + (20/21 Pr))}{Pr \cdot G_x}\right]^{1/4} \rightarrow \delta\,(x) \sim \left(\frac{x}{T_w - T_\infty}\right)^{1/4}$$

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_0 = \frac{2k\,(T_w - T_\infty)}{\delta\,(x)} = h\,(T_w - T_\infty)$$

$$Nu_x = \frac{h_x x}{k} = \frac{2x}{\delta\,(x)} = \left[\frac{Pr}{15\,((20/21) + Pr)}\right]^{1/4} \cdot Ra_x^{1/4} \tag{9.26}$$

$$Nu_x \cong 0.413 Ra_x^{1/4} \quad \text{for } Pr = 0.733 \tag{9.27}$$

Note: $Nu_x = 0.359 Ra_x^{1/4}$ for $Pr = 0.733$ by using the exact similarity solution.

$$Ra_x = \frac{g\beta\,(T_w - T_\infty)\,x^3}{\nu\alpha} < 10^8 - 10^9 - \text{Laminar natural convection}$$

Remarks

In the undergraduate-level heat transfer, we have heat transfer correlations of external natural convection for a vertical plate, an inclined plate, a horizontal plate, a vertical tube, and a horizontal tube as well as heat transfer correlations of internal natural convection for a horizontal tube, between two parallel plates, and inside a rectangular cavity with various aspect ratios. These correlations are important for many real-life engineering applications such as electronic components.

In the intermediate-level heat transfer, this chapter focuses on how to analytically solve the external natural convection from a vertical plate at

a uniform surface temperature by using the similarity method as well as the integral method. In advanced heat transfer, these methods can be modified and extended to solve mixed convection (combined natural and forced convection) problems for vertical, horizontal, and inclined plates or tubes, respectively, for various Prandtl number fluids.

PROBLEMS

9.1. Consider the system of boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = R\left(T - T_\infty\right) + v\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial\left(T - T_\infty\right)}{\partial x} + v\frac{\partial\left(T - T_\infty\right)}{\partial y} = \alpha\frac{\partial^2\left(T - T_\infty\right)}{\partial y^2}$$

subject to the BCs

$$y = 0 \quad u = 0, \quad v = 0, \quad q_w = \text{constant}$$
$$y = \infty \quad u = 0 \qquad\qquad\quad T = T_\infty$$

where the quantities R, v, and α are constants. Determine the similarity variables that will transform the equations to two ODEs. Derive the resultant ODEs.

9.2. Consider a natural convection flow over a vertical heated plate at a uniform wall temperature T_0. Let T_∞ be the free-stream temperature. The following correlation holds for the heat transfer coefficient h_x at height x:

$$\frac{h_x x}{k} = 0.443\,(Gr_x\,Pr)^{1/4}$$

The Grashof number is

$$Gr_x = g\beta x^3(T(x) - T_\infty)/v^2$$

where g is the acceleration due to gravity, β is the coefficient of thermal expansion, and v is the kinematic viscosity.
a. Draw a diagram of the system.
b. Sketch a plot of h_x along the plate length.
c. Find the average heat transfer coefficient.
d. Show that the average heat transfer coefficient between heights 0 and L is given by $\overline{Nu} = 0.59(Gr_L\,Pr)^{1/4}$.

9.3. Derive similarity momentum and energy equations shown in Equations 9.9 and 9.10.

9.4. Derive Equations 9.16 and 9.17.

9.5. Derive Equations 9.20 and 9.21.
9.6. Derive Equations 9.22 and 9.23.
9.7. Derive Equations 9.26 and 9.27.

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10

Turbulent Flow Heat Transfer

10.1   Reynolds-Averaged Navier–Stokes (RANS) Equation

When flow transitions into turbulence, both shear stress and heat transfer from the surface increase due to turbulent mixing. However, in a fully turbulent region, both shear stress and heat transfer slightly decrease again due to turbulent boundary-layer thickness growing along the surface. In this section, we discuss external and internal flow and heat transfer problem in a fully turbulence region. Figure 10.1 shows a sketch of a typical 2-D turbulent boundary layer for heated flow over a cooled flat surface and the fairly uniform velocity and temperature profiles across the boundary layer due to turbulent mixing. A laminar sublayer is developed at a very-near-wall region where turbulent mixing is damped due to viscous effect. This laminar sublayer thickness is the major resistance for velocity and temperature changing from the free-stream value to the wall. In a fully turbulent region, velocity and temperature change with time at a given location inside the turbulent boundary layer, that is, $u(x, y, t)$, $v(x, y, t)$, $T(x, y, t)$. These time- and location-dependent behaviors make turbulent flow and heat transfer much harder to analyze as compared to laminar boundary-layer flow and heat transfer problem.

The following is to show how to obtain the Reynolds-averaged Navier–Stokes (RANS) equation for a fully turbulent boundary-layer flow [1–6]. The idea is to treat a fully turbulent flow as purely random motion superimposed on a steady (time-averaged) mean flow. This means that the time-dependent value (such as instantaneous velocity and temperature, etc.) equals the time-averaged value (Reynolds-averaged value) plus the fluctuation value (due to random motion). For example, the steady (time-averaged) x-direction velocity over a period of time can be shown as

$$\bar{u} = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(t)\, \mathrm{d}t$$

and

$$\overline{u'} = \lim_{t \to \infty} \frac{1}{t} \int_0^t u'(t)\, \mathrm{d}t \cong 0$$
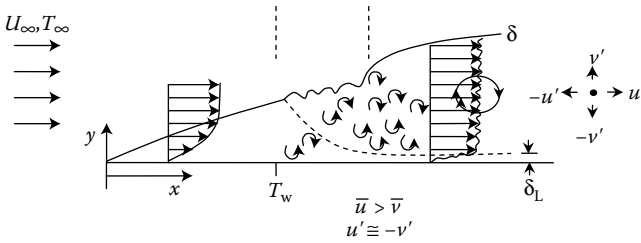
FIGURE 10.1
Typical turbulent boundary-layer flow velocity and temperature profile.

where t is a large enough time interval, time-averaged velocity $\bar{u}$ is not affected by time, and time-averaged fluctuation u is around zero. This can apply to the y-direction velocity as well as to temperature, pressure, and density.

Figure 10.2 shows the sketches for time-dependent (instantaneous) and averaged velocity and temperature over a period of time at a given location (x, y) inside a turbulent boundary layer. The positive and negative fluctuation values (up and down values) is about to cancel each other; averaged u velocity is much greater than averaged v velocity; however, the fluctuating u value is about the same as the fluctuating v value with out of phase (when fluctuating u is positive, fluctuating v is negative); and the fluctuating T value is also about the same as the fluctuating u value. This is under the condition or assumption of an isotropic turbulence case. In general, turbulent fluctuating quantities may not be necessarily the same value.

The following shows the step-by-step method toward the RANS equation.
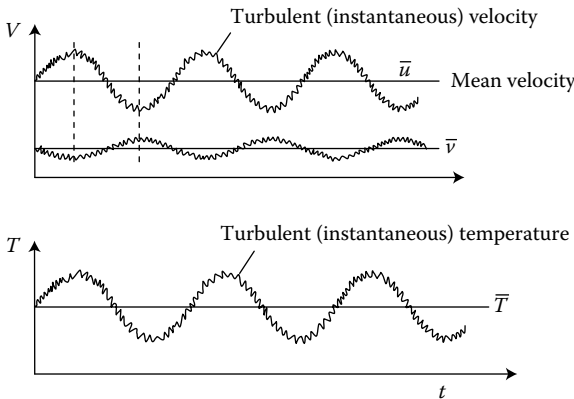
FIGURE 10.2
Instantaneous time-dependent velocity and temperature profiles inside a turbulent boundary layer.

Reynolds time-averaged method:
   Time dependent (instantaneous) = time averaged (Reynolds averaged) + fluctuation.

$$u = \bar{u} + u' \equiv u + u'$$

$$v = \bar{v} + v' \equiv v + v'$$

$$T = \bar{T} + T' \equiv T + T'$$

$$P = \bar{P} + P' \equiv P + P'$$

$$\rho = \bar{\rho} + \rho' \equiv \rho + \rho'$$

Since

$$\int_{t}^{t+\Delta t} u' \, dt = 0$$

$$\int_{t}^{t+\Delta t} v' \, dt = 0$$

$$\int_{t}^{t+\Delta t} P' \, dt = 0$$

$$\int_{t}^{t+\Delta t} T' \, dt = 0$$

Therefore,

$$\overline{v'} = \overline{T'} = \overline{\rho'} = \overline{P'} \cong 0$$

Substitute the above instantaneous quantities into the original conservation of mass, momentum, and energy equation, respectively, and then performing time average over a period of time.

10.1.1 Continuity Equation

$$\lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} \underbrace{\frac{\partial}{\partial t} \rho}_{0 \text{ (steady)}} + \frac{\partial}{\partial x} \left[(\bar{\rho} + \rho')(\bar{u} + u') \right] + \frac{\partial}{\partial y} \left[(\bar{\rho} + \rho')(\bar{v} + v') \right] = 0 \qquad (10.1)$$

$$\frac{\partial}{\partial x} \left(\bar{\rho}\bar{u} + \overline{\rho' u'} \right) + \frac{\partial}{\partial y} \left(\bar{\rho}\bar{v} + \overline{\rho' v'} \right) = 0$$

Therefore,

$$\frac{\partial}{\partial x}(\overline{\rho u}) + \frac{\partial}{\partial y}(\overline{\rho v}) = 0 \tag{10.2}$$

and

$$\frac{\partial}{\partial x}\left(\overline{\rho' u'}\right) + \frac{\partial}{\partial y}\left(\overline{\rho' v'}\right) = 0 \tag{10.3}$$

10.1.2   Momentum Equation: RANS

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{\partial}{\partial t}\rho u + \frac{\partial}{\partial x}[(\overline{\rho}+\rho')(\overline{u}+u')^2] + \frac{\partial}{\partial y}[(\overline{\rho}+\rho')(\overline{u}+u')(\overline{v}+v')] \tag{10.4}$$

0 (steady)

$$= -\frac{\partial}{\partial x}\left(\overline{P}+P'\right) + \left(\overline{\rho}+\rho'\right)\left(\overline{f_x}+f_x'\right) + \mu\left[\frac{\partial^2\left(\overline{u}+u'\right)}{\partial x^2} + \frac{\partial^2\left(\overline{u}+u'\right)}{\partial y^2}\right]$$

Therefore,

$$\frac{\partial}{\partial x}\overline{\rho u}^2 + \frac{\partial}{\partial y}\overline{\rho u v} + \frac{\partial}{\partial x}\overline{\rho u'^2} + \frac{\partial}{\partial y}\overline{\rho u' v'} + \frac{\partial}{\partial x}\overline{\rho' u'^2} + \frac{\partial}{\partial y}\overline{\rho' u' v'} + \frac{\partial}{\partial x}2\overline{u}\overline{\rho' u'}$$

$$+ \frac{\partial}{\partial y}\left(\overline{u}\overline{\rho' v'} + \overline{v}\overline{\rho' u'}\right) = -\frac{\partial\overline{P}}{\partial x} + \overline{\rho}\overline{f_x} + \overline{\rho' f_x'} + \mu\left[\frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2}\right]$$

Rearranging,

$$\left(\overline{\rho u} + \overline{\rho' u'}\right)\frac{\partial\overline{u}}{\partial x} + \left(\overline{\rho v} + \overline{\rho' v'}\right)\frac{\partial\overline{u}}{\partial y} = -\frac{\partial\overline{P}}{\partial x} + \overline{\rho}\overline{f_x} + \overline{\rho' f_x'} + \mu\left[\frac{\partial^2\overline{u}}{\partial x^2} + \frac{\partial^2\overline{u}}{\partial y^2}\right]$$

$$\tag{10.5}$$

$$\left.\begin{array}{l} + \dfrac{\partial}{\partial x}\left(\overline{\rho u'^2} + \overline{\rho' u'^2} + \overline{u}\overline{\rho' u'}\right) \\[2mm] + \dfrac{\partial}{\partial y}\left(\overline{\rho u' v'} + \overline{\rho' u' v'} + \overline{v}\overline{\rho' u'}\right) \end{array}\right\} 6 \text{ Reynolds Stress}$$

For incompressible flow,
$\rho' = 0$
X-Momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial\overline{u'^2}}{\partial x^2} - \frac{\partial\overline{u' v'}}{\partial y} + f_x \tag{10.6}$$

Y-Momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \underbrace{\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}_{\text{viscous forces}} - \underbrace{\frac{\partial \overline{u'v'}}{\partial x} - \frac{\partial \overline{v'^2}}{\partial y}}_{\text{Reynolds stress, Turbulent stress}} + f_y$$

(10.7)

From boundary-layer approximation, *Y*-Momentum equation is not important as compared with *X*-Momentum equation.

10.1.3   Enthalpy/Energy Equation

0 (steady)

$$\lim_{t\to\infty}\frac{1}{t}\int (\overline{\rho} + \rho')c_p\left[\cancel{\frac{\partial}{\partial t}}T + (\overline{u} + u')\frac{\partial}{\partial x}(\overline{T} + T') + (\overline{v} + v')\frac{\partial}{\partial y}(\overline{T} + T')\right]$$

$$= \frac{\partial}{\partial x}k\frac{\partial}{\partial x}\left(\overline{T} + T'\right) + \frac{\partial}{\partial y}k\frac{\partial}{\partial y}\left(\overline{T} + T'\right) + \frac{\partial P}{\partial t} + (\overline{u} + u')\frac{\partial\left(\overline{P} + P'\right)}{\partial x}$$

$$+ (\overline{v} + v')\frac{\partial\left(\overline{P} + P'\right)}{\partial y} + \Phi \tag{10.8}$$

Therefore,

$$c_p\left[(\overline{\rho u} + \overline{\rho'u'})\frac{\partial\overline{T}}{\partial x} + \left(\overline{\rho v} + \overline{\rho'v'}\right)\frac{\partial\overline{T}}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}k\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k\frac{\partial T}{\partial y} + \overline{u}\frac{\partial\overline{P}}{\partial x} + \overline{v}\frac{\partial\overline{P}}{\partial y} + \Phi + \rho\varepsilon \tag{10.9}$$

$$+ c_p\frac{\partial}{\partial x}\left(\overline{\rho u'T'} + \overline{\rho'u'T'} + \overline{u}\overline{\rho'T'}\right) + \left(\frac{\partial\overline{u}}{\partial x}\right)^2 + \dots$$

$$+ c_p\frac{\partial}{\partial y}\underbrace{\left(\overline{\rho v'T'} + \overline{\rho'v'T'} + \overline{v}\overline{\rho'T'}\right)}_{\text{6 Reynolds Flux}} + \left(\frac{\partial\overline{u'}}{\partial x}\right)^2 + \dots$$

where Φ is the viscous dissipation and ε is the turbulent dissipation.

For incompressible flow,

$$\rho' = 0$$

$$\Phi = 0$$

$$\varepsilon = 0$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \underbrace{\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)}_{\text{molecular conduction}} + \underbrace{\frac{-\partial \overline{T'u'}}{\partial x} - \frac{\partial \overline{T'v'}}{\partial y}}_{\text{Reynolds flux, turbulent flux}} \tag{10.10}$$

10.1.4 Concept of Eddy or Turbulent Diffusivity

The following is a summary from the above RANS equation. The continuity equation is no useful. The Y-momentum equation is small as compared with the X-momentum equation. Therefore, only one of six Reynolds stresses and one of six Reynolds fluxes remain in the RANS equation for a 2-D steady, incompressible and constant property fully turbulent boundary-layer flow. In addition to a laminar-type shear stress due to viscous effect, turbulent stress due to random velocity fluctuation plays the major contribution to the total pressure loss over a turbulent boundary layer. Similarly, turbulent flux due to random velocity with temperature fluctuation dominates the total heat transfer over the turbulent boundary layer. The real problem is how to quantify the time-averaged Reynolds stress and Reynolds flux because they are varying with the location (x, y) inside the turbulent boundary. It is assumed that Reynolds stress is proportional to the velocity gradient and the proportional constant (actually is not a constant value) is called eddy or turbulent diffusivity for momentum (turbulent viscosity divided by fluid density); similarly, Reynolds flux is proportional to temperature gradient and the proportional constant (actually is not a constant value) is called eddy or turbulent diffusivity for heat. Therefore, the real turbulent flow problem is how to determine or how to model the turbulent diffusivity for momentum (turbulent viscosity) and turbulent diffusivity for heat because they depend on the location (x, y). It is important to note that molecular Prandtl number is a ratio of fluid kinematic viscosity to thermal diffusivity and is a fluid property depending on what kind of fluid is (e.g., air or water has different molecular Prandtl numbers); however, turbulent Prandtl number is a ratio of turbulent diffusivity for momentum to turbulent diffusivity for heat and is a flow structure behavior depending on how turbulent flow is (e.g., air and water have the same turbulent Prandtl number at the same turbulent flow condition). For a simple turbulent flow problem, turbulent Prandtl number is about one (say 0.9 for most of the models), which implies that one can solve for turbulent diffusivity for heat if turbulent diffusivity for momentum has been determined/modeled. Therefore, the first question is how to determine

or model turbulent diffusivity for momentum (turbulent viscosity divided by fluid density). Once turbulent viscosity is given, one almost solves the turbulent flow and heat transfer problem.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y} - \rho\overline{u'v'}\right) = \frac{1}{\rho}\frac{\partial}{\partial y}(\tau_{\text{viscous}} + \tau_{\text{turb.}}) \quad (10.11)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho C_p}\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y} - \rho C_p\overline{v'T'}\right) = \frac{1}{\rho C_p}\frac{\partial}{\partial y}\left(q''_{\text{molecular}} + q''_{\text{turb.}}\right)$$

$$\quad (10.12)$$

$$\tau_{\text{total}} = \tau_{\text{m}} + \tau_{\text{t}} = \mu\frac{\partial u}{\partial y} + \rho\varepsilon_{\text{m}}\frac{\partial u}{\partial y} \quad (10.13)$$

where $-\rho\overline{u'v'} = \rho\varepsilon_{\text{m}}(\partial u/\partial y)$, eddy diffusivity for momentum $\varepsilon_{\text{m}} = v_{\text{t}} = (\mu_{\text{t}}/\rho)$, turbulence viscosity $\mu_{\text{t}} = \rho\varepsilon_{\text{m}}$

$$q''_{\text{total}} = q''_{\text{m}} + q''_{\text{t}} = k\frac{\partial T}{\partial y} + \rho C_p\varepsilon_H\frac{\partial T}{\partial y} \quad (10.14)$$

where $-\rho C_p\overline{v'T'} = \rho C_p\varepsilon_H(\partial T/\partial y)$, eddy diffusivity for heat $\varepsilon_H = \alpha_{\text{t}}$

$$\tau_{\text{total}} = \rho\left(v + \varepsilon_{\text{m}}\right)\frac{\partial u}{\partial y} \quad (10.15)$$

$$q''_{\text{total}} = \rho C_p(\alpha + \varepsilon_H)\frac{\partial T}{\partial y} \quad (10.16)$$

Turbulence Prandtl number

$$Pr_{\text{t}} = \frac{v_{\text{t}}}{\alpha_{\text{t}}} = \frac{\varepsilon_{\text{m}}}{\varepsilon_H} \sim 1 \quad (10.17)$$

So the unknowns are

- Turbulent viscosity μ_{t}
- Turbulent diffusivity $v_{\text{t}} = (\mu_{\text{t}}/\rho) = \varepsilon_{\text{m}}$
- Turbulent diffusivity for heat $\varepsilon_H = \alpha_{\text{t}}$

The following shows how to determine the turbulent diffusivity for momentum.

From force balance in a circular tube as shown in Figure 10.3,

$$\frac{dP}{dx} = \frac{1}{r}\frac{\partial}{\partial r}(r\tau)$$

$$\int r\frac{dP}{dx} = \int \frac{d}{dr}(r\tau)$$

$$\int r\frac{dP}{dx}\,dr = \int \frac{d}{dr}(r\tau)\,dr$$

Therefore,

$$\tau = \frac{1}{2}r\frac{dP}{dx} \sim r$$

From BCs,

$$r = 0, \quad \tau = 0,$$
$$r = R, \quad \tau = \tau_w$$

One obtains

$$\tau = \frac{r}{R}\tau_w \tag{10.18}$$

$$\tau = \frac{r}{R}\tau_w = \frac{R-y}{R}\tau_w = \left(1 - \frac{y}{R}\right)\tau_w = \frac{\partial u}{\partial y}\rho(v + v_t)$$

$$v\frac{\partial u}{\partial y}\left(1 + \frac{v_t}{v}\right) = \left(1 - \frac{y}{R}\right)\frac{\tau_w}{\rho}$$

$$\left(1 - \frac{y^+}{R^+}\right) = \left(1 + \frac{v_t}{v}\right)\frac{\partial u}{\partial y}\frac{v}{\sqrt{(\tau_w/\rho)}\sqrt{\tau_w/\rho}} = \left(1 + \frac{v_t}{v}\right)\frac{du^+}{dy^+}$$

where

$$u^+ = \frac{u}{u^*}, \quad y^+ = \frac{yu^*}{v}, \quad R^+ = \frac{Ru^*}{v}, \quad u^* = \sqrt{\frac{\tau_w}{\rho}}$$
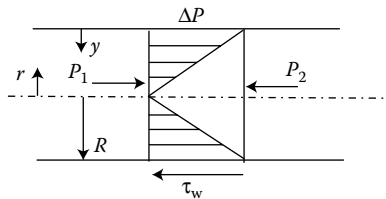
FIGURE 10.3
Force balance in a circular tube.
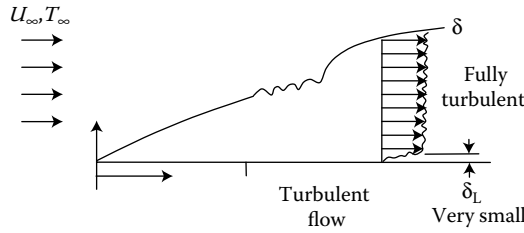
FIGURE 10.4
Concept of 2-D turbulent boundary layer flow.

Therefore,

$$\frac{v_t}{v} = \frac{\varepsilon_m}{v} = \frac{1 - (y^+/R^+)}{(du^+/dy^+)} - 1 \tag{10.19}$$

Similarly, the turbulent diffusivity for momentum for a boundary-layer flow can be obtained by replacing R^+ by δ^+ (where δ is turbulent boundary-layer thickness as shown in Figure 10.4) as

$$\frac{v_t}{v} = \frac{\varepsilon_m}{v} = \frac{1 - (y^+/\delta^+)}{(du^+/dy^+)} - 1 \tag{10.20}$$

where

$$\delta^+ = \frac{\delta u^*}{v}$$

10.1.5 Reynolds Analogy for Turbulent Flow

The following outlines the simple Reynolds analogy between momentum and heat transfer for a turbulent boundary-layer flow.

$$\frac{\tau}{q''} = \frac{\rho(v + \varepsilon_m)(\partial u/\partial y)}{\rho C_p(\alpha + \varepsilon_H)(\partial T/\partial y)} = \frac{(\partial u/\partial y)}{C_p(\partial T/\partial y)} \tag{10.21}$$

If $v = \alpha \Rightarrow Pr = (v/\alpha) \approx 1$, then $\varepsilon_m \approx \varepsilon_H \Rightarrow Pr_t = (\varepsilon_m/\varepsilon_H) \approx 1$
The turbulence Prandtl number is the flow structure.
Assume a linear velocity and temperature profile,

$$\frac{\tau_w}{q''_w} = \frac{1}{C_p} \frac{\Delta u}{\Delta T} \tag{10.22}$$

$$\frac{q''_w}{C_p(T_w - T_\infty)} = \frac{\tau_w}{U_\infty}$$

where

$$\tau_w = \frac{1}{2}\rho U_\infty^2 \cdot C_f$$

$$Nu = \frac{1}{2}C_f \frac{\rho U_\infty x}{\mu} \cdot \frac{\mu C_p}{k}$$

$$\frac{Nu}{Re \cdot Pr} = \frac{1}{2}C_f \quad \Rightarrow \quad \frac{1}{2}C_f = St \tag{10.23}$$

$$\frac{1}{2}C_f = St \cdot Pr^{2/3} \tag{10.24}$$

For $0.7 \le Pr \le 60$ for air, water, and oil.
where

$$St = \frac{Nu}{Re \cdot Pr} = \frac{(hx/k)}{(\rho U_\infty x/\mu) \cdot (\mu C_p/k)} = \frac{h}{\rho C_p U_\infty} = \frac{q_w''}{\rho C_p U_\infty (T_w - T_\infty)} \tag{10.25}$$

There are two kinds of problems:

1. For given C_f to determine St or h;
2. For given h or St, to determine C_f

For a turbulent pipe flow

$$C_f = \frac{0.046}{Re_D^{0.2}} \tag{10.26}$$

$$\frac{1}{2}\frac{0.046}{Re_D^{0.2}} = \frac{Nu}{Re \cdot Pr}Pr^{2/3}$$

$$Nu_D = \frac{hD}{k} = 0.023\,Re_D^{0.8}Pr^{1/3} \quad \text{for cooling} \tag{10.27}$$

$$Nu_D = \frac{hD}{k} = 0.023\,Re_D^{0.8}Pr^{0.4} \quad \text{for heating} \tag{10.28}$$

For a turbulent boundary-layer flow,

$$C_f = \frac{0.0592}{Re_x^{0.2}} \tag{10.29}$$

$$\frac{1}{2}\frac{0.0592}{Re_x^{0.2}} = \frac{Nu_x}{Re_x Pr}Pr^{2/3}$$

$$Nu_x = \frac{hx}{k} = 0.0296\,Re_x^{0.8}Pr^{1/3} \tag{10.30}$$

10.2 Prandtl Mixing Length Theory and Law of Wall for Velocity and Temperature Profiles

In a laminar boundary-layer flow, universal velocity and temperature profiles can be obtained by solving conservation equations for mass, momentum, and energy using the similarity method. In the turbulent flow boundary layer, we hope to obtain universal velocity and temperature profiles too. The following outlines step by step how Prandtl mixing length theory can be applied to achieve the law of the wall for velocity and temperature profiles (a kind of universal velocity and temperature profiles for turbulent boundary-layer flow). Unlike the laminar boundary, however, there is no complete analytical solution for the turbulent boundary layer due to turbulent random motion as indicated before. The law of wall for the velocity profile still requires a predetermined shear stress (or the friction factor from the experimental data) for a given turbulent flow problem. Therefore, we can only obtain a semitheoretical (or semiempirical) velocity profile for turbulent boundary-layer flow. Figure 10.5 shows the analytical universal velocity profile for a laminar boundary layer and the semiempirical law of the wall velocity profile for the turbulent boundary layer.

The following is the details of Prandtl mixing length theory and the law of the wall for velocity and temperature profiles [3–6]. First, we define the dimensionless x-direction velocity and y-direction wall coordinate.

$$u^+ = \frac{u}{u^*} \tag{10.31}$$

$$y^+ = \frac{yu^*}{\nu} \tag{10.32}$$

$$u^* = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\frac{(1/2)C_f\rho U_\infty^2}{\rho}} = U_\infty\sqrt{\frac{1}{2}C_f} \tag{10.33}$$

where u^* is the friction velocity and C_f is the predetermined friction factor by experiment.

For example,

$$C_f = 0.046\,Re_D^{-0.2}, \quad \text{for turbulent flow in a tube}$$

$$C_f = 0.0592\,Re_x^{-0.2}, \quad \text{for the turbulent flow over a flat plate}$$

and y^+ is the dimensionless wall coordinate or the roughness Reynolds number.
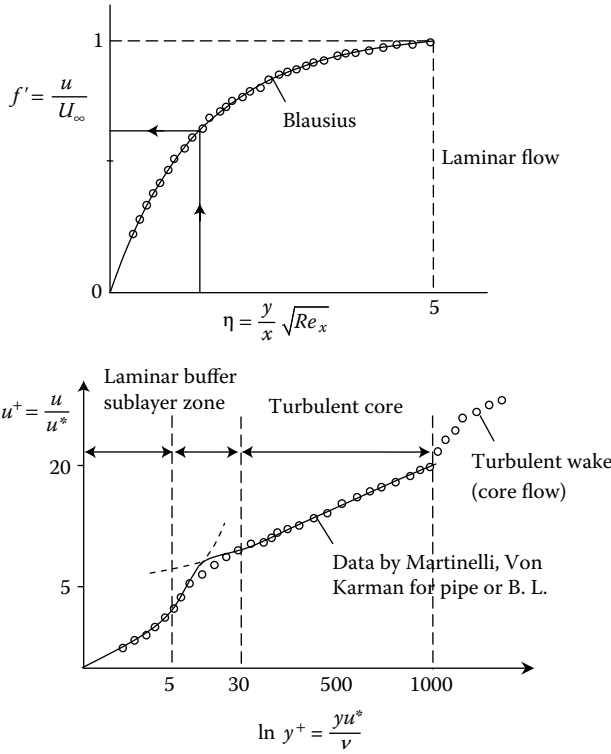
FIGURE 10.5
Analytical universal velocity profile for laminar boundary layer and semiempirical law of wall velocity profile for turbulent boundary layer.

Consider a laminar sublayer region, very close to the wall region where viscosity is dominated and turbulence is damped at the wall.

$$\tau = \rho v \frac{du}{dy} \tag{10.34}$$

$$\frac{\tau_w}{\rho} = v \left. \frac{du}{dy}\right|_w \approx v \frac{u}{y}$$

$$\sqrt{\frac{\tau_w}{\rho}} \sqrt{\frac{\tau_w}{\rho}} = v \frac{du}{dy}$$

$$u^* u^* = v \frac{du}{dy}$$

Therefore,

$$\frac{u}{u^*} = \frac{yu^*}{v}$$

and

$$u^+ = y^+ \tag{10.35}$$

Now, consider a turbulent region, as sketched in Figure 10.6, away from the wall where turbulence is dominated. From Prandtl mixing length theory, assuming velocity fluctuation is proportional to velocity gradient and the mixing length is linearly increasing with distance from the wall,

$$u' \sim \frac{\partial u}{\partial y} = l\frac{\partial u}{\partial y} \tag{10.36}$$

$$l \approx y = \kappa y \tag{10.37}$$

where $\kappa = 0.4$, a universal constant from the experimental data by Von Karman.

And

$$v' = -u' = -\kappa y\frac{\partial u}{\partial y} \tag{10.38}$$

Since wall shear is dominated by turbulence and can be approximated as

$$\tau_w \approx -\rho u'v' = \rho\kappa^2 y^2 \left(\frac{\partial u}{\partial y}\right)^2 \tag{10.39}$$

Therefore, from the above Prandtl mixing length assumption,

$$\sqrt{\frac{\tau_w}{\rho}} = \kappa y\frac{\partial u}{\partial y}$$

$$u^* = \kappa y\frac{\partial u}{\partial y}$$

$$\frac{du}{u^*} = \int \frac{1}{\kappa y}\, dy$$

$$\int du^+ = \int \frac{1}{\kappa y}\, dy = \int \frac{1}{\kappa y^+}\, dy^+$$
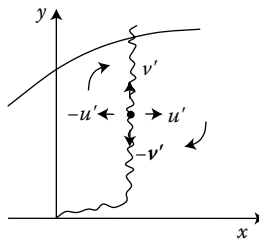
FIGURE 10.6
Concept of turbulence in 2-D turbulent boundary-layer flow.

Therefore, the law of wall velocity profile can be obtained as

$$u^+ = \frac{1}{\kappa} \ln y^+ + C \tag{10.40}$$

From the experimental data curve fitting for $y^+ \geq 30$,
$\kappa = 0.4$ or 0.41

$$u^+ = 2.5 \ln y^+ + 5.0 \tag{10.41}$$

Or

$$u^+ = 2.44 \ln y^+ + 5.5 \tag{10.42}$$

Then, consider the buffer zone between the laminar sublayer and the turbulence region, $5 \leq y^+ \leq 30$, the velocity profile can be obtained as

$$u^+ = 5 \ln y^+ - 3.05 \tag{10.43}$$

It is important to mention that the above three-region velocity profile has been validated and can be applied for turbulent flow over a flat plate or in a tube with air, water, or oil as the working fluid. In addition, before showing the law of wall for temperature profile, it is interesting to point out, just like for the laminar boundary-layer case, the law of wall for temperature profile is identical to the law of wall for velocity profile if Prandtl number is unity ($Pr = 1$) by replacing the above dimensionless velocity with the appropriate dimensionless temperature with the same dimensionless y-direction wall coordinate. The effect of Prandtl number on the law of wall for temperature profile will be discussed in Section 10.3.

10.3   Turbulent Flow Heat Transfer

Consider a fully turbulent flow in a circular tube with uniform wall heat flux ($q''_w = C$) as the thermal BC [4–6], as sketched in Figure 10.7. From the energy equation,

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left[r(\alpha + \varepsilon_H) \frac{\partial T}{\partial r} \right] + \alpha \frac{\partial^2 T}{\partial x^2} \tag{10.44}$$

$$y = R - r$$
$$r = R - y$$
$$dr = -dy$$
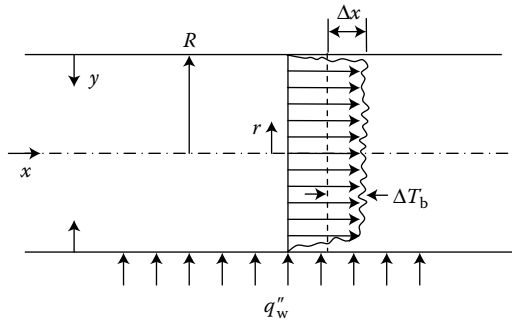
FIGURE 10.7
Turbulent flow heat transfer in a circular tube.

For a fully developed flow $(v = 0)$ and assuming $(\partial^2 T/\partial x^2) \ll (1/r)$ $(\partial/\partial r)[r(\alpha + \varepsilon_H)(\partial T/\partial r)]$, the energy equation becomes

$$u\frac{\partial T}{\partial x} = \frac{1}{R - y}\frac{\partial}{(-\partial y)}\left[(R - y)(\alpha + \varepsilon_H)\frac{\partial T}{(-\partial y)}\right]$$

Assume

$$u = \overline{V}$$

$$\frac{\partial T}{\partial x} = \frac{\partial T_b}{\partial x}$$

From energy balance,

$$q_w'' 2\pi R\, dx = \rho C_p \overline{V} A_c\, dT_b$$

$$\overline{V}\frac{dT_b}{dx} = \frac{q_w'' 2\pi R}{\rho C_p \pi R^2} = \frac{2q_w''}{\rho C_p R} \equiv C$$

And applying BCs,

$$\begin{cases} y = 0 & T = T_w \\ y = R & \frac{\partial T}{\partial y} = 0 \end{cases}$$

The energy equation becomes

$$\overline{V}\frac{dT_b}{dx}(R - y)\, d(-y) = d\left[(R - y)(\alpha + \varepsilon_H)\frac{\partial T}{-\partial y}\right]$$

$$\overline{V}\frac{dT_b}{dx}\frac{1}{2}(R - y)^2 = (R - y)(\alpha + \varepsilon_H)\frac{dT}{-dy} + C$$

$$\frac{1}{2}(R-y)\overline{V}\frac{\mathrm{d}T_\mathrm{b}}{\mathrm{d}x} = (\alpha + \varepsilon_H)\frac{\mathrm{d}T}{-\mathrm{d}y} + \frac{C}{R-y}$$

where $C = 0$ at $R - y = 0$, $(\mathrm{d}T/\mathrm{d}y) = 0$.
Therefore,

$$\mathrm{d}T = \frac{1}{2}\overline{V}\frac{\mathrm{d}T_\mathrm{b}}{\mathrm{d}x}\frac{y-R}{\alpha+\varepsilon_H}\,\mathrm{d}y$$

$$\int_{T_w}^{T}\mathrm{d}T = \frac{1}{2}\overline{V}\frac{\mathrm{d}T_\mathrm{b}}{\mathrm{d}x}\int_{0}^{y}\frac{y-R}{\alpha+\varepsilon_H}\,\mathrm{d}y$$

$$T - T_\mathrm{w} = \frac{1}{2}\overline{V}\frac{\partial T_\mathrm{b}}{\partial x}\int_{0}^{y}\frac{y-R}{\alpha+\varepsilon_H}\,\mathrm{d}y = \frac{q_\mathrm{w}''}{\rho C_p\sqrt{\tau_\mathrm{w}/\rho}}\int_{0}^{y^+}\frac{1-(y^+/R^+)}{(1/Pr)+(\varepsilon_H/\nu)}\,\mathrm{d}y^+$$

Therefore,

$$T^+ \equiv \frac{T_\mathrm{w}-T}{(q_\mathrm{w}''/\rho C_p u^*)} = \int_{0}^{y^+}\frac{1-(y^+/R^+)}{(1/Pr)+(\varepsilon_H/\nu)}\,\mathrm{d}y^+ \qquad (10.45)$$

The above equation can be integrated if one assumes

$$\frac{\varepsilon_H}{\nu} \approx \frac{\varepsilon_m}{\nu} \quad \text{or} \quad Pr_\mathrm{t} = \frac{\varepsilon_m}{\varepsilon_H} \approx 1$$

And from Equation 10.19,

$$\frac{\varepsilon_m}{\nu} = \frac{1-(y^+/R^+)}{(\mathrm{d}u^+/\mathrm{d}y^+)} - 1$$

For the laminar sublayer region,

$$0 \le y^+ \le 5, \quad y^+ \ll R^+, \quad u^+ = y^+, \quad \frac{\mathrm{d}u^+}{\mathrm{d}y^+} = 1, \quad \frac{\varepsilon_m}{\nu} = 0$$

For the buffer zone,

$$5 \le y^+ \le 30, \quad u^+ = 5\ln y^+ - 3.05, \quad \frac{\mathrm{d}u^+}{\mathrm{d}y^+} = \frac{5}{y^+}$$

For the turbulent region

$$y^+ \ge 30, \quad u^+ = 2.5\ln y^+ + 5.0, \quad \frac{\mathrm{d}u^+}{\mathrm{d}y^+} = \frac{2.5}{y^+}$$

The above energy integral equation 10.45 can be obtained in the following three-regions and the Prandtl number effect is shown in Figure 10.8.

$$T^+ = \int_0^{y^+} \frac{1 - (y^+/R^+)}{(1/Pr) + (\varepsilon_H/\nu)} \, dy^+$$

$$\frac{\varepsilon_H}{\nu} = \frac{\varepsilon_m}{\nu}$$

For the region $0 \leq y^+ \leq 5$ with $y^+ \approx 0$ and $(\varepsilon_m/\nu) \approx 0$,

$$T^+ = Pr \int_0^{y^+} dy^+ = Pr\, y^+$$

$$u^+ = y^+, \quad T^+ = Pr\, y^+ \qquad (10.46)$$

For the region $5 \leq y^+ \leq 30$ with $u^+ = 5 \ln y^+ - 3.05$ and $(y^+/R^+) \approx 0$,

$$\frac{\varepsilon_m}{\nu} = \frac{\varepsilon_H}{\nu} = \frac{y^+}{5} - 1$$

$$T^+ - T_5^+ = \int_5^{y^+} \frac{1}{(1/Pr) + (y^+/5) - 1} \, dy^+$$

$$= 5 \int_5^{y^+} \frac{1}{(1/Pr) + (y^+/5) - 1} \, d\left(\frac{1}{Pr} + \frac{y^+}{5} - 1\right)$$
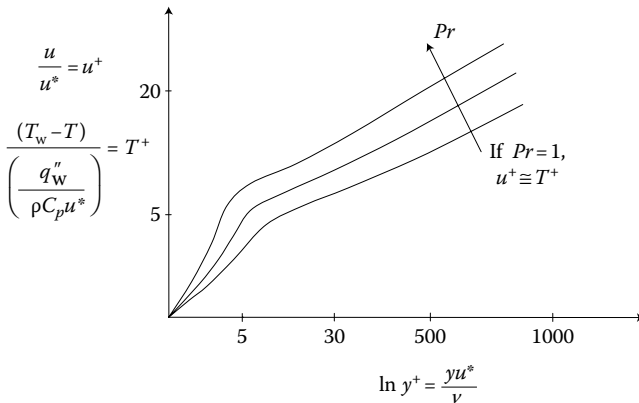
FIGURE 10.8
Law of wall temperature profile for turbulent boundary layer.

Therefore,

$$T^+ - T_5^+ = 5 \ln \left(\frac{1}{Pr} + \frac{y^+}{5} - 1 \right) - 5 \ln \left(\frac{1}{Pr} + \frac{5}{5} - 1 \right)$$

$$= 5 \ln \frac{(1/Pr) + (y^+/5) - 1}{(1/Pr)}$$

$$= 5 \ln \left(1 + \frac{Pr\, y^+}{5} - Pr \right) \tag{10.47}$$

For the region $30 \le y^+$ with $u^+ = 2.5 \ln y^+ + 5.0$, $(du^+/dy^+) = (2.5/y^+)$ and

$$\frac{\varepsilon_H}{\nu} = \frac{\varepsilon_m}{\nu} = \frac{1 - (y^+/R^+)}{(2.5/y^+)} - 1$$

$$T^+ - T_{30}^+ = \int_{30}^{y^+} \frac{1 - (y^+/R^+)}{(1/Pr) + \{[1 - (y^+/R^+)]/(2.5/y^+)\} - 1} \, dy^+$$

$$= \int_{30}^{y^+} \frac{2.5}{y^+} \, dy^+ = 2.5 \ln y^+ \Big|_{30}^{y^+}$$

Therefore,

$$T^+ - T_{30}^+ = 2.5 \ln y^+ - 2.5 \ln 30 \tag{10.48}$$

The next question is how to determine heat transfer coefficient from the law of wall for temperature profile. This is shown in the following.

One assumes the simple velocity and temperature profiles as

$$\frac{u}{U_{\max}} \cong \left(\frac{y}{R} \right)^{1/7} = \left(1 - \frac{r}{R} \right)^{1/7}$$

$$\frac{T - T_w}{T_c - T_w} \cong \left(\frac{y}{R} \right)^{1/7} = \left(1 - \frac{r}{R} \right)^{1/7}$$

Therefore,

$$u_b \quad \text{or} \quad \overline{V} = \frac{\int u 2\pi r \, dr}{\int 2\pi r \, dr} = 0.82 U_{\max}$$

$$T_w - T_b = \frac{\int_0^R (T_w - T_c)(1 - (r/R))^{1/7} U_{\max}(1 - (r/R))^{1/7} \cdot r \, dr}{\int_0^R U_{\max}(1 - (r/R))^{1/7} \cdot r \, dr}$$

$$\cong \frac{15}{18}(T_w - T_c)$$

$$= 0.833(T_w - T_c)$$

where

$$(T_w - T_c) = (T_w - T_5) + (T_5 - T_{30}) + (T_{30} - T_c)$$

$$= \frac{q''}{\rho C_p u^*}\left[5Pr + 5\ln(5Pr + 1) + 2.5\ln\left(\frac{R^+}{30}\right)\right]$$

Therefore,

$$h = \frac{q''_w}{T_w - T_b} = \frac{q''_w}{0.833(q''/\rho C_p u^*)[5Pr + 5\ln(5Pr + 1) + 2.5\ln(Ru^*/v30)]}$$

The final heat transfer coefficient and the Nusselt number are expressed as

$$Nu_D \equiv \frac{hD}{k} = \frac{Re \cdot Pr\sqrt{C_f/2}}{0.833\left[5Pr + 5\ln(5Pr + 1) + 2.5\ln\left((Re_D\sqrt{C_f/2})/60\right)\right]} \quad (10.49)$$

where

$$C_f = \frac{0.046}{Re_D^{0.2}} \quad \text{or} \quad C_f = \frac{0.079}{Re_D^{0.25}}$$

For given Re_D, and Pr, the above prediction is fairly close to the following experimental correlation:

$$Nu_D \equiv \frac{hD}{k} \approx 0.023\,Re_D^{0.8}Pr^{0.3} \quad \text{or} \quad \approx 0.023\,Re_D^{0.8}Pr^{0.4}$$

Consider a fully turbulent boundary-layer flow over a flat plate with uniform wall temperature as the thermal BC ($T_w = C$) [4–6]. From energy equation,

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[(\alpha + \varepsilon_H)\frac{\partial T}{\partial y}\right] \quad (10.50)$$

For a fully turbulent boundary-layer flow ($v = 0$) and assume

$$\frac{\partial T}{\partial x} \sim \frac{\partial T_w}{\partial x} \approx 0$$

One can obtain the law of wall for temperature profile as

$$T^+ = \int_0^{y^+} \frac{1}{(1/Pr) + (\varepsilon_H/v)}\,dy^+$$

Follow a similar procedure as in the previous case; one can obtain three-region temperature profile.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \underbrace{\frac{\partial}{\partial y}[(\alpha + \varepsilon_H)(\partial T/\partial y)]}_{\text{const.}=q_w''/\rho C_p} = 0$$

$$(\alpha + \varepsilon_H)\frac{\partial T}{\partial y} = c$$

where $c = (-q_w''/\rho C_p)$.
  Therefore,

$$(\alpha + \varepsilon_H)\frac{\partial T}{\partial y} = \frac{-q_w''}{\rho C_p}$$

$$\int \frac{\partial T}{\partial y} = \frac{-q_w''/\rho C_p}{\alpha + \varepsilon_H}$$

$$T - T_w = \frac{-q_w''}{\rho C_p}\int_0^y \frac{1}{v((1/Pr) + (\varepsilon_H/v))}\, dy \frac{u^*}{u^*}$$

$$T_w - T = \frac{q_w''}{\rho C_p u^*}\int_0^y \frac{1}{((1/Pr) + (\varepsilon_H/v))}\, dy^+ \qquad (10.51)$$

where $\varepsilon_H \simeq \varepsilon_m$.
  And from Equation 10.20,

$$\frac{\varepsilon_m}{v} = \frac{1 - (y^+/\delta^+)}{(du^+/dy^+)} - 1$$

Follow the same procedure as outlined before:

$$0 < y^+ < 5, \quad u^+ = y^+$$

$$T^+ = Pr\, y^+ \qquad (10.52)$$

$$5 < y^+ \leq 30, \quad u^+ = 5\ln y^+ - 3.05$$

$$T^+ - T_5^+ = 5\ln\left(1 + \frac{Pr\, y^+}{5} - Pr\right) \qquad (10.53)$$

$$30 \leq y^+, \quad u^+ = 2.5\ln y^+ + 5.0$$

$$T^+ - T_{30}^+ = 2.5\ln y^+ - 2.5\ln 30 \qquad (10.54)$$

Note that $\Delta T^+ = \Delta u^+$ at $y^+ \geq 30$

$$T^+ - T_{30}^+ = u^+ - u_{30}^+$$

$$T_\infty^+ - T_{30}^+ = u_\infty^+ - u_{30}^+$$

where

$$u_{30}^+ = 14, \quad u_\infty^+ = \frac{u_\infty}{u^*} = \frac{1}{\sqrt{C_f/2}}$$

Therefore,

$$T_w - T_\infty = (T_w - T_5) + (T_5 - T_{30}) + (T_{30} - T_\infty)$$

$$= \frac{q_w''}{\rho C_p u^*}[5Pr + 5 \ln(5Pr + 1) + (T_\infty^+ - T_{30}^+)]$$

$$h = \frac{q_w''}{T_w - T_\infty} = \frac{q_w''}{(q_w''/\rho C_p u^*)[5Pr + 5 \ln(5Pr + 1) + ((1/\sqrt{C_f/2}) - 14)]}$$

The final heat transfer coefficient and the Stantan number can be obtained as

$$St = \frac{Nu_x}{Re_x Pr} = \frac{0.0295 Re_x^{-0.2}}{1 + 0.172 Re_x^{-0.1}[5Pr + 5 \ln(5Pr + 1) - 14]} \approx 0.0295 Re_x^{-0.2}$$

(10.55)

where $Nu_x = (hx/k)$, $Re_x = (\rho U_\infty x/\mu)$, $C_f = \left(0.0592/Re_x^{1/5}\right)$.

For given Re_x and Pr, the above predict Nu_x value is very close to the following experimental correlation:

$$Nu_x = 0.0296 Re_x^{0.8} Pr^{1/3}$$

Remarks

In undergraduate heat transfer, students are expected to know how to calculate heat transfer coefficients (Nusselt numbers) for turbulent flows over a flat plate at uniform surface temperature and inside a circular tube at uniform surface heat flux, by using heat transfer correlations from experiments, that is, Nusselt numbers relate to Reynolds numbers and Prandtl numbers. There are many engineering applications involving turbulent flow conditions. These turbulent flow heat transfer correlations are very useful for basic heat transfer calculations such as for heat exchangers design.

In intermediate-level heat transfer, this chapter focuses on how to derive RANS equation; introduce the concept of turbulent viscosity and turbulent Prandtl number; Reynolds analogy; Prandtl mixing length theory; law of wall for velocity and temperature profiles; and turbulent flow heat transfer coefficients derived from law of wall velocity and temperature profiles and

their comparisons with the heat transfer correlations from experiments. This classic turbulent flow theory is important to provide students with fundamental background in order to handle advanced turbulence models.

In advanced turbulent heat transfer, students will learn many more topics such as flow transition and transitional flow heat transfer; unsteady high turbulence flow and heat transfer; surface roughness effect and heat transfer enhancement; rotating flow, and heat transfer; high-speed flow and heat transfer; and advanced turbulence models including the two-equation model and the Reynolds stress model.

PROBLEMS

10.1. Consider a steady low-speed, constant-property, fully turbulent boundary-layer flow over a flat surface at constant wall temperature. Based on the Reynolds time-averaged concept, the following momentum and energy equations are listed for reference:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[(v + \varepsilon_M)\frac{\partial u}{\partial y}\right]$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{v}{Pr} + \frac{\varepsilon_M}{Pr_t}\right)\frac{\partial T}{\partial y}\right]$$

a. Define dimensionless parameters, u^+, T^+, and y^+, respectively, for universal velocity and temperature profiles for turbulent flow and heat transfer problems.

b. Based on the Prandtl's mixing length theory, derive and plot (u^+ versus y^+) the following universal velocity profile (make necessary assumptions):
   $u^+ = y^+$ for a viscous sub-layer region
   $u^+ = 1/\kappa \ell n y^+ + C$ for a turbulent layer (the law of the wall region)

c. Based on the heat and momentum transfer analogy, derive and plot (T^+ versus y^+) for various Pr the universal temperature profile (i.e., temperature law of the wall, $T^+(y^+, Pr)$) for a turbulent boundary layer on a flat plate. Make necessary assumptions.

10.2. Consider a fully developed turbulent pipe flow with a uniform q''_w. If a two-layer universal velocity profiles can be assumed as

$$u^+ = y^+ \quad \text{for } 0 \le y^+ < 13.6$$

$$u^+ = 5.0 + 2.44\ell n y^+ \quad \text{for } 13.6 \le y^+$$

Derive the corresponding two-layer university temperature profiles.

Also, predict the local u and T at $y = 0.05$ cm from the pipe wall under the following conditions: friction velocity $u^* = 10\,\text{m/s} \cong$

$\sqrt{\tau_w/\rho}$, friction temperature $T^* = 3°C \cong q_w''/(\rho c_p u^*)$, pipe wall temperature $T_w=100°C$, air flow Prandtl $Pr = 0.7$

10.3. Consider the Von Kaŕman–Martinelli heat–momentum analogy for a turbulent pipe flow:

a. If a two-layer universal velocity profile will be employed, that is,

$$\bar{u}^+ = y^+ \quad \text{for } 0 < y^+ < 10$$

$$\bar{u}^+ = 5.0 + 2.5\,\ell\,\text{n}\,y^+ \text{ for } 10 < y^+$$

For a constant wall heat flux, determine the universal temperature profiles at the corresponding two-layer region. Then determine the Nu_D where $Nu_D = $ function (Re_D, Pr, f).

b. If $Re_D = 10^4$, $Pr = 1$, calculate Nu_D from (a) and then compare it with correlation $Nu_D = 0.023\,Re_D^{0.8} \cdot Pr^{0.4}$. If velocity increases, the laminar sublayer thickness will be increased or decreased? Why? How about Nu_D?

10.4. Consider a fully developed turbulent flow between two parallel plates with a gap of b and a uniform wall heat flux $(q/A)_w$. Using the Kaŕman–Martinelli analogy, determine the turbulent heat transfer coefficient. The result should be in a format such as

$$Nu = \frac{h2b}{k} = \text{function of } (Re, Pr, f)$$

If $Re = (\bar{V}2b/\upsilon) = 2 \times 10^3$, 2×10^4, 2×10^5, and $Pr = 0.7$, compare your result of Nu to those of semiempirical correlations, such that $Nu = (h2b/k) = 0.023Re^{0.8}Pr^{0.4}$

10.5. Consider the turbulent flow heat transfer.

a. Derive the following momentum and energy equations for a turbulent boundary-layer flow, a 2-D flat plate, incompressible, constant properties:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\frac{\partial u}{\partial y}\right]$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\nu}{Pr} + \frac{\varepsilon_M}{Pr_t}\right)\frac{\partial T}{\partial y}\right]$$

b. Derive the following momentum and energy equations for a fully developed turbulent flow in a circular tube, incompressible, constant properties:

$$\frac{1}{r}\frac{d}{dr}\left(r(\nu + \varepsilon_M)\frac{du}{dr}\right] = \frac{1}{\rho}\frac{dP}{dx}$$

$$u\frac{\partial T}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left[r(\alpha + \varepsilon_H)\frac{\partial T}{\partial r}\right]$$

10.6. Consider a fully developed turbulent pipe flow in a circular tube with a 5.0 cm I-D, constant properties.

a. Draw the velocity distribution from Martinelli Universal Velocity Profile (i.e., law of the wall) for air flow at 1 atm, 25°C, and $Re_D = 30,000$.

b. Calculate the laminar sublayer thickness.

10.7. Consider the turbulent boundary-layer flow heat transfer: Air at 300 K, 1 atm, flows at 12 m/s along a flat plate maintained at 600 K. Plot the temperature profile $T(y)$ across the boundary layer for the following two cases:

a. At a location $x = 0.1$ m for a laminar boundary layer.

b. At a location $x = 1.0$ m if the transition Reynolds number is 10^5.

Plot both profiles on the same graph to show significant differences.

c. Determine C_{fx}, τ_s, u^*, St_x, Nu_x, h_x, and q_s'' for case (b).

10.8. Consider a 2-D incompressible turbulent flow in a pipe.

a. Specialize (simplify) the given continuity and Navier–Stokes equations for a fully developed turbulent flow in a pipe. Write appropriate BCs to solve the flow equations for a fully developed turbulent flow. Do not attempt to solve the problem.

b. The velocity profile $(u(r))$ for a fully developed turbulent flow in a pipe is given by

$$\frac{u}{u_{\max}} = \left(\frac{R - r}{R}\right)^{1/7}$$

where R is the pipe radius, r is the radial distance measured from the pipe axis, and $U_{\max}$ is the maximum velocity. Calculate mean or the bulk velocity (U_m) for a fully developed flow in terms of $U_{\max}$.

c. Obtain an expression for the skin friction coefficient (Fanning friction factor) for a fully developed turbulent flow in terms of Re_D, where Re_D is the Reynolds number based on the pipe hydraulic diameter.

10.9. Consider a steady low-speed, constant-property, fully turbulent boundary-layer flow over a flat surface at constant wall temperature.

a. Based on the Reynolds time-averaged concept, derive the following momentum and energy equations (make necessary assumptions):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[(v + \varepsilon_M)\frac{\partial u}{\partial y}\right]$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{v}{Pr} + \frac{\varepsilon_M}{Pr_t}\right)\frac{\partial T}{\partial y}\right]$$

b. Define dimensionless parameters, u^+, T^+, and y^+, respectively, for universal velocity and temperature profiles for turbulent flow and heat transfer problems. Based on the Prandtl's

mixing length theory, the following universal velocity profile has been derived:

$u^+ = y^+$ for the viscous sublayer region.

$u^+ = (1/\kappa) \ln y^+ + C$ for the turbulent layer (the law of the wall region).

Now, based on the heat and momentum transfer analogy, <u>derive</u> and <u>plot</u> (T^+ versus y^+ for various Pr) the universal temperature profile (i.e., temperature law of the wall, $T^+(y^+, Pr)$] for a turbulent boundary layer on a flat plate? Make the necessary assumptions.

10.10. Consider a steady low-speed, constant-property, fully turbulent boundary-layer flow over a flat surface at constant wall temperature. Based on the Reynolds time-averaged concept, the following momentum and energy equations can be derived:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \varepsilon_{\mathrm{M}})\frac{\partial u}{\partial y}\right]$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left[\left(\frac{\nu}{Pr} + \frac{\varepsilon_{\mathrm{M}}}{Pr_{\mathrm{t}}}\right)\frac{\partial T}{\partial y}\right]$$

a. Explain why the turbulent viscosity and turbulent Prandtl number should be included in the above equations. Explain the physical meaning and the importance of the turbulent viscosity and turbulent Prandtl number, respectively.

b. Based on the Prandtl's mixing length theory, the following universal velocity profile has been obtained:

$u^+ = y^+$ for the viscous sublayer region.

$u^+ = (1/\kappa) \ln y^+ + C$ for the turbulent layer (the law of the wall region).

Based on the heat and momentum transfer analogy, derive and plot (T^+ versus y^+ for various Pr) the universal temperature profile for a turbulent boundary layer on a flat plate. Make necessary assumptions.

References

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11

Fundamental Radiation

![section divider]

11.1 Thermal Radiation Intensity and Emissive Power

Any surface can emit energy as long as its surface temperature is greater than absolute zero. Thermal radiation can refer to (1) surface radiation and (2) gas or volume radiation. Surface radiation is that radiation which comes from an opaque material surface (such as a solid or liquid surface,1 μm thick from the surface). Gas radiation is that radiation which comes from a volume of gas (such as, CO_2, H_2O, CO, or NH_3). But gas radiation does not occur from a volume of air (air cannot emit or absorb radiation energy). Modern theory describes the nature of radiation in terms of electromagnetic waves that travel at the speed of light. The various forms of radiation differ only in terms of wavelength. In this chapter, the discussion will be confined to thermal radiation [1–4], as shown in Figure 11.1. Thermal radiation primarily depends on wavelength (spectral distribution, 0.1–100 μm, from visible light to infrared (IR)), direction (directional distribution, θ, φ), and material temperature (absolute temperature, °K or °R) and material radiation properties. Figure 11.2 shows the nature of spectral and directional distributions.

What follows try to establish the relation between surface radiation flux (i.e., radiation rate per unit surface area, or called emissive power) and radiation intensity. Here we assume that radiation intensity from a surface is given. The later section will discuss how to obtain the radiation intensity value from Planck. Figure 11.3 shows the conceptual view of a hemispheric radiation from a surface (consider radiation from the upper surface only) [4]. Monochromatic directional radiation intensity (function of wavelength and temperature in all directions) is defined as differential radiation rate per unit surface area and unit solid angle,

$$I_\lambda(\theta, \varphi, \lambda, T) = \frac{dq}{dA_n \, d\omega} \tag{11.1}$$

where dω is the unit solid angle,

$$d\omega = \frac{dA_n}{r^2} = \frac{r d\theta \cdot r \sin\theta \cdot d\varphi}{r^2} = \sin\theta \, d\theta \, d\varphi$$
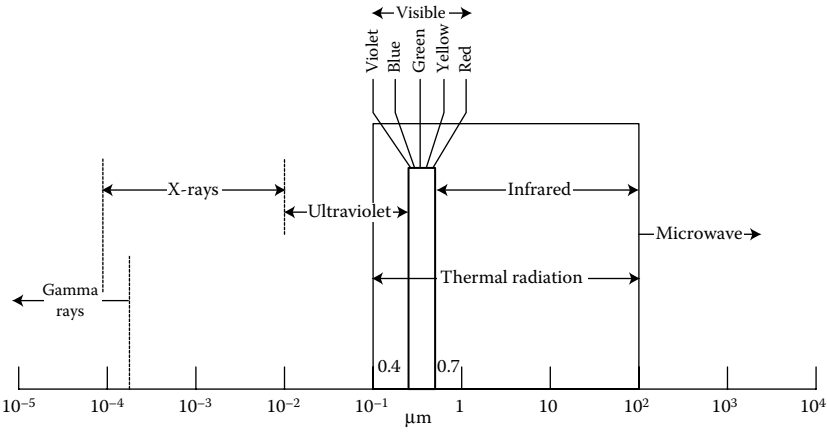
FIGURE 11.1
Conceptual view of hemispheric radiation.

and

$$\mathrm{d}A_n = \mathrm{d}A \cos \theta$$

Therefore, differential surface radiation flux (or differential emissive power) becomes

$$\frac{\mathrm{d}q}{\mathrm{d}A} \equiv \mathrm{d}E_\lambda = I_\lambda(\theta, \varphi, \lambda, T) \sin \theta \cos \theta \, \mathrm{d}\theta \, \mathrm{d}\varphi$$

After integration in all directions, the monochromatic hemispherical emissive power is

$$E_\lambda = \int_0^{2\pi} \int_0^{\pi/2} I_\lambda(\theta, \varphi, \lambda, T) \sin \theta \cos \theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \tag{11.2}$$
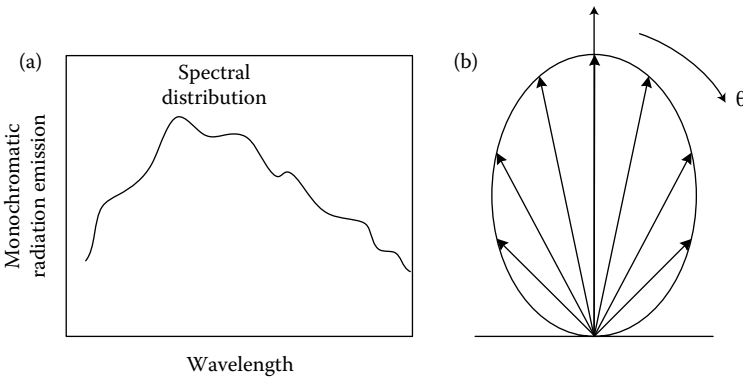
FIGURE 11.2
(a) Spectral radiation varies with wavelength. (b) Directional distribution.
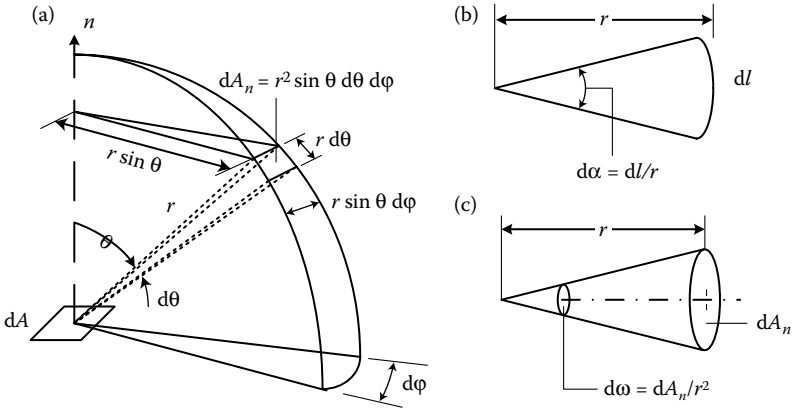
FIGURE 11.3
(a) Conceptual view of hemispheric radiation from a differential element area. (b) Definition of plane angle. (c) Definition of solid angle.

The total hemispherical emissive power (integration over wavelength) is

$$E(T) = \int_0^\infty E_\lambda(\lambda, T)\, d\lambda \tag{11.3}$$

For the isotropic surface (independent of circumferential direction angle), for most of the surfaces,

$$E_\lambda = 2\pi \int_0^{\pi/2} I_\lambda(\theta, \lambda, T) \sin\theta \cos\theta\, d\theta$$

For the isotropic and black surface (independent of vertical direction angle),

$$E_{b,\lambda} = 2\pi I_{b,\lambda}(\lambda, T) \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta = \pi I_{b,\lambda} \tag{11.4}$$

11.2 Surface Radiation Properties for Blackbody and Real-Surface Radiation

Total emissive power for black surface (ideal surface)

$$E_b = \pi I_b = \sigma T^4 \tag{11.5}$$

Total emissive power for real surface

$$E = \pi I = \varepsilon \sigma T^4 \tag{11.6}$$

Monochromatic emissivity, a surface radiation property, is defined as monochromatic emissive power from a real surface to an ideal surface.

$$\varepsilon_{\lambda,\theta}(\theta, \varphi, \lambda, T) = \frac{E(\theta, \varphi, \lambda, T)}{E_b(\lambda, T)} \tag{11.7}$$

Therefore, the total hemispherical emissivity can be obtained as a ratio of total emissive power from a real surface to an ideal surface. The total emissivity varies from 0 to 1.

$$\varepsilon(T) = \frac{E(T)}{E_b(T)} = 0 \sim 1 \tag{11.8}$$

Gray surface is defined as if surface emissivity is independent of wavelength as sketched in Figure 11.4. Diffuse surface is defined as if surface emissivity is independent of direction as sketched in Figure 11.4. In general, the experimental data show that normal emissivity of nonconductive materials (about 0.9, such as nonmetals) is much higher than conductive materials (about 0.1, such as metals). Emissivity is the most important radiation property and slightly depends on temperature. Emissivity is measured from experiments and is available for various materials from any heat transfer textbook.

In addition to emission, a surface can reflect, absorb, or transmit any oncoming radiation energy (irradiation). Figure 11.5 sketches an energy balance between irradiation (radiation coming to the surface) and reflection, absorption, and transmission. Absorptivity, reflectivity, and transmissivity are defined as the portions of irradiation that are absorbed, reflected, and transmitted, respectively. For many engineering gray and diffuse materials, we can assume that surface absorptivity is the same as surface emissivity ($\varepsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}, \varepsilon_\lambda = \alpha_\lambda$, then $\varepsilon = \alpha$, but in general $\varepsilon \neq \alpha$), and transmissivity is

Gray surface $\varepsilon \cong \varepsilon_\lambda = \text{const}$ Diffuse surface $\varepsilon \cong \varepsilon_\theta = \text{const}$
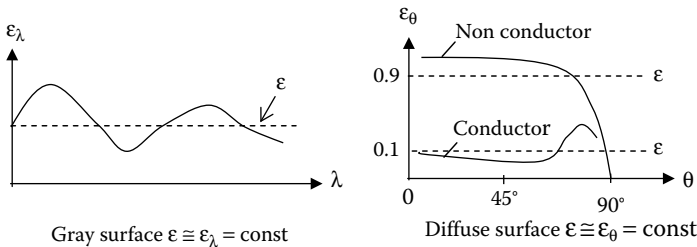
FIGURE 11.4
Definition of gray surface and diffuse surface.
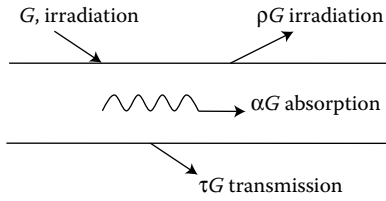
FIGURE 11.5
Radiation energy balance on a surface.

about zero (except window glasses); therefore, reflectivity can be found from emissivity too.

Absorptivity $\alpha = G_a/G$
Reflectivity $\rho = G_\rho/G$
Transmissivity $\tau = G_\tau/G$
From radiation energy balance, $\alpha + \rho + \tau = 1$
If $\tau = 0$, and assume $\alpha = \varepsilon$, then $\rho = 1 - \alpha \approx 1 - \varepsilon$

Blackbody radiation is the maximum radiation from an ideal surface. Blackbody is a diffuse surface and can emit the maximum radiation and can absorb the maximum radiation (i.e., $\alpha = 1$ and $\varepsilon = 1$). Therefore, blackbody radiation intensity is not a function of direction ($\neq (\theta, \varphi)$), but a function of wavelength and temperature ($= (\lambda, T)$). Planck obtained blackbody radiation intensity from quantum theory as

$$I_{\lambda,b}(\lambda, T) = \frac{2hC_0^2}{\lambda^5 \left[\exp\left(hC_0/\lambda kT\right) - 1\right]} \tag{11.9}$$

where h is the Planck's constant $= 6.626 \times 10^{-34}$ J s, C_0 is the speed of light in vacuum $= 2.998 \times 10^8$ m/s, k is the Boltzmann's constant $= 1.381 \times 10^{-23}$ J/K, and T is the absolute temperature, $°K$ or $°R$. Or, it can be shown as follows:

$$E_{\lambda,b} = \pi I_{\lambda,b} = \frac{C_1 \lambda^{-5}}{e^{(C_2/\lambda T)} - 1}$$

where $C_1 = 2\pi h C_0^2 = 3.742 \times 10^8 \, W \, \mu m^4/m^2$

$$C_2 = hC_0/k = 1.4389 \times 10^4 \, \mu m \, K$$

Therefore, Planck emissive power for the black surface can be shown as

$$E_{\lambda,b}(\lambda, T) = \pi I_{\lambda,b}(\lambda, T) = \pi \frac{2hC_0^2}{\lambda^5[\exp\left(hC_0/\lambda kT\right) - 1]} = \frac{C_1}{\lambda^5[\exp(C_2/\lambda T) - 1]}$$

Performing integration over the entire wavelength, one obtains Stefan–Boltzmann law for blackbody radiation as

$$E_b(T) = \int_0^\infty E_{\lambda,b}(\lambda, T)\, d\lambda = \int_0^\infty \pi I_\lambda(\lambda, T)\, d\lambda = \sigma T^4 \, \text{W/m}^2 \qquad (11.10)$$

with the Stefan–Boltzmann constant

$$\sigma = f(C_1, C_2) = 5.67 \times 10^{-8} \text{W/m}^2\, \text{K}^4$$

However, for a real surface the emissive power is lower than Planck's blackbody radiation (because the emissivity for the real surface is less than unity). Therefore, the emissive power for the real surface is

$$E = \varepsilon \sigma T^4 \qquad (11.11)$$

Figure 11.6 shows emissive power versus wavelength over a wide range of temperatures [1–4]. In general, emissive power increases with absolute temperature; emissive power for the black surface (solid lines) is greater than that for the real gray surface (lower than the solid lines, depending on emissivity) for a given temperature.
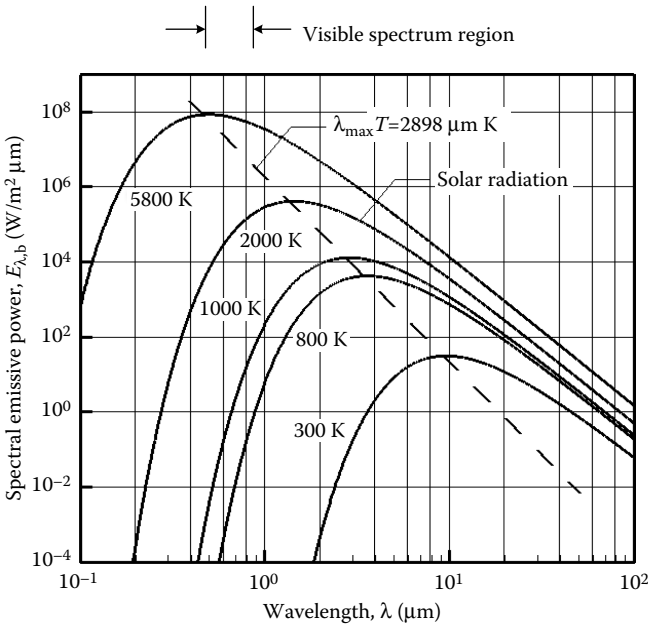
FIGURE 11.6
Spectral blackbody emissive power.
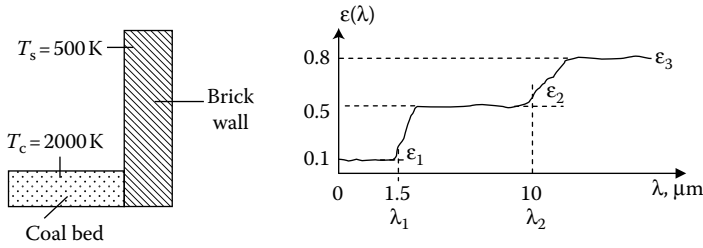
FIGURE 11.7
Radiation between hot coal bed and cold brick wall with nongray.

From Figure 11.6 we see that the blackbody emissive power distribution has a maximum and that the corresponding wavelength λ_{max} depends on temperature. Taking a derivation on Equation 11.9 with respect to λ and setting the result as equal to zero, we obtain Wien's displacement law as

$$\lambda_{max}T = C_3 = 2898 \ \mu m \ K \tag{11.12}$$

The focus of Wien's displacement law is also shown in Figure 11.5. According to this result, the maximum emissive power is displaced to shorter wavelengths with increasing temperature. For example, the maximum emission is in the middle of the visible spectrum ($\lambda_{max} \approx 0.5 \ \mu m$) for solar radiation at 5800 K; the peak emission occurs at $\lambda_{max} = 1 \ \mu m$ for a tungsten filament lamp operating at 2900 K emitting white light, although most of the emission remains in the IR region.

There are many engineering surfaces with diffuse but not gray behaviors. In this case, surface emissivity is a function of wavelength and is not the same as absorptivity. Figure 11.7 shows the radiation problem between a hot coal bed and a cold brick wall with an emissivity function of wavelength [4]. To determine emissive power from the cold brick wall, one needs to determine the average emissivity from the brick wall first. The following outlines a method to determine average emissivity and absorptivity.

The following shows how to determine $\varepsilon(T_s)$, $E(T_s)$, and $\alpha(T_s)$. Average emissivity can be determined by adding three regions of wavelength shown in Figure 11.7. Then treat each region as a product of constant emissivity and fraction of blackbody emissive power to total blackbody emissive power as shown in Figure 11.8. The fraction value is a function of wavelength and temperature, and can be obtained from integration in each region (e.g., see Figure 11.9 or Table 11.1) [4].

$$\varepsilon(T_s) = \frac{\int_0^\infty \varepsilon(\lambda)E_b \, d\lambda}{E_b} \tag{11.13}$$

$$= \varepsilon_1 \frac{\int_0^{\lambda_1} E_b \, d\lambda}{E_b} + \varepsilon_2 \frac{\int_{\lambda_1}^{\lambda_2} E_b \, d\lambda}{E_b} + \varepsilon_1 \frac{\int_{\lambda_2}^{\lambda_3} E_b \, d\lambda}{E_b} \tag{11.14}$$
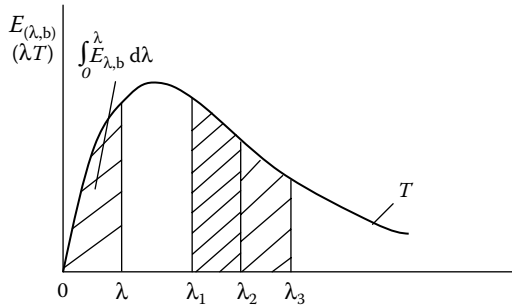
FIGURE 11.8
Concept of fraction method from a blackbody.

$$= \varepsilon_1 F_{0-\lambda_1} + \varepsilon_2[F_{0-\lambda_2} - F_{0-\lambda_1}] + \varepsilon_3[1 - F_{0-\lambda_2}] \qquad (11.15)$$
$$= 0.1 \times 0 + 0.5 \times 0.634 + 0.8 \times (1 - 0.634)$$

where

$$F_{0-\lambda} = \frac{\int\limits_0^\lambda E_{\lambda,b}\, d\lambda}{\int\limits_0^\infty E_{\lambda,b}\, d\lambda} = \frac{\int_0^\lambda E_{\lambda,b}\, d\lambda}{\sigma T^4} = \int\limits_0^{\lambda T} \frac{E_{\lambda,b}\, d(\lambda T)}{\sigma T^5} = f(\lambda T) \qquad (11.16)$$

$$F_{\lambda_1-\lambda_2} = \frac{\int_0^{\lambda_2} E_{\lambda,b}\, d\lambda - \int_0^{\lambda_1} E_{\lambda,b}\, d\lambda}{\sigma T^4} = F_{0-\lambda_2} - F_{0-\lambda_1} \qquad (11.17)$$

From Table 11.1 or from Figure 11.9, $F_{0-\lambda}$ is a function of $\lambda T(\mu m\, K)$, with $T = T_s = 500\,K$, emission from the brick wall.
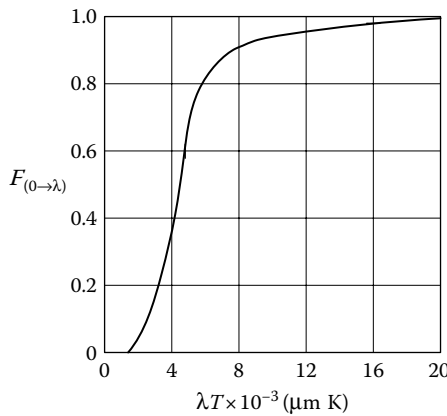
FIGURE 11.9
Fraction of the total blackbody emission in the spectral band from 0 to λ as a function of λT.

TABLE 11.1

Blackbody Radiation Functions

$\lambda T (\mu m \cdot K)$	$F_{(0 \to \lambda)}$
200	0.000000
400	0.000000
600	0.000000
800	0.000016
1000	0.000321
1200	0.002134
1400	0.007790
1600	0.019718
1800	0.039341
2000	0.066728
2200	0.100888
2400	0.140256
2600	0.183120
2800	0.227897
2898	0.250108
3000	0.273232
3200	0.318102
3400	0.361735
3600	0.403607
3800	0.443382
4000	0.480877
4200	0.516014
4400	0.548796
4600	0.579280
4800	0.607559
5000	0.633747
5200	0.658970
5400	0.680360
5600	0.701046
5800	0.720158
6000	0.737818
6200	0.754140
6400	0.769234
6600	0.783199
6800	0.796129
7000	0.808109
7200	0.819217
7400	0.829527

continued

TABLE 11.1 (continued)

Blackbody Radiation Functions

$\lambda T(\mu m \cdot K)$	$F_{(0\rightarrow\lambda)}$
7600	0.839102
7800	0.848005
8000	0.856288
8500	0.874608
9000	0.890029
9500	0.903085
10,000	0.914199
10,500	0.923710
11,000	0.931890
11,500	0.939959
12,000	0.945098
13,000	0.955139
14,000	0.962898
15,000	0.969981
16,000	0.973814
18,000	0.980860
20,000	0.985602
25,000	0.992215
30,000	0.995340
40,000	0.997967
50,000	0.998953
75,000	0.999713
100,000	0.999905

Source: Data from F. Incropera and D. Dewritt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.

The radiation constants used to generate these blackbody functions are

$$C_1 = 3.7420 \times 10^8 \mu m^4/m^2$$
$$C_2 = 1.4388 \times 10^4 \mu m \cdot K$$
$$\sigma = 5.670 \times 10^{-8} W/m^2 \cdot K^4$$

Therefore, the average emissivity can be calculated as

$$\varepsilon(T_s) = 0.61$$

and the total emissive power is

$$E(T_s) = \varepsilon(T_s)\sigma T_s^4 = 2161 \, W/m^2$$

The brick wall is not a gray surface, and hence $\alpha(T_s) \neq \varepsilon(T_s)$. But it is a diffuse surface, and hence $\alpha(\lambda) = \varepsilon(\lambda)$. The irradiation from the black coal bed (at temperature $T_c = 2000\,K$) to the brick wall is $G(\lambda) \propto E_b$. The following shows a similar way to determine brick wall absorptivity.

$$\alpha(T_s) \equiv \frac{\int_0^\infty \alpha(\lambda)G(\lambda)\,d\lambda}{\int_0^\infty G(\lambda)\,d\lambda} = \frac{\int_0^\infty \varepsilon(\lambda)E_b\,d\lambda}{E_b} \tag{11.18}$$

$$= \varepsilon_1 F_{0-\lambda_1} + \varepsilon_2[F_{0-\lambda_2} - F_{0-\lambda_1}] + \varepsilon_3[1 - F_{0-\lambda_2}] \tag{11.19}$$

$$= 0.1 \times 0.275 + 0.5 \times (0.986 - 0.273) + 0.8 \times (1 - 0.986)$$

where the fraction value can be found from Table 11.1, or from Figure 11.9, with $T = T_c = 2000\,K$, irradiation from the black coal bed.

Therefore, average absorptivity can be calculated as

$$\alpha(T_s) = 0.395 < 0.61 \text{ for } \varepsilon(T_s)$$

11.3 Solar and Atmospheric Radiation

Solar radiation is essential to all life on earth. Through the thermal and photovoltaic process, solar radiation is important for the design of solar collectors, air-conditioning systems for buildings and vehicles, temperature control systems for spacecrafts, and photocells for electricity. The sun is approximate as a spherical radiation source with a diameter of 1.39×10^9 m and is located around 1.50×10^{11} m from the earth. The average solar flux (solar constant) incident on the outer edge of the Earth's atmosphere is about $1353\,W/m^2$. Assuming a blackbody radiation, the sun's temperature can be estimated at about $5800\,K$. Figure 11.10 shows the spectral distribution of solar radiation [2]. The radiation is concentrated in the low-wavelength region $(0.2 \leq \lambda \leq 3\,\mu m)$ with the peak value of $0.50\,\mu m$.

The magnitude spectral and directional distributions of solar flux change significantly as solar radiation passes through the Earth's atmosphere. The change is due to absorption and scattering of the radiation by the atmosphere particles and gases. The effect of absorption by the atmosphere gases O_3 (ozone), H_2O vapor, O_2, and CO_2 is shown by the lower curve in Figure 11.10. Absorption by ozone is strong in the UV region, providing considerable attenuation below 0.3–$0.4\,\mu m$. In the visible light region (0.4–$0.7\,\mu m$), absorption is contributed by O_3 and O_2; in the IR region (0.7–$3.0\,\mu m$), absorption is due to H_2O vapor and CO_2. The effect of scattering by particles and gases is that about half goes back to atmosphere and half comes to the earth surface. Therefore, the average solar flux incident on the Earth's surface is reduced to about 300–$800\,W/m^2$, depending on the time of the day, the season, the latitude, and the weather conditions.
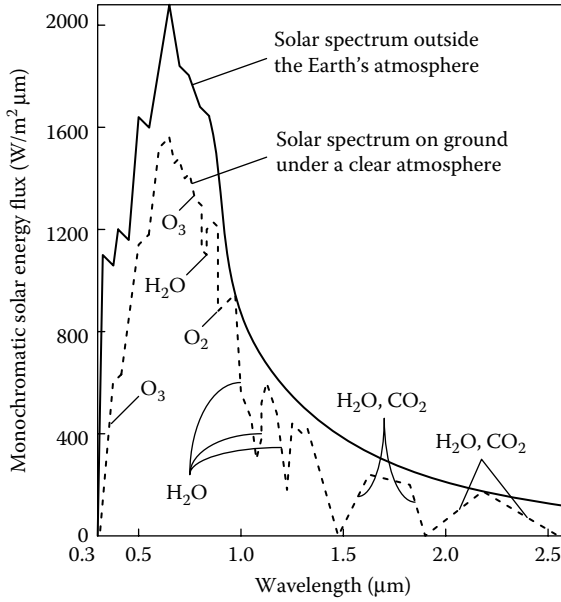
FIGURE 11.10
Solar spectra outside the Earth's atmosphere and on the ground.

It is known that H_2O vapor and CO_2 gas in the atmosphere not only absorb solar radiation but also absorb radiation from the Earth's surface at around 300 K that give radiation of wavelengths from 10 to 20 μm with an emissivity about 1.0. In addition, H_2O vapor and CO_2 gas in the atmosphere (sky) can also emit energy at wavelengths of 5–10 μm at the effective sky temperature around 250–270 K (assume an emissivity of about 0.8–1.0).

Figure 11.11 shows a typical setup for a solar collector. A special glass is used as a cover for the collector and a specialized coating is used on the collector plate and tubes where the solar energy is collected to maximize the performance of the collector.
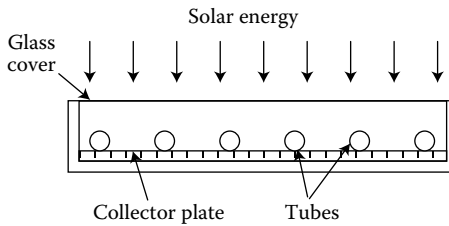
FIGURE 11.11
A typical setup for a solar collector.
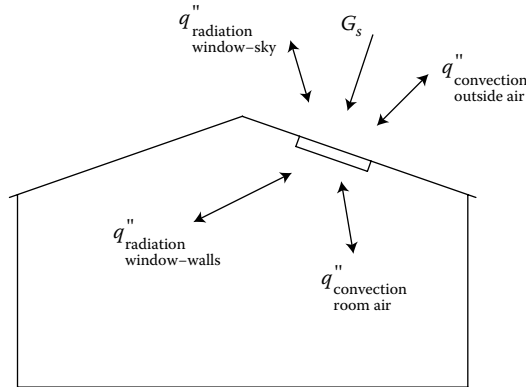
FIGURE 11.12
A typical design for a house with the skylight.

Figure 11.12 shows a typical design for a house with the skylight. The thin glass of the skylight of a house has a specific spectral emissivity or absorptivity distribution. For a given solar flux, atmospheric emission flux, interior surface emission flux, inside and outside house convection conditions, the thin glass temperature or the inside house temperature can be predicted.

Examples

11.1. A simple solar collector plate without the cover glass has a selective absorber surface of high absorptivity α_1 (for $\lambda < 1\,\mu m$) and low absorptivity α_2 (for $\lambda > 1\,\mu m$). Assume that solar irradiation flux $= G_s$, the effective sky temperature $= T_{sky}$, the absorber surface temperature $= T_s$, and the ambient air temperature $= T_\infty$; determine the useful heat removal flux (q''_{useful}) from the collector under these conditions. What is the correspondent efficiency (η) of the collector?

SOLUTION

Performing an energy balance on the absorber plate per unit surface area, we obtain

$$q''_{useful} = \alpha_s G_s + \alpha_{sky} G_{sky} - q''_{conv} - E$$

$$\eta = \frac{q''_{useful}}{G_s}$$

where $\alpha_s = \alpha_1$

$$\alpha_{sky} = \alpha_2$$

$$G_{sky} = \sigma T_{sky}^4$$

$$E = \varepsilon \sigma T_s^4$$

$$\varepsilon \cong \alpha_2$$

$$q''_{conv} = h(T_s - T_\infty)$$

11.2. A thin glass is used on the roof of a greenhouse. The glass is totally transparent for $\lambda < 1\,\mu m$, and opaque with an absorptivity $\alpha = 1$ for $\lambda > 1\,\mu m$. Assume that solar flux $= G_s$, atmospheric emission flux $= G_{atm}$, thin glass temperature $= T_g$, and interior surface emission flux$=G_i$, where G_{atm} and G_i are concentrated in the far IR region ($\lambda > 10\,\mu m$), and determine the temperature of the greenhouse ambient air (i.e., inside room air temperature, $T_{\infty,i}$).

SOLUTION

Performing an energy balance on the thin glass plate per unit surface area, and considering two convection processes (inside and outside greenhouse), two emissions (inside and outside the glass plate), and three absorbed irradiations (from solar, atmospheric, interior surface), we obtain $T_{\infty,i}$ from

$$\alpha_s G_s + \alpha_{atm} G_{atm} + h_o(T_{\infty,o} - T_g) + h_i(T_{\infty,i} - T_g) + \alpha_i G_i - 2\varepsilon\sigma T_g^4 = 0$$

where α_s = solar absorptivity for absorption of $G_{\lambda,s} \sim E_{\lambda,b}(\lambda, 5800\,K)$

$$= \alpha_1 F_{0-1\,\mu m} + \alpha_2[1 - F_{0-1\,\mu m}$$

$$= 0 \times 0.72 + 1.0[1 - 0.72]$$

$$= 0.28(\text{Note that from Table 11.1}, \lambda T = 1\,\mu m \times 5800\,K, F_{0-1\,\mu m} = 0.72)$$

α_{atm} = absorptivity for $\lambda > 10\,\mu m = 1$
  α_i = absorptivity for $\lambda > 10\,\mu m = 1$
  $\varepsilon = \alpha_\lambda$ for $\lambda \gg 1\,\mu m$, emissivity of the glass for long wavelength emission $= 1$
  h_o = convection heat transfer coefficient of the outside roof
  h_i = convection heat transfer coefficient of the inside room

Remarks

This chapter covers the same topics as in the undergraduate-level heat transfer. These include spectrum thermal radiation intensity and emissive power for a blackbody as well as a real surface at elevated temperatures; surface radiation properties such as spectral emissivity and absorptivity for real-surface radiation; how to obtain the total emissivity or absorptivity from the fraction method; how to perform energy balance from a flat surface including radiation and convection; and solar and atmospheric radiation problems. This chapter provides fundamental thermal radiation and surface properties that are useful for many engineering applications such as surface radiators, space vehicles, and solar collectors.

PROBLEMS

11.1. A diffuse surface having the following spectral distributions ($\varepsilon_\lambda = 0.3$ for $0 \leq \lambda \leq 4\,\mu m$, $\varepsilon_\lambda = 0.7$ for $4\,\mu m \leq \lambda$) is maintained at 500 K when situated in a large furnace enclosure whose walls are maintained at 1500 K. Neglecting convection effects,

    a. Determine the surface's total hemispherical emissivity (ε) and absorptivity (α).

    b. What is the net heat flux to the surface for the prescribed conditions?

    Given: $\sigma = 5.67 \times 10^{-8} (W/m^2 K^4)$

11.2. An opaque, gray surface at 27°C is exposed to an irradiation of $1000\,W/m^2$, and $800\,W/m^2$ is reflected. Air at 17°C flows over the surface, and the heat transfer convection coefficient is $15\,W/m^2\,K$. Determine the net heat flux from the surface.

11.3. A diffuse surface having the flowing spectral characteristics ($\varepsilon_\lambda = 0.4$ for $0 \leq \lambda \leq 3\,\mu m$, $\varepsilon_\lambda = 0.8$ for $3\,\mu m \leq \lambda$) is maintained at 500 K when situated in a large furnace enclosure whose walls are maintained at 1500 K:

    a. Sketch the spectral distribution of the surface emissive power E_λ and the emissive power $E_{\lambda,b}$ that the surface would have if it were a blackbody.

    b. Neglecting convection effects, what is the net heat flux to the surface for the prescribed conditions?

    c. Plot the net heat flux as a function of the surface temperature for $500 \leq T \leq 1000$ K. On the same coordinates, plot the heat flux for a diffuse, gray surface with total emissivities of 0.4 and 0.8.

    d. For the prescribed spectral distribution of ε_λ, how do the total emissivity and absorptivity of the surface vary with temperature in the range $500 \leq T \leq 1000$ K?

11.4. The spectral, hemispherical emissivity distributions for two diffuse panels to be used in a spacecraft are as shown.

    For panel A: $\varepsilon_\lambda = 0.5$ for $0 \leq \lambda \leq 3\,\mu m$, $\varepsilon_\lambda = 0.2$ for $3\,\mu m \leq \lambda$.

    For panel B: $\varepsilon_\lambda = 0.1$ for $0 \leq \lambda \leq 3\,\mu m$, $\varepsilon_\lambda = 0.01$ for $3\,\mu m \leq \lambda$.

    Assuming that the backsides of the panels are insulated and that the panels are oriented normal to the solar flux at $1300\,W/m^2$, determine which panel has high steady-state temperature.

11.5. From a heat transfer and engineering approach, explain how a glass greenhouse, which is used in the winter to grow vegetables, works. Include sketches of both the system showing energy flows and balances, and of radiation property data (radioactive properties versus wavelengths) for greenhouse components (glass and the contents inside the greenhouse). When applicable, show the appropriate equations and properties to explain the greenhouse phenomenon. When finished with the above for a glass greenhouse, extend your explanation to global warming, introducing new radioactive properties and characteristics if needed.

11.6. The glass of the skylight of a house as shown in Figure 11.12 has a spectral emissivity, $\varepsilon_\lambda(\lambda)$ [or absorptivity, $\alpha_\lambda(\lambda)$] distribution as shown: $\varepsilon_\lambda = 0.9$ for $0 \leq \lambda \leq 0.3\,\mu$m, $\varepsilon_\lambda = 0$ for $0.3 \leq \lambda \leq 2.0\,\mu$m, $\varepsilon_\lambda = 0.9$ for $\lambda \geq 2.0\,\mu$m. During an afternoon when the solar flux is $900\,\text{W/m}^2$, the temperature of the glass is $27°$C. The interior surfaces of the walls of the house and the air in the house are at $22°$C, and the heat transfer coefficient between the glass of the skylight and the air in the house is $5\,\text{W/(m}^2\,\text{K)}$.

   a. What is the overall emissivity, ε, of the skylight?

   b. What is the overall absorptivity, α, of the skylight for solar irradiation? You may assume that the sun emits radiation as a blackbody at 6000 K.

   c. What is the convective heat flux on the outer surface of the skylight, $q''_{\text{convection outside air}}$, in W/m^2? Is the temperature of the outside air higher or lower than the temperature of the skylight? Please assume that the sky is at $0°$C.

11.7. Consider a typical setup for a solar collector as shown in Figure 11.11. A special glass is used as a cover for the collector and a specialized coating is used on the collector plate and tubes, where the solar energy is collected, to maximize the performance of the collector.

   a. If you had to specify the value of the glass transmissivity, τ_λ, as a function of λ to maximize the performance of the collector, what would you choose and why? Explain. Use illustrations or sketches if needed to help explain your answer.

   b. If you had to specify the value of the collector plate and tube absorptivity, α_λ, as a function of λ to maximize the performance of the collector, what would you choose and why? Explain.

   c. A manufacturing process calls for heating a long aluminum rod that is coated with a thin film with an emissivity of ε. The rod is placed in a large convection oven whose surface is maintained at T_w (K). Air at T_∞ (K) circulates in the oven at a velocity of u (m/s) across the surface of the rod and produces a convective heat transfer coefficient of h [$\text{W/(m}^2\,\text{K)}$]. The rod has a small diameter of d (m) and has an initial temperature of T_i (K). Here, $T_i < T_\infty < T_w$. What is the rate of change of the rod temperature (K/s) when the rod is first placed in the oven?

11.8. A solar collector consists of an insulating back layer, a fluid conduit through which a water–glycol solution flows to remove heat, an absorber plate and a glass cover plate. The external temperatures T_{air}, T_{sky}, and T_{ground} are known. Solar radiation of intensity q''_s (W/m^2) is incident on the collector and collected heat q''_c (W/m^2) is removed by the fluid. The absorber plate is painted black with

an average solar absorptivity of 0.8 and an average emissivity of 0.8. Assume that the collector plate is so large that you may treat the problems as 1-D heat flow with heat sources and/or sinks.

a. Identify and label all significant heat transfer resistances and flows and draw the steady-state thermal network diagram for the collector.

b. Write the heat balance equations needed to solve for q_c''. Do not solve.

c. If the absorber plate is replaced with a black chrome surface with an average solar absorptivity of 0.95 and an average emissivity of 0.1, what values will change in the thermal network diagram? How will q_c'' change and why?

11.9. A solar collector consists of an insulating back layer, a fluid conduit through which a water–glycol solution flows to remove heat, an absorber plate and a glass cover plate. The external temperatures T_{air}, T_{sky}, and T_{ground} are also shown in the diagram. Solar radiation of intensity q_s'' (W/m^2) is incident on the collector and collected heat q_c'' (W/m^2) is removed by the fluid. The absorber plate is painted black with an average solar absorptivity of 0.9 and an average emissivity of 0.9. Assume that the collector is so large that you may treat the problems as 1-D heat flow with heat sources and/or sinks.

a. Identify and label all significant heat transfer resistances and flows and draw the steady-state thermal network diagram for this collector.

b. Write the heat balance equations needed to solve for q_c''. Do not solve.

c. If the absorber plate is replaced with a black chrome surface with an average solar absorptivity of 0.9 and an average emissivity of 0.1, what values will change in the thermal network diagram? How will q_c'' change and why?

11.10. An opaque, gray surface at 27°C is exposed to an irradiation of 1000 W/m^2, and 800 W/m^2 is reflected. Air at 17°C flows over the surface and the heat transfer convection coefficient is 15W/m^2 K. Determine the net heat flux from the surface.

11.11. Consider an opaque, horizontal plate with an electrical heater on its backside. The front side is exposed to ambient air that is at 20°C and provides a convection heat transfer coefficient of 10 W/m^2 K, a solar irradiation (at 5800°K) of 600 W/m^2, and an effective sky temperature of −40°C. What is the electrical power (W/m^2) required to maintain the plate surface temperature at $T_s = 60$°C (steady state) if the plate is diffuse and has designated spectral, hemispherical reflectivity (reflectivity $= 0.2$ for wavelength less than $2\,\mu$m, reflectivity $= 0.7$ for wavelength greater than $2\,\mu$m)?

References

1. W. Rohsenow and H. Choi, *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.
2. A. Mills, *Heat Transfer*, Richard D. Irwin, Inc., Boston, MA, 1992.
3. K.-F. Vincent Wong, *Intermediate Heat Transfer*, Marcel Dekker, Inc., New York, NY, 2003.
4. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, Fifth Edition, John Wiley & Sons, New York, NY, 2002.

12

View Factor

12.1 View Factor

In addition to surface radiation properties such as emissivity, reflectivity, and absorptivity, view factor is another important parameter for determining radiation heat transfer between two surfaces. View factor is defined as the fraction of radiation energy (the so-called radiosity including emission and reflection) from a given surface that can be seen (viewed) by the other surface. It is purely a geometric parameter, depending on the relative geometric configuration between two surfaces (depending on how the surfaces can see each other). A view factor is also called an angle factor or shape factor. The following shows how to define the view factor between two surfaces [1–4]. Figure 12.1 shows radiation exchange between the two diffuse isothermal surfaces (i.e., each surface has uniform emission and reflection from the surface). The differential radiation rate (including emission and reflection) from unit surfaces i to j is proportional to its intensity and the unit solid angle as discussed before.

$$dq_{i-j} = I_i dA_i \cos \theta_i \, dw_{j-i} = I_i \cos \theta_i \frac{\cos \theta_j}{R^2} \, dA_j \, dA_i \qquad (12.1)$$

$$= \pi I_i \frac{\cos \theta_i \cos \theta_j}{\pi R^2} \, dA_i \, dA_j$$

Performing integration over surface area i and surface area j, one obtains radiation rate from surface i to surface j as

$$q_{i-j} = J_i \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} \, dA_j \, dA_i \qquad (12.2)$$

where $\pi I_i \equiv J_i$ = radiosity (emission plus reflection).

If the radiosity J_i is uniform, that is, diffuse reflection and isothermal emission, then

$$F_{ij} = \frac{\text{Engery intercepted by } A_j}{\text{Radiosity leaving } A_i} = \frac{q_{ij}}{A_i J_i} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} \, dA_j \, dA_i \quad (12.3)$$
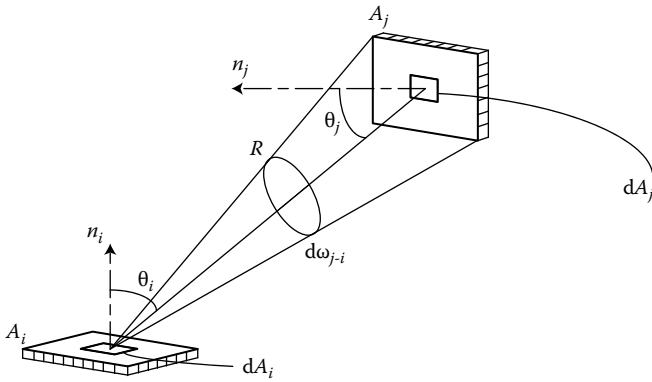
FIGURE 12.1
Radiation exchange between two diffuse isothermal surfaces.

Similarly,

$$F_{ji} = \frac{1}{A_j} \int\limits_{A_j} \int\limits_{A_i} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_i \, dA_j \qquad (12.4)$$

Therefore, we obtain the Reciprocity Rule

$$A_i F_{ij} = A_j F_{ji} \qquad (12.5)$$

For enclosure N surfaces, as shown in Figure 12.2,

$$\sum_{j=1}^{n} F_{ij} = 1 \qquad (12.6)$$
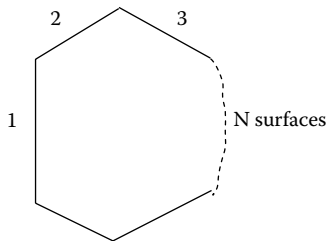
FIGURE 12.2
View factor for a N-surface enclosure.

That is,

$$F_{11} + F_{12} + F_{13} + \cdots = 1$$
$$F_{21} + F_{22} + F_{23} + \cdots = 1$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$F_{N1} + F_{N2} + F_{N3} + \cdots = 1$$

Figure 12.3 shows the differential view factor between two differential areas i and j, and between area i and differential area j. The differential view factor between differential area i and differential area j can be obtained as

$$dF_{dAi-dAj} = \frac{dq_{ij}}{J_i \, dA_i} = \frac{\cos\theta_i \cos\theta_j \, dA_j}{\pi R^2};$$

$$dF_{dAj-dAi} = \frac{dq_{ji}}{J_j \, dA_j} = \frac{\cos\theta_i \cos\theta_j \, dA_i}{\pi R^2}$$
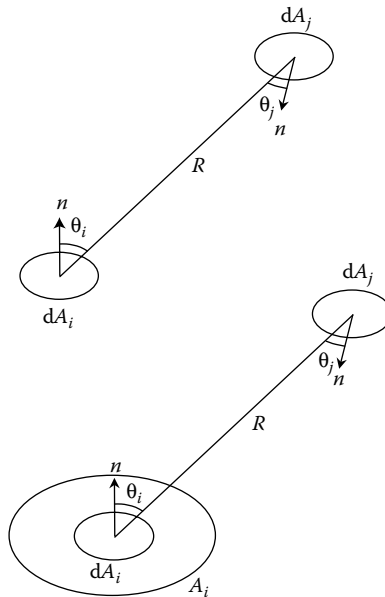
FIGURE 12.3
Concept of view factor between two differential areas.

Similarly, the differential view factor between area i and differential area j is

$$dF_{Ai-dAj} = \frac{\int J_i(\cos\theta_i \cos\theta_j/\pi R^2)\, dA_i\, dA_j}{J_i A_i} = \frac{dA_j}{A_i}\int_{A_i} \frac{\cos\theta_i \cos\theta_j}{\pi R^2}\, dA_i$$

And the view factor between differential area j and area i is

$$F_{dAj-Ai} = \frac{\int J_j(\cos\theta_i \cos\theta_j/\pi R^2)\, dA_i\, dA_j}{J_j\, dA_j} = \int_{A_i} \frac{\cos\theta_i \cos\theta_j}{\pi R^2}\, dA_i$$

From reciprocity rules for diffuse and isothermal surfaces,

$$A_i F_{Ai-Aj} = A_j F_{Aj-Ai}$$

$$dA_i\, dF_{dAi-dAj} = dA_j\, dF_{dAj-dAi}$$

$$A_i\, dF_{Ai-dAj} = dA_j F_{dAj-Ai}$$

$$dA_i\, dF_{dAi-Aj} = A_j\, dF_{Aj-dAi}$$

Example 12.1

Determine the view factor between two parallel discs as shown in Figure 12.4. Assume that $A_i \ll A_j$, the distance between two surfaces is L, and the larger disc has a diameter D.
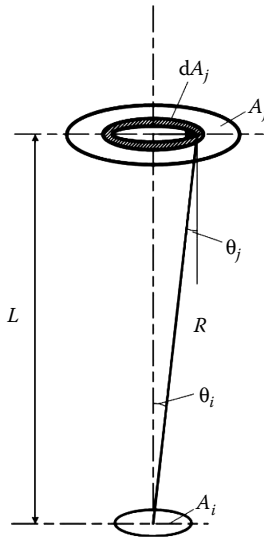
FIGURE 12.4
View factor between two parallel disks.

From Equation 12.3,

$$F_{ij} = \frac{1}{A_i} \int\limits_{A_i} \int\limits_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_i \, dA_j$$

$$= \int\limits_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_j$$

where $A_i = \int_{A_i} dA_i$,
with $\theta_i = \theta_j = \theta$, $R^2 = r^2 + L^2$, $\cos\theta = L/R$, and $dA_j = 2\pi r \, dr$,

$$F_{ij} = \int\limits_{A_j} \frac{\cos^2\theta}{\pi R^2} \, dA_j$$

$$= 2L^2 \int\limits_{0}^{D/2} \frac{r \, dr}{(r^2 + L^2)^2} = \frac{D^2}{D^2 + 4L^2}$$

12.2 Evaluation of the View Factor

View factor is a function of geometry only. The following shows several well-known methods to obtain the view factor for common geometry for radiation heat transfer applications [3].

Elongated surfaces—use Hottel's String method for 2-D geometry.

Direct integration—need to perform double-area integration (difficulty).

Contour integration—use Stoke's theorem to transform area-to-line integration.

Algebraic method—determine the unknown view factor from the known value.

12.2.1 Method 1—Hottel's Crossed-String Method for 2-D Geometry

Hottel proposed the following view factor between surface i and surface j for 2-D geometry (with surfaces elongated in the direction normal to the paper as shown in Figure 12.5.
Find

(1)

$$F_{1-2} = \frac{L_1 + L_2 - L_{ac}}{2L_1} \tag{12.7}$$
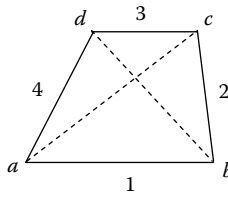
FIGURE 12.5
Concept of Hotell's cross-string method for 2-D geometry.

(2)

$$F_{1-4} = \frac{L_1 + L_4 - L_{bd}}{2L_1} \tag{12.8}$$

(3)

$$F_{1-3} = \frac{L_{ac} + L_{bd} - (L_2 + L_4)}{2L_1} \tag{12.9}$$

(1) $F_{1-2} + F_{1-ac} = 1$

$$F_{1-2} = 1 - F_{1-ac}$$

$$= 1 - \frac{L_{ac}}{L_1} F_{ac-1}$$

$$= 1 - \frac{L_{ac}}{L_1}(1 - F_{ac-2})$$

$$= 1 - \frac{L_{ac}}{L_1} + \frac{L_{ac}}{L_1} \cdot \frac{L_2}{L_{ac}} \cdot F_{2-ac}$$

$$= 1 - \frac{L_{ac}}{L_1} + \frac{L_2}{L_1} \cdot (1 - F_{2-1})$$

$$= 1 - \frac{L_{ac}}{L_1} + \frac{L_2}{L_1} - \frac{L_2}{L_1} \cdot \frac{L_1}{L_2} \cdot F_{1-2}$$

$$= 1 - \frac{L_{ac}}{L_1} + \frac{L_2}{L_1} - F_{1-2}$$

$$\therefore F_{1-2} = \frac{L_1 + L_2 - L_{ac}}{2L_1}$$

(2) Similarly, $F_{1-4} = ((L_1 + L_4 - L_{bd})/2L_1)$
(3) $F_{1-2} + F_{1-3} + F_{1-4} = 1$

$$\therefore F_{1-3} = 1 - F_{1-2} - F_{1-4}$$

$$= 1 - \frac{L_1 + L_2 - L_{ac}}{2L_1} - \frac{L_1 + L_4 - L_{bd}}{2L_1}$$

$$= \frac{2L_1 - L_1 - L_2 + L_{ac} - L_1 - L_4 + L_{bd}}{2L_1}$$

$$= \frac{L_{ac} + L_{bd} - L_2 - L_4}{2L_1}$$

The concept of the following examples (2) through (4) comes from [3].

Example 12.2

Determine the view factor between two parallel plates with partial blockages as shown in Figure 12.6.

$$\text{Length of each crossed string} = \sqrt{l^2 + c^2}$$

$$\text{Length of each uncrossed string} = 2\sqrt{b^2 + \left(\frac{c}{2}\right)^2}$$

From Hottel's crossed-string method, the view factor can be determined as

$$F_{1-2} = \frac{\sqrt{l^2 + c^2} - 2\sqrt{b^2 + (c/2)^2}}{l} = \sqrt{1 + \left(\frac{c}{l}\right)^2} - \sqrt{\left(\frac{2b}{l}\right)^2 + \left(\frac{c}{l}\right)^2}$$

Example 12.3

Determine the view factor between two opposite circular tubes as shown in Figure 12.7. From Hottel's cross-string method, the view factor is

$$F_{1-2} = \frac{2L_1 - 2L_2}{2A_1} = \frac{L_1 - L_2}{\pi R}$$

where

$$L_1 - \text{crossed string abcde}$$

$$L_2 - \text{uncrossed string ef } L_2 = D + 2R$$
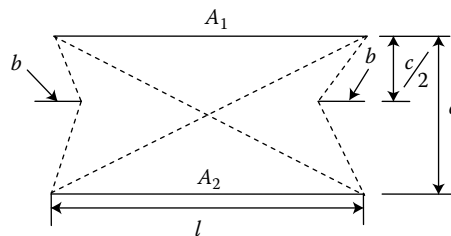
FIGURE 12.6
View factor between two parallel plates with partial blockages.
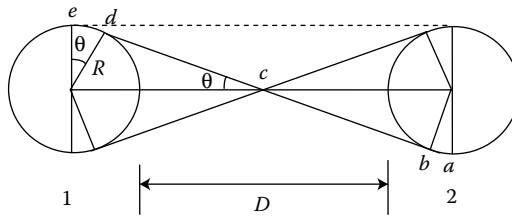
FIGURE 12.7
View factor between two opposite circular tubes.

Let $X = 1 + D/(2R)$

$$F_{1-2} = \frac{2}{\pi} \left[(x^2 - 1)^{1/2} + \sin^{-1}\left(\frac{1}{X}\right) - X \right]$$

$$F_{1-2} = \frac{2}{\pi} \left[(x^2 - 1)^{1/2} + \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{X}\right) - X \right]$$

Example 12.4

Determine the view factor between two circular tubes with partial blockage as shown in Figure 12.8. From Hottel's cross-string method, the view factor can be determined as follows:

The sum of the length of crossed strings: $L_{A-B-D-G-I} + L_{H-C-D-E-F}$
The sum of the length of uncrossed strings: $L_{A-F} + L_{H-C-D-G-I}$

Therefore,

$$L_{A-B-C-H}F_{1-2} = 2 \frac{L_{A-B-D-G-I} + L_{H-C-D-E-F} - L_{A-F} + L_{H-C-D-G-I}}{2}$$
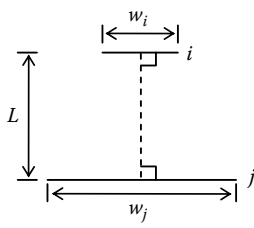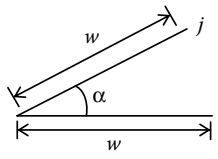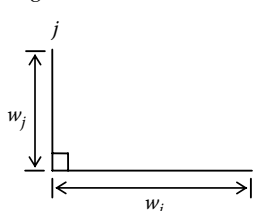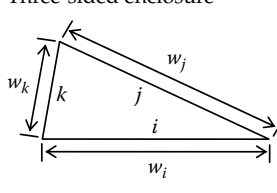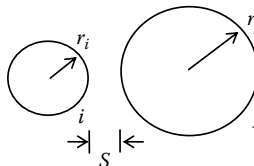
Table 12.1 shows many useful view factors for 2-D geometries that can be determined by using Hottel's cross-string method [2,4].
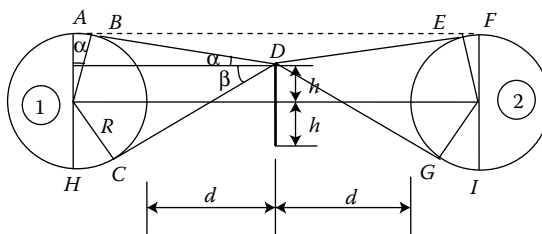
FIGURE 12.8
View factor between two circular tubes with partial blockage.

TABLE 12.1
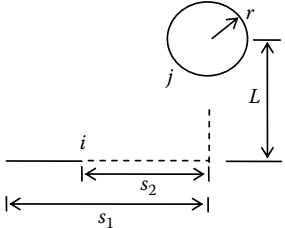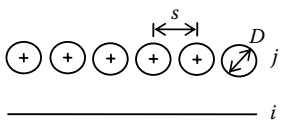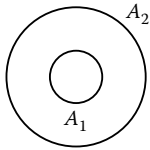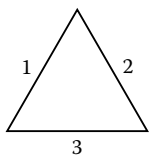
View Factors for 2-D Geometries

Geometry	Relation
Parallel plates with midlines connected by perpendicular	$F_{ij} = \dfrac{[(W_i + W_j)^2 + 4]^{1/2} - [(W_j - W_i)^2 + 4]^{1/2}}{2W_i}$ $W_i = w_i/L, W_j = w_j/L$

Geometry	Relation
Inclined parallel plates of equal width and a common edge	$F_{ij} = 1 - \sin\left(\dfrac{\alpha}{2}\right)$

Geometry	Relation
Perpendicular plates with a common edge	$F_{ij} = \dfrac{1 + (w_j/w_i) - [1 + (w_j/w_i)^2]^{1/2}}{2}$

Geometry	Relation
Three-sided enclosure	$F_{ij} = \dfrac{w_i + w_j - w_k}{2w_i}$

Geometry	Relation
Parallel cylinders of different radii	$F_{ij} = \dfrac{1}{2\pi}\left\{\pi + [C^2 - (R+1)^2]^{1/2} - [C^2 - (R-1)^2]^{1/2}\right.$ $\left. +(R-1)\cos^{-1}\left[\left(\dfrac{R}{C}\right) - \left(\dfrac{1}{C}\right)\right]\right.$ $\left. -(R+1)\cos^{-1}\left[\left(\dfrac{R}{C}\right) + \left(\dfrac{1}{C}\right)\right]\right\}$ $R = r_j/r_i, S = s/r_i$ $C = 1 + R + S$

TABLE 12.1   (continued)
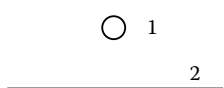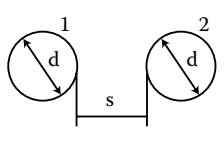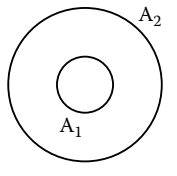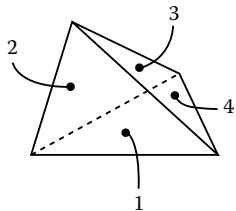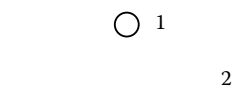
View Factors for 2-D Geometries

Geometry	Relation
Cylinder and parallel rectangle	$F_{ij} = \dfrac{r}{s_1 - s_2}\left[\tan^{-1}\dfrac{s_1}{L} - \tan^{-1}\dfrac{s_2}{L}\right]$

Infinite plane and row of cylinders	$F_{ij} = 1 - \left[1 - \left(\dfrac{D}{s}\right)^2\right]^{1/2} + \left(\dfrac{D}{s}\right)\tan^{-1}\left(\dfrac{s^2 - D^2}{D^2}\right)^{1/2}$

Concentric cylinders	$F_{12} = 1; \quad F_{21} = \dfrac{A_1}{A_2}$ $F_{22} = 1 - F_{21} = 1 - \dfrac{A_1}{A_2}$

Long duct with equilateral triangular cross-section	$F_{12} = F_{13} = \dfrac{1}{2}$

Long parallel plates of equal width	$F_{12} = F_{21} = \left[1 + \left(\dfrac{c}{a}\right)^2\right]^{1/2} - \left(\dfrac{c}{a}\right)$
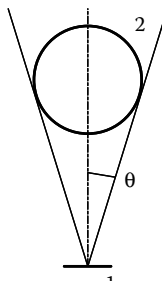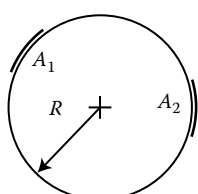
TABLE 12.1   (continued)

View Factors for 2-D Geometries

Geometry	Relation
Long cylinder parallel to a large plane area	$F_{12} = \frac{1}{2}$
Long adjacent parallel cylinders of equal diameters	Let $X = 1 + \dfrac{s}{d}$, then $$F_{12} = F_{21} = \frac{1}{\pi}\left[(X^2 - 1)^{1/2} + \sin^{-1}\frac{1}{X} - X\right]$$
Concentric spheres	$F_{12} = 1; \quad F_{21} = \dfrac{A_1}{A_2}$ $F_{22} = 1 - F_{21} = 1 - \dfrac{A_1}{A_2}$
Regular tetrahedron	$F_{12} = F_{13} = F_{14} = \dfrac{1}{3}$
Sphere near a large plane area	$F_{12} = \dfrac{1}{2}$

TABLE 12.1   (continued)

View Factors for 2-D Geometries

Geometry	Relation
Small area perpendicular to the axis of a surface of revolution	$F_{12} = \sin^2 \theta$

| Area on the inside of a sphere | $F_{12} = \dfrac{A_2}{4\pi R^2}$ |

12.2.2   Method 2—Double-Area Integration

Use direct integration to determine the view factor between two adjacent surface areas *i* and *j* as shown in Figure 12.9.

$$F_{Ai-Aj} = \frac{1}{A_i} \int\limits_{A_i} \int\limits_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} \, dA_i \, dA_j$$

where

$$R^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2$$

$$\cos \theta_i = \frac{1}{R} \left[l_i(x_j - x_i) + m_i(y_j - y_i) + n_i(z_j - z_i) \right]$$

$$\cos \theta_j = \frac{1}{R} \left[l_j(x_i - x_j) + m_j(y_i - y_j) + n_j(z_i - z_j) \right]$$

Therefore, the view factor can be determined by performing the following integration:

$$F_{Ai-Aj} = \frac{1}{bc} \int\limits_0^b dx_j \int\limits_0^b dx_i \int\limits_0^a z_j \, dz_j \int\limits_0^c \frac{y_i \, dy_i}{\pi [(x_i - x_j)^2 + y_i^2 + z_j^2]^2} \tag{12.10}$$
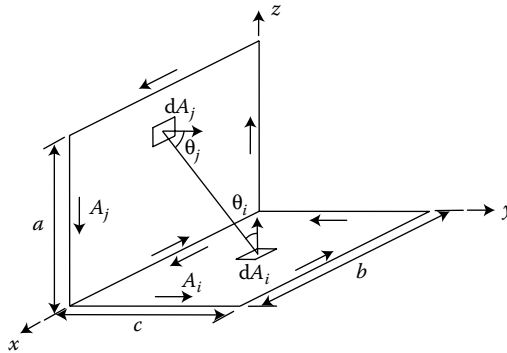
FIGURE 12.9
View factor between two adjacent surfaces.

12.2.3 Method 3—Contour Integration

Use Stokes' theorem to transform area to line integration.

Determine the view factor between two adjacent surfaces as shown in Figure 12.9.

$$F_{Ai-Aj} = \frac{1}{2\pi A_i} \left[\oint_{c_i} \oint_{c_j} \left(\ln R \, dx_i \, dx_j + \ln R \, dy_i \, dy_j + \ln R \, dz_i \, dz_j \right) \right] \tag{12.11}$$

Apply Stokes' theorem to reduce quadric to double integrations as

$$
\begin{aligned}
F_{Ai-Aj} = \frac{1}{2\pi bc} & \left[\int_0^b dx_j \left\{ \int_0^b \ln \left[(x_i - x_j)^2 + a^2 \right]^{1/2} dx \right. \right. \\
& \left. + \int_b^0 \ln \left[(x_i - x_j)^2 + c^2 + a^2 \right]^{1/2} dx_i \right\} \\
& + \int_b^0 dx_j \left\{ \int_0^b \ln \left[(x_i - x_j)^2 + 0^2 \right]^{1/2} dx \right. \\
& \left. \left. + \int_b^0 \ln \left[(x_i - x_j)^2 + c^2 + 0^2 \right]^{1/2} dx_i \right\} \right] \tag{12.12}
\end{aligned}
$$

The following shows how to use Stokes' theorem to determine the view factor between two opposite surfaces [3] as shown in Figure 12.10.

$$
F_{Ai-Aj} = \frac{1}{2\pi A_i} \left[\oint_{c_i} \oint_{c_j} (\ln R \, dx_i \, dx_j + \ln R \, dy_i \, dy_j + \ln R \, dz_i \, dz_j) \right]
$$

$$
F_{Ai-Aj} = \frac{1}{2\pi bc} \oint_{c_i} \left\{ \int_0^c \ln \left[x_i^2 + (y_j - y_i)^2 + a^2 \right]^{1/2} dy_j \right\} dy_i
$$

$$
+ \frac{1}{2\pi bc} \oint_{c_i} \left\{ \int_0^b \ln \left[(x_j - x_i)^2 + (c - y_i)^2 + a^2 \right]^{1/2} dx_j \right\} dx_i
$$

$$
+ \frac{1}{2\pi bc} \oint_{c_i} \left\{ \int_c^0 \ln \left[(b - x_i)^2 + (y_j - y_i)^2 + a^2 \right]^{1/2} dy_j \right\} dy_i
$$

$$
+ \frac{1}{2\pi bc} \oint_{c_i} \left\{ \int_b^0 \ln \left[(x_j - x_i)^2 x_i^2 + y_i^2 + a^2 \right]^{1/2} dx_j \right\} dx_i
$$

$$
= \frac{1}{2\pi bc} \int_c^0 \int_0^c \left\{ \ln \left[(y_j - y_i)^2 + a^2 \right]^{1/2} + \ln \left[b^2 + (y_j - y_i)^2 + a^2 \right]^{1/2} \right\} dy_j \, dy_i
$$

+ other integrals

$$
= \frac{2a^2}{\pi bc} \left\{ \ln \left[\frac{(1 + (b/a)^2)(1 + (c/a)^2)}{1 + (b/a)^2 + (c/a)^2} \right] \right.
$$

$$
+ \frac{b}{a} [1 + (c/a)^2]^{1/2} \tan^{-1} \left[\frac{b/a}{[1 + (c/a)^2]^{1/2}} \right]
$$

$$
+ \frac{c}{a} \left(1 + \left(\frac{b}{a} \right)^2 \right)^{1/2} \tan^{-1} \left[\frac{c/a}{[1 + (b/a)^2]^{1/2}} \right]
$$

$$
\left. - \frac{b}{a} \tan^{-1} \left(\frac{b}{a} \right) - \frac{c}{a} \tan^{-1} \left(\frac{c}{a} \right) \right\}
\tag{12.13}
$$

Table 12.2 shows several useful view factors for 3-D geometries [4] that can be determined by using Stoke's theorem to transform area-to-line integration.
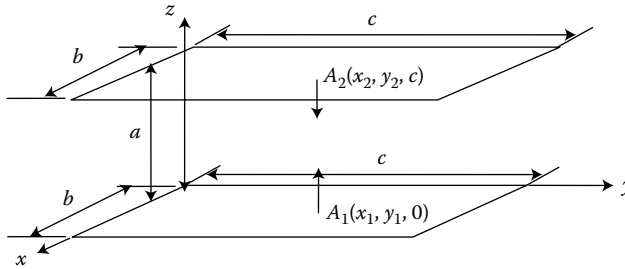
FIGURE 12.10
View factor between two opposite surfaces.

Example 12.5

Determine view factors F_{1-2} and F_{2-1} for the following geometries shown in Figure 12.11.

For geometries (a) and (b),

$$F_{1-1} + F_{1-2} + F_{1-3} = 1$$

$$F_{1-1} = 0$$

$$\therefore F_{1-2} = 1 - F_{1-3} \,(\text{can be obtained from Table 12.2})$$

$$A_1 F_{1-2} = A_2 F_{2-1}$$

$$\therefore F_{2-1} = \frac{A_1}{A_2} F_{1-2}$$

12.2.4 Method 4—Algebraic Method

Determine the unknown view factor between surface areas A_1 and A_2, as shown in Figure 12.12, from the known value of view factors.

$$A_1 F_{1-2} = A_1 F_{1-j} - A_1 F_{1-4}$$

$$= A_j(F_{j-i} - F_{j-3}) - A_4(F_{4-i} - F_{4-3}) \qquad (12.14)$$

where $F_{j-i}, F_{j-3}, F_{4-i},$ and F_{4-3}, are available from formulas or charts.

Example 12.6

Determine view factors F_{1-4} and F_{4-1} from Figures 12.13a and b:

$$A_1 F_{1-4} = A_2 F_{2-3}$$
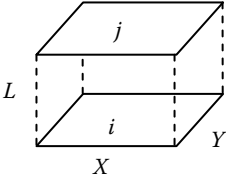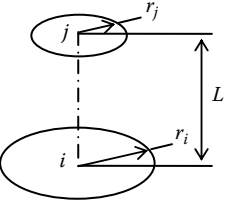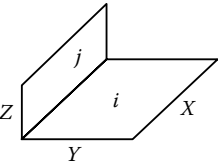
$$A_i F_{i-j} = A_1 F_{1-j} + A_2 F_{2-j}$$

$$= A_1 F_{1-3} + A_1 F_{1-4} + A_2 F_{2-3} + A_2 F_{2-4}$$

$$= A_1 F_{1-3} + 2 A_1 F_{1-4} + A_2 F_{2-4}$$

TABLE 12.2

View Factors for 3-D Geometries

Geometry	Relation
Aligned parallel rectangles	$\overline{X} = X/L, \overline{Y} = Y/L$ $F_{ij} = \dfrac{2}{\pi \overline{XY}} \left\{ \ln \left[\dfrac{(1+\overline{X^2})(1+\overline{Y^2})}{1+\overline{X}^2+\overline{Y}^2} \right]^{1/2} \right.$ $+ \overline{X}(1+\overline{Y^2})^{1/2} \tan^{-1} \dfrac{\overline{X}}{(1+\overline{Y^2})^{1/2}}$ $+ \overline{Y}(1+\overline{X^2})^{1/2} \tan^{-1} \dfrac{\overline{Y}}{(1+\overline{X^2})^{1/2}}$ $\left. - \overline{X} \tan^{-1} \overline{X} - \overline{Y} \tan^{-1} \overline{Y} \right\}$
Coaxial parallel disks	$R_i = r_i/L, R_j = r_j/L$ $S = 1 + \dfrac{1+R_j^2}{R_i^2}$ $F_{ij} = \dfrac{1}{2}\{S - [S^2 - 4(r_j/r_i)^2]^{1/2}\}$
Perpendicular rectangles with a common edge	$H = Z/X, W = Y/X$ $F_{ij} = \dfrac{1}{\pi W} \left(W \tan^{-1} \dfrac{1}{W} + H \tan^{-1} \dfrac{1}{H} \right.$ $-(H^2+W^2)^{1/2} \tan^{-1} \dfrac{1}{(H^2+W^2)^{1/2}}$ $+ \dfrac{1}{4} \ln \left\{ \dfrac{(1+W^2)(1+H^2)}{1+W^2+H^2} \left[\dfrac{W^2(1+W^2+H^2)}{(1+W^2)(W^2+H^2)} \right]^{W^4} \right.$ $\left. \left. \times \left[\dfrac{H^2(1+H^2+W^2)}{(1+H^2)(H^2+W^2)} \right]^{H^2} \right\} \right)$

Source: F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, John Wiley & Sons, Fifth Edition, New York, NY, 2002.

Hence,

$$A_1 F_{1-4} = \frac{1}{2}[A_i F_{i-j} - A_1 F_{1-3} - A_2 F_{2-4}] \qquad (12.15)$$

where F_{i-j}, F_{1-3}, and F_{2-4}, are available from Table 12.2 formulas or charts.
   And, $A_1 F_{1-4} = A_4 F_{4-1}$.

(a)

1 (top)

2 (side)

$L = D$

3 (bottom)

(b)

L

L

L

1 (top)

2 (side)

3 (bottom)

D
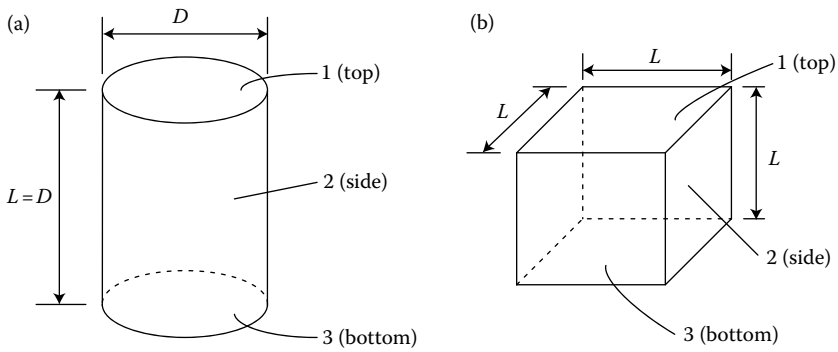
FIGURE 12.11
(a) A cylindrical furnace. (b) A cubic furnace.

A_1

A_3

A_i

A_4

A_2

A_j
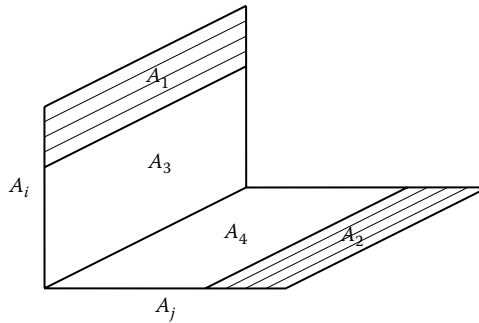
FIGURE 12.12
Algebraic method.

A_3

A_4

A_1

A_2

A_3

A_4

A_1

A_2
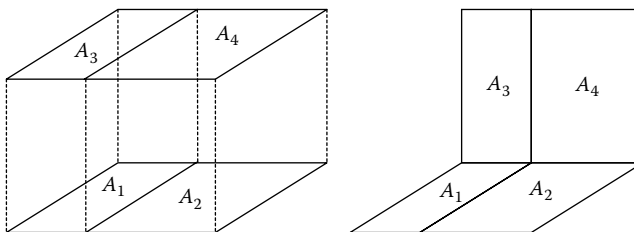
FIGURE 12.13
Applications of shape factor algebra to opposing and adjacent rectangles.

Remarks

This chapter covers the same information as in the undergraduate-level heat transfer. In the undergraduate-level heat transfer, students are expected to know how to use those view factors available from tables or charts in order to

calculate radiation heat transfer between two surfaces for many engineering applications. However, at the intermediate-level heat transfer, students are expected to focus more on how to derive view factors instead of how to use them. In particular, students are expected to know how to determine the view factors by using Hottel's string method for many 2-D geometries. For 3-D geometry view factors, we do not go into much detail because of the required double-area integrations that belong to advanced radiation heat transfer.

PROBLEMS

12.1   Determine the view factors for Examples 1, 2, 3, and 4, respectively.
12.2   Determine the view factors shown in Table 12.1.
12.3   Determine the view factors shown in Table 12.2.

References

1. W. Rohsenow and H. Choi, *Heat, Mass, and Momentum Transfer*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1961.
2. A. Mills, *Heat Transfer*, Richard D. Irwin, Inc., Boston, MA, 1992.
3. K.-F.V. Wong, *Intermediate Heat Transfer*, Marcel Dekker, Inc., New York, NY, 2003.
4. F. Incropera and D. Dewitt, *Fundamentals of Heat and Mass Transfer*, John Wiley & Sons, Fifth Edition, New York, NY, 2002.

13

Radiation Exchange in a Nonparticipating Medium

13.1 Radiation Exchange between Gray Diffuse Isothermal Surfaces in an Enclosure

Since we know how to get surface radiation properties such as emissivity, reflectivity, and absorptivity and how to determine the view factor between two surfaces, the following shows how to determine radiation heat transfer between surfaces in an enclosure [1–4]. Assume that there are N gray diffuse and isothermal surfaces. This implies that each surface at T_i has uniform radiosity J_i (emission plus reflection). Figure 13.1 shows an energy balance on each surface i and an energy balance between surface i and the rest of enclosure surfaces j.

Based on the assumptions, each surface radiation properties can be given as follows.

$\alpha_i = \varepsilon_i$, for gray and diffuse surfaces
$\tau_i = 0$, for the opaque body

$$\rho_i = 1 - \alpha_i = 1 - \varepsilon_i$$

There are three types of radiation problems for electric furnace applications.

1. Given each surface temperature T_i to determine each surface heat flux $(q/A)_i =$?
2. Given each surface heat flux $(q/A)_i$ to determine each surface temperature $T_i =$?
3. Combination of 1 and 2, some surfaces given temperature but heat flux unknown, and some surfaces given heat flux but temperature unknown.

Perform energy balance on the i surface: net energy $=$ energy out (radiosity) $-$ energy in (irradiation)

$$q_i = A_i(J_i - G_i) \tag{13.1}$$
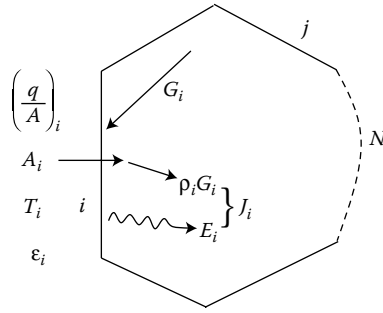
FIGURE 13.1
Radiation heat transfer between N surfaces in an enclosure.

also

$$J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i) G_i \qquad (13.2)$$

Therefore,

$$q_i = A_i \left(J_i - \frac{J_i - \varepsilon_i E_{bi}}{1 - \varepsilon_i} \right) = A_i \left(\frac{J_i - \varepsilon_i J_i - J_i + \varepsilon_i E_{bi}}{1 - \varepsilon_i} \right)$$

$$= A_i \left(\frac{\varepsilon_i (E_{bi} - J_i)}{1 - \varepsilon_i} \right)$$

So that energy from surface i

$$\Rightarrow \quad q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/(\varepsilon_i A_i)} \qquad (13.3)$$

Then perform energy exchange between surface i and the rest of surfaces j:
net energy = energy out (radiosity) − energy in (irradiation)

$$q_i = A_i (J_i - G_i)$$

where

$$A_i G_i = \sum_{j=1}^{N} F_{ji} A_j J_j = \sum_{j=1}^{N} F_{ij} A_i J_j \qquad (13.4)$$

Therefore,

$$q_i = A_i \left(J_i - \sum_{j=1}^{N} F_{ij} J_j \right)$$

By using $\sum_{j=1}^{N} F_{ij} = 1$, and multiplying to J_i,

$$q_i = A_i \left(\sum_{j=1}^{N} F_{ij}J_i - \sum_{j=1}^{N} F_{ij}J_j \right)$$

So that energy transfer between surface i and the rest of enclosure surfaces j becomes

$$q_i = \sum_{j=1}^{N} A_i F_{ij}(J_i - J_j) = \sum_{j=1}^{N} q_{ij} \tag{13.5}$$

Combining Equations 13.3 and 13.5, we have

$$q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} = \sum_{j=1}^{N} A_i F_{ij}(J_i - J_j) = \sum_{j=1}^{N} q_{ij}$$

$$q_i = \underbrace{\frac{E_{bi} - J_i}{(1 - \varepsilon_i)/(\varepsilon_i A_i)}}_{\substack{\text{surface resistance} \\ \text{due to emissivity}}} = \sum_{j=1}^{N} \underbrace{\frac{J_i - J_j}{(1/A_i F_{ij})}}_{\substack{\text{geometrical resistance} \\ \text{due to view factor}}} = \sum_{j=1}^{N} q_{ij} \tag{13.6}$$

In addition, combining Equations 13.2 and 13.4, we get

$$J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i) \sum_{j=1}^{N} F_{ij}J_j \tag{13.7}$$

$$= \text{emission from surface } i + \text{reflection from surface } i.$$

13.1.1   Method 1: Electric Network Analogy

Electric network analogy [3] can be used to solve the aforementioned radiation heat transfer problem as shown in Figure 13.2. The following shows a few special cases for radiation heat transfer applications.

Special case 1—Radiation between a two-surface enclosure as shown in Figure 13.3: (a) hemicylinder, (b) parallel plates, (c) rectangular channel, (d) long concentric cylinders, (e) concentric spheres, (f) small convex object in a large enclosure:

$$q_1 = q_{12} = -q_2 = \frac{\sigma(T_1^4 - T_2^4)}{(1 - \varepsilon_1)/(A_1\varepsilon_1) + 1/(A_1 F_{12}) + (1 - \varepsilon_2)/(A_2\varepsilon_2)} \tag{13.8}$$

If for a blackbody, $\varepsilon_1 = \varepsilon_2 = 1$, then

$$q_1 = A_1 F_{12} \sigma(T_1^4 - T_2^4) \tag{13.9}$$
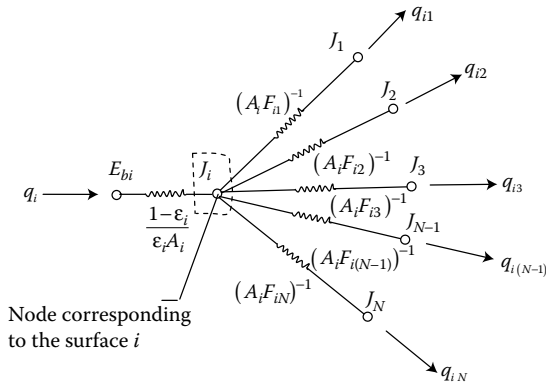
FIGURE 13.2
Network representation of the radiative exchange between surface i and the remaining surfaces of an enclosure.
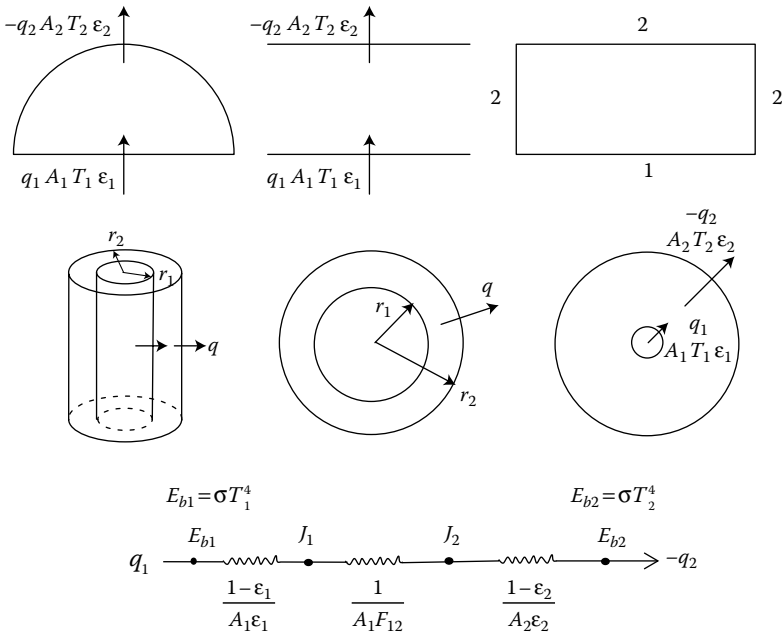
FIGURE 13.3
Radiation between a two-surface enclosure.

Special case 2—Radiation between two parallel surfaces with middle shields as shown in Figure 13.4:

$$q_1 = q_{12} = \frac{A_1\sigma(T_1^4 - T_2^4)}{\begin{array}{c}(1 - \varepsilon_1)/\varepsilon_1 + (1/F_{13}) + (1 - \varepsilon_{31})/\varepsilon_{31} \\ +(1 - \varepsilon_{32})/\varepsilon_{32} + (1/F_{32}) + (1 - \varepsilon_2)/\varepsilon_2\end{array}} \qquad (13.10)$$

$$= \frac{A_1\sigma(T_1^4 - T_2^4)}{(1/\varepsilon_1) + (1 - \varepsilon_{31})/\varepsilon_{31} + (1 - \varepsilon_{32})/\varepsilon_{32} + (1/\varepsilon_2)} = -q_2$$

where $F_{13} = F_{32} = 1$.

To cut down radiation heat loss, ε_{31} and ε_{32} should be small, that is, ρ_{31} is large.

Special case 3—Reradiating surfaces (insulated surface, $q_R = 0$): The following electric furnaces, as shown in Figure 13.5, can be modeled as radiation heat transfer between two opposite surfaces (hot and cold) with a third reradiation side surface (perfect reflection and perfect insulation).
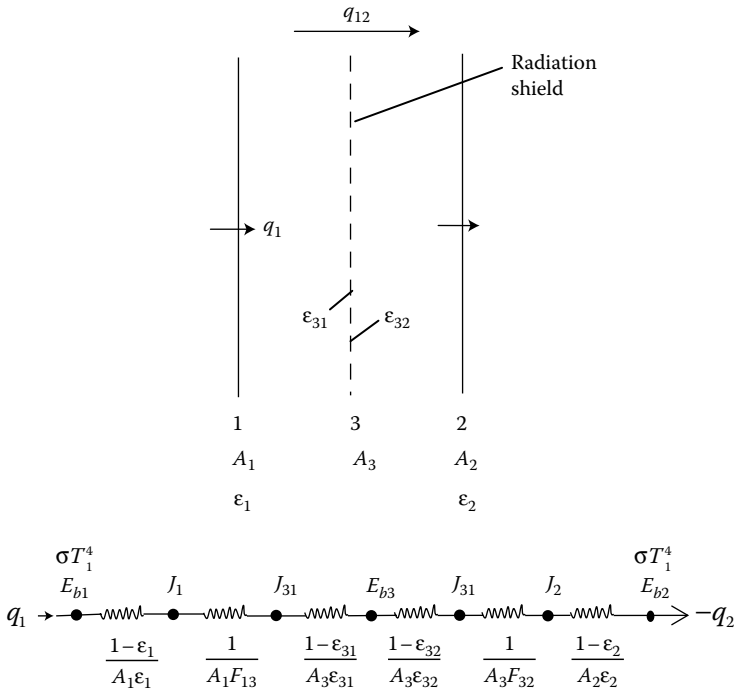
FIGURE 13.4
Radiation between two parallel surfaces with middle shield.
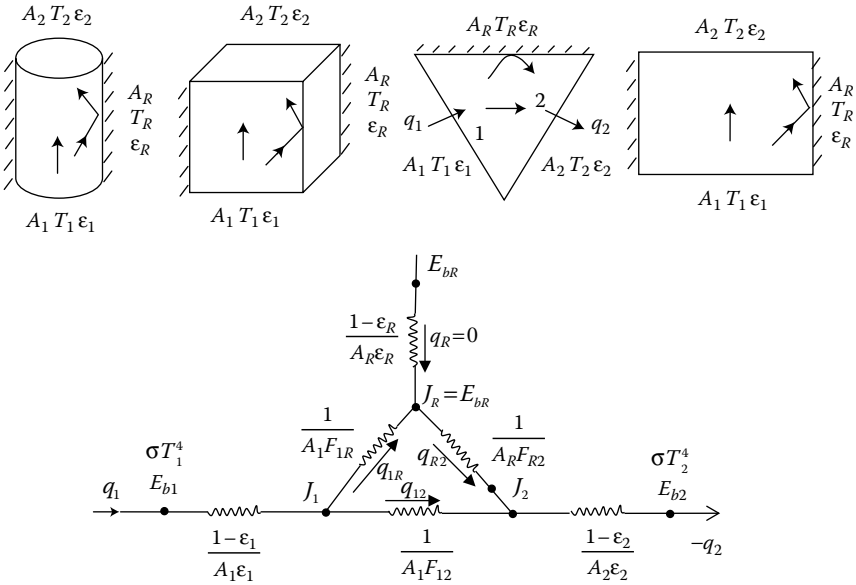
FIGURE 13.5
Electric furnaces with a reradiating surface.

$\because q_R = 0$, therefore, $q_1 = -q_2$

$$q_1 = -q_2 = \frac{\sigma T_1^4 - \sigma T_2^4}{(1 - \varepsilon_1)/A_1\varepsilon_1 + 1/(A_1F_{12} + 1/[(1/A_1F_{1R}) + (1/A_2F_{2R})]) \atop +(1 - \varepsilon_2)/A_2\varepsilon_2}$$

(13.11)

where T_1, and T_2 are given

$$A_R F_{R2} = A_2 F_{2R}$$

And surface emissivity, area, and view factors are also given or predetermined.

If $q_1 = -q_2$ is determined from above and if $q_R = 0$, how to determine reradiation surface temperature $T_R =$?

From energy balance on the reradiation surface,

$$q_{1R} = \frac{J_1 - J_R}{1/A_1F_{1R}} = q_{R2} = \frac{J_R - J_2}{1/A_R F_{R2}}$$

and

$$q_1 = \frac{E_{b1} - J_1}{(1 - \varepsilon_1)/A_1\varepsilon_1} \Rightarrow J_1 = \sigma T_1^4 - q_1\frac{1 - \varepsilon_1}{A_1\varepsilon_1}$$

$$q_2 = \frac{J_2 - E_{b2}}{(1 - \varepsilon_2)/A_2\varepsilon_2} \Rightarrow J_2 = q_2\frac{1 - \varepsilon_2}{A_2\varepsilon_2} + \sigma T_2^4$$
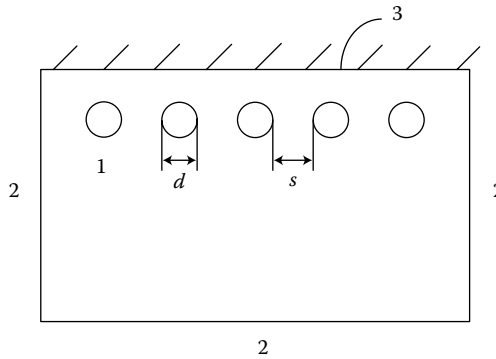
FIGURE 13.6
A radiant heater panel model.

from

$$\frac{J_1 - J_R}{1/A_1 F_{1R}} = \frac{J_R - J_2}{1/A_R F_{R2}} \Rightarrow J_R = \frac{A_1 F_{1R} J_1 + A_R F_{R2} J_2}{A_1 F_{1R} + A_R F_{R2}}$$

Therefore,

$$J_R = E_{b,R} = \sigma T_R^4 \Rightarrow T_R = \left[\frac{J_R}{\sigma}\right]^{1/4} \tag{13.12}$$

Special case 4—A radiant heater panel problem: A long radiant heater panel consists of a row of cylindrical electrical heating elements, as shown in Figure 13.6.

The above Equations 13.11 and 13.12 can be used to determine heat transfer rate $q_1 = -q_2$, and T_R, respectively. However, we need to calculate view factors F_{11}, F_{12}, and F_{1R}(or F_{13}). In Table 12.1,

$$F_{11} = \frac{1}{\pi}\left[(X^2 - 1)^{1/2} + \sin^{-1}\frac{1}{X} - X\right]$$

with $X = 1 + (s/d)$.

Assume $F_{12} \cong F_{13}$ for symmetry and $F_{11} + F_{12} + F_{13} = 1$.

Therefore, $F_{12} = 1/2(1 - F_{11})$.

13.1.2 Method 2: Matrix Linear Equations

Applying energy balance on each surface, one can obtain N radiosity linear equations for N surfaces in an enclosure. The matrix and its inverse matrix can be used to solve these N radiosity linear algebraic equations. The following shows how to solve this type of problems for either given surface temperature or surface heat flux in an enclosure with N surfaces [3–4].

Case A—Given each surface temperature to determine its heat flux, T_i *given,* $\Rightarrow$ $q_i = ?$

Use energy balance on surface i and energy exchange between surface i and the rest of surfaces j, from Equation 13.6,

$$\frac{E_{bi} - J_i}{(1 - \varepsilon_i)/(\varepsilon_i A_i)} = \sum_{j=1}^{N} \frac{J_i - J_j}{1/A_i F_{ij}} \tag{13.13}$$

Applying the above equation to each surface $(1, 2, 3, \ldots,$ and $N)$, respectively, one obtains the following N radiosity linear equations (after rearranging them).

$$a_{11}J_1 + a_{12}J_2 + \cdots + a_{1N}J_N = c_1$$
$$a_{21}J_1 + a_{22}J_2 + \cdots + a_{2N}J_N = c_2$$
$$\vdots$$
$$a_{N1}J_1 + a_{N2}J_2 + \cdots + a_{NN}J_N = c_N$$

Then coefficient matrix $[A]$, column matrix $[J]$, and column matrix $[C]$ can be formed to satisfy the N linear equations. Therefore, the unknown radiosity matrix $[J]$ can be determined by solving the given inverse matrices $[A]$ and $[C]$.

$$[A][J] = [C]$$
$$[J] = [A]^{-1}[C] = \cdots$$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \quad [J] = \begin{bmatrix} J_1 \\ J_2 \\ \cdots \\ J_N \end{bmatrix} \quad [C] = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_N \end{bmatrix}$$

Once the unknown radiosity J from each surface i has been solved from the aforementioned matrix relation, radiation heat transfer from each surface can be shown from Equation 13.6 as

$$q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} = \frac{\sigma T_i^4 - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} \tag{13.14}$$

Special example for a three-surface enclosure problem: If surface temperatures shown in Figure 13.7 are given (T_1, T_2, T_3), how to determine surface heat transfer rates (q_1, q_2, q_3)?

From Equation 13.13,

$$\frac{E_{bi} - J_i}{(1 - \varepsilon_i)/A_i \varepsilon_i} = \sum_{j=1}^{N} \frac{J_i - J_j}{1/A_i F_{ij}}$$
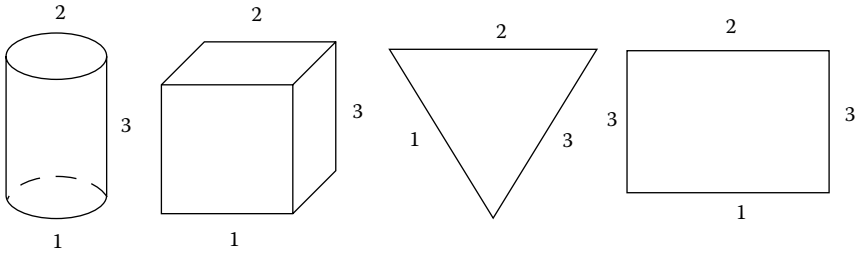
FIGURE 13.7
Radiation heat transfer among a three-surface enclosure.

Apply for surface 1:

$$\frac{E_{b1} - J_1}{(1 - \varepsilon_1)/A_1\varepsilon_1} = \frac{J_1 - J_1}{1/A_1F_{11}} + \frac{J_1 - J_2}{1/A_2F_{12}} + \frac{J_1 - J_3}{1/A_1F_{13}}$$

$$\frac{A_1\varepsilon_1}{1 - \varepsilon_1}E_{b1} - \left(\frac{A_1\varepsilon_1}{1 - \varepsilon_1}\right)J_1 = (A_2F_{12})J_1 + (-A_2F_{12})J_2 + (A_1F_{13})J_1 + (-A_1F_{13})J_3$$

$$\therefore \left(\frac{A_1\varepsilon_1}{1 - \varepsilon_1} + A_2F_{12} + A_1F_{13}\right)J_1 + (-A_2F_{12})J_2 + (-A_1F_{13})J_3$$

$$= \frac{A_1\varepsilon_1}{1 - \varepsilon_1}E_{b1} = \frac{A_1\varepsilon_1}{1 - \varepsilon_1}\sigma T_1^4$$

$$\Rightarrow a_{11}J_1 + a_{12}J_2 + a_{13}J_3 = c_1$$

where $a_{11} = (A_1\varepsilon_1/(1 - \varepsilon_1) + A_2F_{12} + A_1F_{13})$, $a_{12} = -A_2F_{12}, a_{13} = -A_1F_{13}$, $c_1 = (A_1\varepsilon_1/(1 - \varepsilon_1))\sigma T_1^4$.

Apply for surface 2:

$$\frac{E_{b2} - J_2}{(1 - \varepsilon_2)/A_2\varepsilon_2} = \frac{J_2 - J_1}{1/A_2F_{12}} + \frac{J_2 - J_2}{1/A_2F_{22}} + \frac{J_2 - J_3}{1/A_2F_{23}}$$

$$\frac{A_2\varepsilon_2}{1 - \varepsilon_2}E_{b2} - \left(\frac{A_2\varepsilon_2}{1 - \varepsilon_2}\right)J_2 = (A_2F_{21})J_2 + (-A_2F_{21})J_1 + (A_2F_{23})J_2 + (-A_2F_{23})J_3$$

$$\therefore (-A_2F_{21})J_1 + \left(\frac{A_2\varepsilon_2}{1 - \varepsilon_2} + A_2F_{21} + A_2F_{23}\right)J_2 + (-A_2F_{23})J_3$$

$$= \frac{A_2\varepsilon_2}{1 - \varepsilon_2}E_{b2} = \frac{A_2\varepsilon_2}{1 - \varepsilon_2}\sigma T_2^4$$

$$\Rightarrow a_{21}J_1 + a_{22}J_2 + a_{23}J_3 = c_2$$

where $a_{21} = -A_2F_{21}$, $a_{22} = ((A_2\varepsilon_2/1 - \varepsilon_2) + A_2F_{21} + A_2F_{23})$, $a_{23} = -A_2F_{23}$, $c_2 = (A_2\varepsilon_2/(1 - \varepsilon_2))\sigma T_2^4$.

Apply for surface 3:

$$\frac{E_{b3} - J_3}{(1 - \varepsilon_3)/A_3\varepsilon_3} = \frac{J_3 - J_1}{1/A_3F_{31}} + \frac{J_3 - J_2}{1/A_3F_{32}} + \frac{J_3 - J_3}{1/A_3F_{33}}$$

$$\frac{A_3\varepsilon_3}{1 - \varepsilon_3}E_{b3} - \left(\frac{A_3\varepsilon_3}{1 - \varepsilon_3}\right)J_3 = (A_3F_{31})J_3 + (-A_3F_{31})J_1 + (A_3F_{32})J_3 + (-A_3F_{32})J_2$$

$$\therefore (-A_3F_{31})J_1 + (-A_3F_{32})J_2 + \left(\frac{A_3\varepsilon_3}{1 - \varepsilon_3} + A_3F_{31} + A_3F_{32}\right)J_3$$

$$= \frac{A_3\varepsilon_3}{1 - \varepsilon_3}E_{b3} = \frac{A_3\varepsilon_3}{1 - \varepsilon_3}\sigma T_3^4$$

$$\Rightarrow a_{31}J_1 + a_{32}J_2 + a_{33}J_3 = c_3$$

where $a_{31} = -A_3F_{31}$, $a_{32} = -A_3F_{32}$, $a_{33} = (A_3\varepsilon_3/(1 - \varepsilon_3) + A_3F_{31} + A_3F_{32})$, $c_3 = (A_3\varepsilon_3/(1 - \varepsilon_3))\sigma T_3^4$.

From the above three linear equations, the following matrix can be formed:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad J = \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} \quad C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

$$[A][J] = [C]$$

$$[J] = [A]^{-1}[C]$$

Alternatively, we can apply Equation 13.7 to each surface and get

$$J_1 = \varepsilon_1 E_{b1} + (1 - \varepsilon_1)[F_{11}J_1 + F_{12}J_2 + F_{13}J_3]$$

$$J_2 = \varepsilon_2 E_{b2} + (1 - \varepsilon_2)[F_{21}J_1 + F_{22}J_2 + F_{23}J_3]$$

$$J_3 = \varepsilon_3 E_{b3} + (1 - \varepsilon_3)[F_{31}J_1 + F_{32}J_2 + F_{33}J_3]$$

Similarly, the above three linear equations can be rearranged in order to obtain the matrix relation as $[A][J] = [C]$ and then solve for $[J] = [A]^{-1}[C]$.

Once matrix $[J]$ is determined, that is, J_1, J_2, and J_3 have been determined, then use Equation 13.14 to find surface heat transfer rates as

$$\Rightarrow q_1 = \frac{E_{b1} - J_1}{(1 - \varepsilon_1)/(A_1\varepsilon_1)} = \frac{\sigma T_1^4 - J_1}{(1 - \varepsilon_1)/(A_1\varepsilon_1)}$$

$$\Rightarrow q_2 = \frac{E_{b2} - J_2}{(1 - \varepsilon_2)/A_2\varepsilon_2} = \frac{\sigma T_2^4 - J_2}{(1 - \varepsilon_2)/A_2\varepsilon_2}$$

$$\Rightarrow q_3 = \frac{E_{b3} - J_3}{(1 - \varepsilon_3)/A_3\varepsilon_3} = \frac{\sigma T_3^4 - J_3}{(1 - \varepsilon_3)/A_3\varepsilon_3}$$

Case B—Given each surface heat flux to determine its temperature, q_i given, $\Rightarrow$ $T_i = ?$

Use energy exchange between surface i and the rest of surfaces j, from Equation 13.6,

$$q_i = \sum_{j=1}^{N} \frac{J_i - J_j}{1/A_i F_{ij}} \tag{13.15}$$

Apply the above to each surface (1, 2, 3, ..., and N) and obtains N radiosity linear equations as

$$a_{11}J_1 + a_{12}J_2 + \cdots + a_{1N}J_N = c_1$$

$$a_{21}J_1 + a_{22}J_2 + \cdots + a_{2N}J_N = c_2$$

$$\vdots$$

$$a_{N1}J_1 + a_{N2}J_2 + \cdots + a_{NN}J_N = c_N$$

The following matrix can be used to solve for $[J]$:

$$[A][J] = [C]$$

$$[J] = [A]^{-1}[C] = \cdots$$

Once matrix $[J]$ has been solved, use Equation 13.14 on each surface i to determine surface temperature on each surface

$$\Rightarrow q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i} = \frac{\sigma T_i^4 - J_i}{(1 - \varepsilon_i)/\varepsilon_i A_i}$$

or

$$E_{bi} = \sigma T_i^4 = q_i \frac{1 - \varepsilon_i}{A_i \varepsilon_i} + J_i$$

Therefore,

$$T_i = \left(\frac{q_i(1 - \varepsilon_i)/A_i \varepsilon_i + J_i}{\sigma} \right)^{1/4} \tag{13.16}$$

Case C—Combined Case A and Case B

- Some surfaces are given temperatures but heat fluxes are unknown.
- Some surfaces are given heat fluxes but temperatures are unknown.
- Use the same procedure shown for Case A and Case B in order to form matrix $[A]$, column matrix $[J]$, and column matrix $[C]$.
- After determining matrix $[J]$, either heat fluxes or temperatures can be determined.

Special case for the blackbody radiation problem: Use the aforementioned results for any blackbody surface with unity emissivity ($\varepsilon_i = 1$).

13.2 Radiation Exchange between Gray Diffuse Nonisothermal Surfaces

The following shows how to solve radiation heat transfer between non-isothermal surfaces [4–5]. In this case, one shall consider radiation exchange between two differential surfaces (which can be assumed as a uniform temperature over each differential element) as shown in Figure 13.8. Then the aforementioned analysis method can be applied.

Consider radiosity from a differential element i,

$$J_i(\bar{r}_i) = \varepsilon_i \sigma T_i^4(\bar{r}_i) + (1 - \varepsilon_i) G_i(\bar{r}_i)$$

$$= \varepsilon_i \sigma T_i^4(\bar{r}_i) + (1 - \varepsilon_i) \sum_{j=1}^{N} \int_{A_j} J_j(\bar{r}_j) \, dF_{dA_i - dA_j} \qquad (13.17)$$

Define known quantity

$$K(\bar{r}_i, \bar{r}_j) \equiv \frac{dF_{dA_i - dA_j}}{dA_j}$$

Therefore,

$$J_i(\bar{r}_i) = \varepsilon_i \sigma T_i^4(\bar{r}_i) + (1 - \varepsilon_i) \sum_{j=1}^{N} \int_{A_j} J_j(\bar{r}_j) K(\bar{r}_i, \bar{r}_j) \, dA_j \qquad (13.18)$$

For Case A problem, obtain N equations, where T_i is known, solve the integral and $J_i(\bar{r}_i)$ can be obtained.
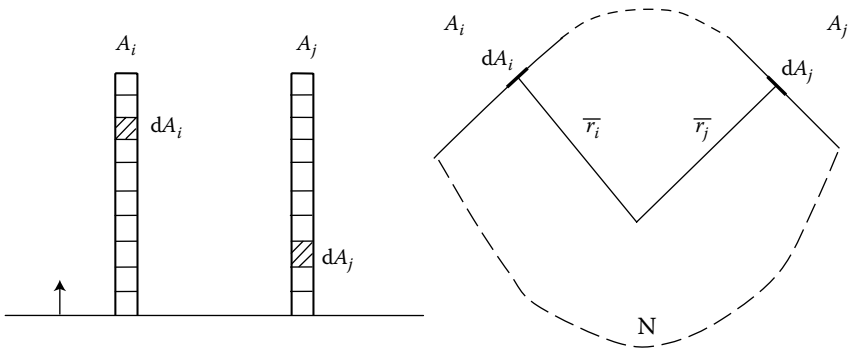
FIGURE 13.8
Radiation exchange between nonisothermal surfaces.

When $J_i(\bar{r}_i)$ is determined by using the matrix method, and for given $T_i(\bar{r}_i)$, then

$$\left(\frac{q}{A}\right)_i (\bar{r}_i) = \frac{\varepsilon_i}{1 - \varepsilon_i} \left[\sigma T_i^4(\bar{r}_i) - J_j(\bar{r}_j)\right] \tag{13.19}$$

can be solved.

Example

Apply the numerical method—Simpson's rule (Trapezoidal rule) for the nonisothermal surfaces shown in Figure 13.9.
Given: $\varepsilon_a = 0.9$, $T_a = 1000 - 1500°C$

$$\varepsilon_b = 0.2, \quad T_b = 300°C$$

1. $\bar{q}_a = ?$
2. Compare $q_a = q_{a1} + q_{a2} = ?$

For case B problem, if $(q/A)_i$ is given, then

$$\left(\frac{q}{A}\right)_i = \sum_{j=1}^{N} \frac{J_i(\bar{r}_i) - J_j\left(\bar{r}_j\right)}{1/dF_{dA_i - dA_j}}$$

$$\left(\frac{q}{A}\right)_i = \sum_{j=1}^{N} \left(J_i(\bar{r}_i) - J_j\left(\bar{r}_j\right)\right) K\left(\bar{r}_i, \bar{r}_j\right) dA_j$$

when $J_i(\bar{r}_i)$ is determined by the matrix method, and for given $(q/A)_i(\bar{r}_i)$, then

$$\sigma T_i^4(\bar{r}_i) = \frac{1 - \varepsilon_i}{\varepsilon_i}\left(\frac{q}{A}\right)_i(\bar{r}_i) + J_i(\bar{r}_i)$$
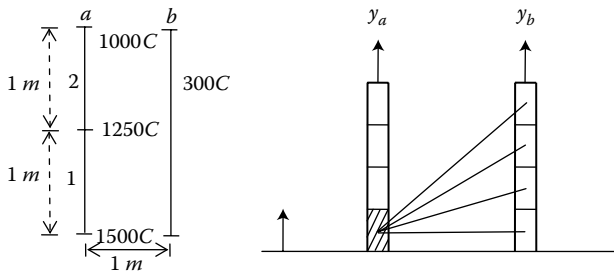
FIGURE 13.9
Radiation between nonisothermal surfaces.

13.3 Radiation Exchange between Nongray Diffuse Isothermal Surfaces

The following (a) integral model or (b) band model can be used to determine radiation exchange among isothermal diffuse nongray surfaces [4,5] as sketched in Figure 13.10.

a. Integral model

$$\left(\frac{q}{A}\right)_i = \int_0^\infty \left(\frac{q}{A}\right)_i d\lambda \tag{13.20}$$

b. Band model

$$\left(\frac{q}{A}\right)_i = \sum_{k=1}^N \left(\frac{q}{A}\right)_i \Delta\lambda_k \tag{13.21}$$

13.4 Radiation Interchange among Diffuse and Nondiffuse (Specular) Surfaces [4,5]

Exchange factor by using the image method, as shown in Figure 13.11,

$$E_{A1-A4} = \underbrace{F_{A1-A4}}_{\text{diffuse}} + \underbrace{\rho_3^s F_{A1(3)-A4}}_{\text{specular reflection}} \tag{13.22}$$

$$E_{A2-A4} = F_{A2-A4} + \rho_3^s F_{A2(3)-A4} \tag{13.23}$$

$$E_{A1-A1} = \underset{0}{F_{A1\downarrow A1}} + \rho_3^s F_{A1(3)-A1} \tag{13.24}$$

Reciprocity Rules $A_i E_{Ai-Aj} = A_j E_{Aj-Ai}$
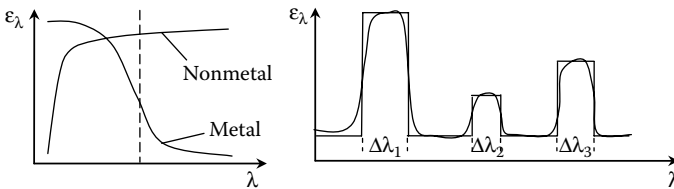
FIGURE 13.10
Radiation exchange among diffuse, isothermal, nongray surfaces.
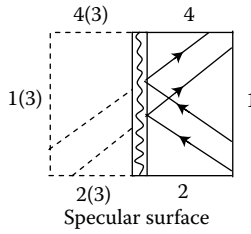
FIGURE 13.11
Radiation exchange among diffuse and specular surfaces.

13.5 Energy Balance in an Enclosure with Diffuse and Specular Surface

N_d is the diffuse reflection surface,

$N - N_d$ the specular reflection surface.

Assume all surfaces shown in Figure 13.12 are diffuse emitting, gray, and isothermal; then

$$J_i = \varepsilon_i \sigma T_i^4 + (1 - \varepsilon_i)G_i$$

where $G_i = G_i^d + G_i^s$.

From the diffuse surface,

$$G_i^d = \sum_{j=1}^{Nd} J_j E_{Ai-Aj} \qquad (13.25)$$

where

$$E_{Ai-Aj} = F_{Ai-Aj} + \sum_k \rho_k^s F_{Ai(k)-Aj}$$
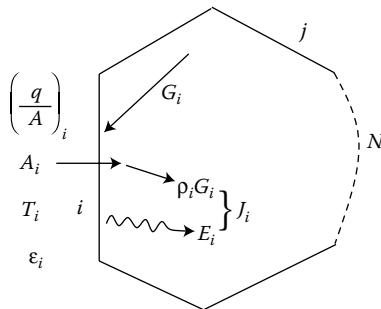
FIGURE 13.12
Radiation heat transfer between N surfaces (including diffuse and specular) in an enclosure.

From the specular surface,

$$G_i^s = \sum_{j=Nd+1}^{N} \varepsilon_j \sigma T_j^4 E_{Ai-Aj} \tag{13.26}$$

for given T_i, from the above equations, J_i can be solved.

If T_i is given, G_i can be solved ($G_i = G_i^d + G_i^s$).

For the N_d diffuse surfaces,

$$\left(\frac{q}{A}\right)_i = \frac{1}{(1 - \varepsilon_i)/\varepsilon_i} \left(\sigma T_i^4 - J_i\right) \tag{13.27}$$

or for the $N - N_d$ specular surfaces,

$$\left(\frac{q}{A}\right)_i = \varepsilon_i \left(\sigma T_i^4 - G_i\right) \tag{13.28}$$

Remarks

In undergraduate-level heat transfer, students are expected to know how to calculate radiation heat transfer between two surfaces or between two surfaces with a third reradiating surface by using the electric network analogy method for many engineering applications such as electric heaters, radiation shields, and electric furnaces with insulating side walls, and so on. In intermediate-level heat transfer, this chapter focuses on how to analyze and solve radiation heat transfer problems in an N-surfaces enclosure by using the matrix linear equations method for more complicated electric or combustion furnaces applications. Students are expected to know how to set up a matrix from linear equations by applying energy balance on each of N-surfaces with given surface temperatures or surface heat fluxes BCs. Here we assume that each N-surface has gray and diffuse properties and keeps at isothermal condition. We do not go into much details for any N-surface behaving as nongray, nondiffuse (specular), or nonisothermal condition. These require more complex mathematics and belong to advanced radiation topics.

PROBLEMS

13.1. A rectangular oven is 1 m wide, 0.5 m tall, and 2 m deep into the paper and is used to bake a carbon-fiber cloth with an electric heater at the top. All vertical walls are reradiating (reflectory and insulated). Take $\varepsilon_1 = 0.7, \varepsilon_2 = 0.9$, and $\varepsilon_3 = 0.8$. The heater temperature is 650°C when 20 kW of power is supplied. Convection is negligible.

Given:

$$\sigma = 5.67 \times 10^{-8} \frac{W}{m^2 K^4}$$

    a. Based on the analogy of electric resistance network, draw radiation heat transfer network from surface 1 to surface 2.

    b. Determine the cloth (surface 2) temperature.

13.2. A rectangular oven is 1 m wide, 0.5 m tall, and very deep into the paper and is used to bake a carbon-fiber cloth with an electric heater at the top. All vertical walls are reradiating (reflectory and insulated). Take $\varepsilon_1 = 0.7, \varepsilon_2 = 0.9$, and $\varepsilon_3 = 0.8$. When 20 kW of power is supplied, the heater temperature is 650°C. Neglecting convection, what is the cloth temperature?

13.3. A cubic furnace (1 m × 1 m × 1 m). During the steady-state operation, the top surface is cooled at 250°C and the bottom floor is heated at 1000°C. The side walls are insulated refractory surfaces. The view factor between the top and bottom surfaces is 0.2.

    a. Determine the net radiation transfer between the top and bottom surfaces.

    b. Determine the temperature of the insulated refractory surfaces.

    c. Comment on what effect changing the values of emissivities of top, bottom, and refractory surfaces would have on the results of (a) and (b).

13.4. Consider two aligned, parallel, square planes (0.5 m × 0.5 m) spaced 0.5 m apart and maintained at $T_1 = 500$ K and $T_2 = 1000$ K. Calculate the net radiative heat transfer from surface 1 for the following special conditions:

    a. Both planes are black and the surroundings are at 0 K.

    b. Both planes are black with connecting, reradiating walls.

    c. Both planes are diffuse and gray with $\varepsilon_1 = 0.6, \varepsilon_2 = 0.8$, and the surroundings at 0 K.

    d. Both planes are diffuse and gray ($\varepsilon_1 = 0.6$ and $\varepsilon_2 = 0.8$) with connecting, reradiating walls.

13.5. A room is 3 m square and 3 m high. The walls can be taken as adiabatic and isothermal. The ceiling is at 35°C and has an emittance of 0.8, while the floor is at 20°C and has an emittance of 0.9. Denote the ceiling as surface 1, the floor 2, and the walls 3.

    a. Set up the radiosity equations. Determine and evaluate all the shape factors, and tabulate as a 3 × 3 array. Solve these equations to determine the heat flow into the floor, q_2.

    b. Draw the radiation network. Use the network to obtain an expression for q_2, and solve for q_2 again.

13.6. A thin plate (surface area A_1, emissivity ε_1, absorptivity α_1) is mounted horizontally facing above a larger horizontal surface (area A_2, emissivity ε_2, absorptivity α_2).

    a. Give the corresponding thermal radiation network associated to the problem.

    b. Develop an expression without the radiosities for the radiation heat transfer rate from 1 to 2.

    c. What is the limit of this expression when the second surface is infinite?

    d. Let surface 2 be the sky which is a blackbody at a temperature 15°C cooler than ambient air, which is at 2°C. The

plate is well insulated at the bottom. Let h be the average convective heat transfer coefficient between the surface and ambient air. Given $h = 10\,\text{W}/\text{m}^2\,\text{K}$, $\varepsilon_1 = 0.54 = \alpha_1$, $\sigma = 5.67 \times 10^{-8}\,\text{W}/\text{m}^2\,\text{K}^4$. Determine the equilibrium plate temperature.

13.7. Consider two very large parallel plates with diffuse, gray surfaces. Determine the irradiation and radiosity for the upper plate (at $T_1 = 1000°\text{K}$, $\varepsilon_1 = 1$). What is the radiosity for the lower plate (at $T_2 = 500°\text{K}$, $\varepsilon_2 = 0.8$)? What is the net radiation exchange between the plates per unit area of the plates?

13.8. A 0.25-m-diameter sphere (surface 1) is located inside of a 0.5-m-diameter sphere (surface 2). Surface 1 is 200°C and surface 2 is 100°C. Determine all of the view factors and calculate the net heat transfer rate (W) between the two spherical surfaces. Show all work and list all assumptions. (*Note*: $\sigma = 5.67 \times 10^{-8}\,\text{W}/\text{m}^2\,\text{K}^4$)

13.9. For a three-surface enclosure problem as shown in Figure 13.7, do the following problems by using the matrix method.
   a. Given T_1, T_2, T_3, determine q_1, q_2, q_3.
   b. Given q_1, q_2, q_3, determine T_1, T_2, T_3.
   c. Given T_1, T_2, q_3, determine q_1, q_2, T_3.
   d. Given T_1, q_2, q_3, determine q_1, T_2, T_3.
   e. Given q_1, q_2, T_3, determine T_1, T_2, q_3.

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14

Radiation Transfer through Gases

14.1 Gas Radiation Properties

A volume of gases, such as CO_2, H_2O (water vapor), CO, NO, NH_3, SO_2, HCl, the hydrocarbons, and the alcohols, can emit and absorb energy at a given temperature and pressure. It is important to determine gas radiation properties such as emissivity and absorptivity for combustion furnace designs [1–5]. The combustion products (CO_2, water vapor, CO, NO, and NH_3) have radiation properties but air (oxygen and nitrogen gases), helium, and hydrogen have no radiation properties (transparent to radiation). Assume that there is no scattering effect, in order to simplify the analysis. In general, gas emissivity and absorptivity increase with pressure and volume, but decrease with temperature; gas emissivity and absorptivity vary with wavelength [1] as shown in Figure 14.1. Gases absorb and emit radiation in rather narrow wavelength bands rather than in the continuous spectrum exhibited by solid surfaces. For real gas, gas absorptivity is not the same as emissivity. But under gray gas assumption, absorptivity can be equivalent to emissivity.

Figure 14.2 shows that gas radiation properties (carbon dioxide, water vapor) increase with their partial pressure and geometric mean bean length (four times volume divided by surface area), decrease with temperature [1]. The results were obtained by applying hemispherical gas radiation to an element area at the center of the base, as shown in Figure 14.3. Several other geometries that contain gases are also sketched in the figure. In general, gas emissivity and absorptivity are relatively low, approximately equivalent to an order of magnitude 0.1.

$$\varepsilon_c = \varepsilon_c(P_cL, T_g) \cong 0.1 \tag{14.1}$$

$$\varepsilon_w = \varepsilon_w(P_wL, T_g) \cong 0.1. \tag{14.2}$$

At 1 atm total pressure, CO_2 has partial pressure P_c and water vapor has partial pressure P_w. L, geometric mean bean length, is defined as

$$L = \frac{4V}{A_s} \cong 0.9\frac{4V}{A_s} \tag{14.3}$$

where V is the volume and A_s is surface area.
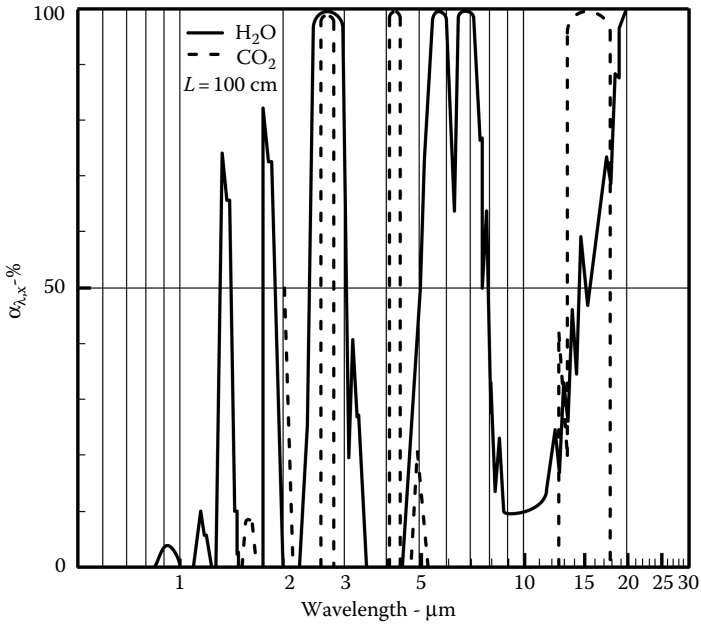
FIGURE 14.1
Band emission of carbon dioxide and water vapor.
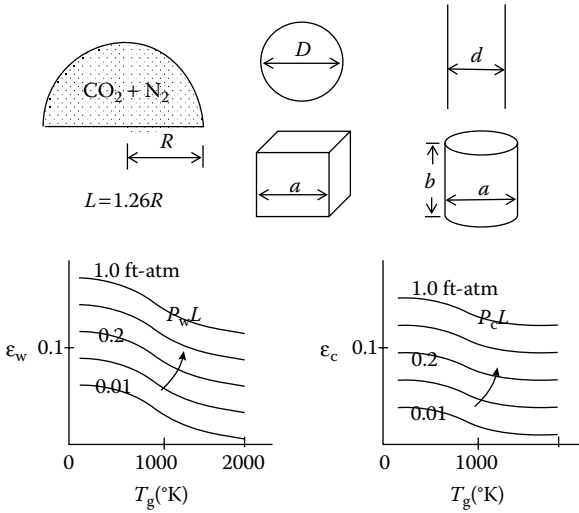
FIGURE 14.2
Gas radiation properties for water vapor and carbon dioxide.
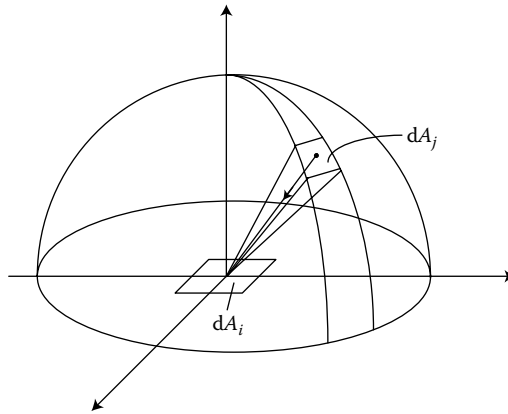
FIGURE 14.3
Hemispherical gas radiation to an element area at the center of base.

The concept of geometric mean beam length for other gas mass geometries will be discussed in a later section.

The total gas emissivity for combined CO_2 and H_2O can be obtained as

$$\varepsilon_g = \varepsilon_c + \varepsilon_w - \Delta\varepsilon \qquad (14.4)$$

where $\Delta\varepsilon \cong 0.01$ is a correction factor of emissivity for overlap of λ for CO_2 and H_2O.

For gray gas,

$$\varepsilon_g = \alpha_g \qquad (14.5)$$

14.1.1 Volumetric Absorption

Consider radiation heat transfer between two parallel plates at T_1 and T_2, filled with absorption gas at a uniform temperature T_g. Spectral radiation absorption in a gas is proportional to the absorption coefficient $k_\lambda(1/m)$ and the thickness L of the gas. The radiation intensity decreases with increasing distance due to absorption [3], as shown in Figure 14.4.

$$dI_\lambda(x) = -k_\lambda I_\lambda(x)\,dx \qquad (14.6)$$

If K_λ is a constant value for a given gas, we obtain

$$\frac{dI_\lambda(x)}{I_\lambda(x)} = -k_\lambda\,dx$$
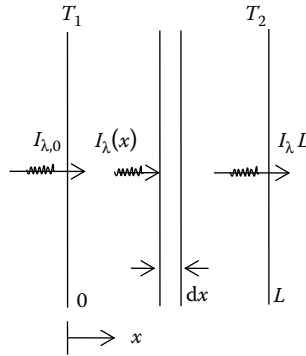
FIGURE 14.4
Absorption in a gas.

Performing integration, we obtain

$$\ln I_\lambda(x) = -K_\lambda x + C_1$$

$$I_\lambda(x) = e^{(-K_\lambda x + C_1)} = C e^{-K_\lambda x}$$

at $x = 0, C = I_{\lambda,0}$

$$I_\lambda(x) = I_{\lambda,0} e^{-K_\lambda x}$$

at $x = L$

$$I_{\lambda,L} = I_{\lambda,0} e^{-K_\lambda L} \tag{14.7}$$

This exponential decay is called Beer's law. One can define the transmissivity as

$$\tau_\lambda = \frac{I_{\lambda,L}}{I_{\lambda,0}} = e^{-K_\lambda L} \tag{14.8}$$

The absorptivity is

$$\alpha_\lambda = 1 - \tau_\lambda = 1 - e^{-K_\lambda L}.$$

For gases, $\alpha = \varepsilon_\lambda =$ emissivity.

If we consider both gas emission and the absorption effect, the intensity of the beam is attenuated due to absorption and is augmented due to gas emission along the distance [2]. Assume a local thermodynamic equilibrium, absorption coefficient will equal emission coefficient, and Equation 14.6 becomes

$$dI_\lambda(x) = [-K_\lambda I_\lambda(x) + K_\lambda I_{b\lambda}] dx \tag{14.9}$$

where $K_\lambda I_{b\lambda}$ is intensity gained due to gas emission, $-K I_\lambda(x)$ is the intensity attenuated due to gas absorption.

Performing integration, we obtain:

$$dI_\lambda(x) = -K_\lambda[I_\lambda(x) - I_{b\lambda}]\,dx$$

$$\frac{d[I_\lambda(x) - I_{b\lambda}]}{[I_\lambda(x) - I_{b\lambda}]} = -K_\lambda\,dx$$

$$\ln[I_\lambda(x) - I_{b\lambda}] = -K_\lambda x + C_1$$

$$I_\lambda(x) - I_{b\lambda} = e^{(-K_\lambda x + C_1)} = C\,e^{-K_\lambda x}$$

at $x = 0, C = I_{\lambda,0} - I_{b\lambda}$

$$I_\lambda(x) = I_{b\lambda} + (I_{\lambda,0} - I_{b\lambda})\,e^{-K_\lambda x}$$

$$I_\lambda(x) = I_{\lambda,0}\,e^{-K_\lambda x} + I_{b\lambda}(1 - e^{-K_\lambda x})$$

at $x = L$, and we obtain

$$I_\lambda(L) = I_{\lambda,0}\,e^{-K_\lambda L} + I_{b\lambda}(1 - e^{-K_\lambda L}) = I_{\lambda,0}\tau_\lambda + I_{b\lambda}\varepsilon_\lambda \qquad (14.10)$$

In general, the absorption coefficient K_λ is strongly dependent on wavelength. If we use the averaged overall wavelength of total properties, we obtain: $K_\lambda = K, \alpha = 1 - \tau, \alpha = \varepsilon, I_\lambda = I$,

$$I_{b\lambda} = I_b = \sigma T_g^4$$

14.1.2 Geometry of Gas Radiation: Geometric Mean Beam Length

The typical gas emissivity data were obtained by applying radiation to hemispherical (of radius R) collection of gases radiating to an element of area at the center of the base, as shown in Figure 14.3. For other furnace shapes, there exists an equivalent mean beam length (L), defined as the radius of a gas hemisphere which radiates to unit area at the center of its base the same as the average radiation over the area from the actual gas volume shape [2]. Consider two surface elements dA_i and dA_j of an enclosure containing an isothermal gray gas at temperature T_g, use total properties, as shown in Figure 14.5. The irradiation dG_i coming to surface dA_i from surface dA_j is

$$dG_i = I_i^- \cos\theta_i\,dw_j \qquad (14.11)$$

where I_i^- is the intensity approaching surface dA_i, using the concept developed in Equation 14.10, replacing L by $R = I_j^+ e^{-KR} + I_{bg}(1 - e^{-KR})$, I_j^+ the intensity leaving surface $dA_j = J_j/\pi$, J_j the radiosity leaving surface $dA_j =$ emission and reflection from surface dA_j, I_{bg} the intensity from blackbody
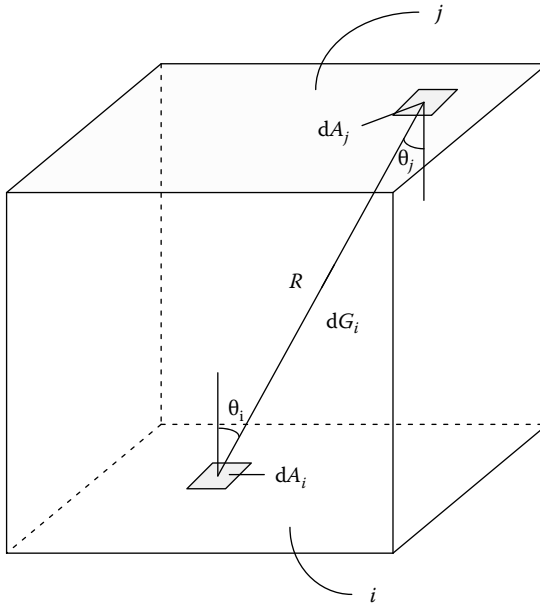
FIGURE 14.5
Elemental surface for radiation in an enclosure containing an isothermal gray gas.

gas emission $= E_{bg}/\pi = \sigma T_g^4/\pi$, E_{bg} the blackbody gas emission $= \sigma T_g^4$, $dw_j = dA_j \cos\theta_j/R^2$, R the distance (beam length) between surface dA_i and dA_j, and K the gas absorption coefficient.

Therefore,

$$dG_i = \int_{A_j} \left[J_j e^{-KR} + E_{bg}(1 - e^{-KR}) \right] \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_j \, dA_i$$

$$G_i = \frac{1}{A_i} \int dG_i \, dA_i$$

$$= \frac{1}{A_i} \int_{A_i} \int_{A_j} \left[J_j e^{-KR} + E_{bg}(1 - e^{-KR}) \right] \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_j \, dA_i \qquad (14.12)$$

The distance (beam length) R varies over the surface. For convenience, we define a mean surface (mean beam length) L_{ij}, such that

$$G_i = \left[J_j e^{-KL_{ij}} + E_{bg}(1 - e^{-KL_{ij}}) \right] \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos\theta_i \cos\theta_j}{\pi R^2} \, dA_j \, dA_i$$

$$= \left[J_j e^{-KL_{ij}} + E_{bg}(1 - e^{-KL_{ij}}) \right] F_{ij} \qquad (14.13)$$

Comparing Equations 14.12 and 14.13, we obtain

$$e^{-KL_{ij}} F_{ij} = \frac{1}{A_i} \int\limits_{A_i} \int\limits_{A_j} e^{-KR} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_j \, dA_i \tag{14.14}$$

If KL_{ij} is small, for the optically thin gases low pressure, $e^{-KL_{ij}} \approx 1 - KL_{ij}$, Equation 14.14 becomes

$$L_{ij} = \frac{1}{A_i F_{ij}} \int\limits_{A_i} \int\limits_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R} dA_j \, dA_i \tag{14.15}$$

$$L_{ij} A_i F_{ij} = L_{ji} A_j F_{ji} \tag{14.16}$$

If a furnace or combustion chamber can be modeled as a single-surface enclosure, that is, with a uniform wall temperature and emission (uniform radiosity J_s), $A_i = A_s, F_{ij} = 1, L_{ij} = L_{ji} = L$ Equations 14.13 and 14.14 become

$$G_s = J_s \, e^{-KL} + E_{bg}(1 - e^{-KL}) \tag{14.17}$$

$$e^{-KL} = \frac{1}{A_s} \int\limits_{A_s} \int\limits_{A_s} e^{-KR} \frac{\cos \theta \cos \theta}{\pi R^2} dA_s \, dA_s \tag{14.18}$$

If KL is small, for the optically thin gases low pressure, $e^{-KL} \cong 1 - KL$, Equation 14.18 becomes

$$L = \frac{1}{A_s} \int\limits_{A_s} \int\limits_{A_s} \frac{\cos \theta \cos \theta}{\pi R} dA_s \, dA_s$$

$$= \frac{4V}{A_s} \tag{14.19}$$

where V is the volume of the gas in the enclosure and A_s is the enclosure surface area.

In general, the geometric mean beam length (L_{ij}) between surfaces i and j of an enclosure should be determined from Equation 14.4, and can be determined by Equation 14.15 for the optically thin gas (i.e., small absorption coefficient K, low pressure, and small enclosure L_{ij}, KL_{ij} is small). In addition, it can be determined from Equations 14.18 and 14.19, respectively, for a single-surface enclosure with a uniform temperature and emissivity. However, in some problems, for the optically thick gases (i.e., KL is not small, high pressure), the geometric mean beam length (L) is less than the above-mentioned values. From experience, the geometric mean beam length has been proved to be a good approximation for the actual mean beam length. For practice, Equation 14.3, $L \cong 0.9(4V/A_s)$, can be used.

14.2　Radiation Exchange between an Isothermal Gray Gas and Gray Diffuse Isothermal Surfaces in an Enclosure

Since we know how to obtain gas radiation properties such as emissivity and absorptivity and how to determine the view factor between two surfaces, the following shows how to determine radiation heat transfer between surfaces in an enclosure with radiation gases. Assume that there are N gray diffuse and isothermal surfaces [2,4,5]. This implies that each surface at T_i has uniform radiosity J_i (emission plus reflection). Also assume that radiation gases are gray gases at uniform pressure and temperature (emissivity = absorptivity) and have no scattering effect. Figure 14.6 shows an energy balance on surface i and an energy balance between surface i and the rest of enclosure surfaces j through gases.

If given surface i temperature (T_i) and gas temperature (T_g), the following shows how to determine heat transfer rate from surface $i(q_i)$ and from radiation gases (q_g). Gas transmissivity is inversely proportional to the gas absorption coefficient as

$$\tau_g = e^{-\kappa L} \tag{14.20}$$

where $\kappa =$ is the total absorption coefficient (predetermined), for example, $\kappa = 0.3\,\text{m}^{-1}, L$ is the mean beam length (predetermined from Equation 14.3).

Since gas absorptivity+gas transmissivity $= 1, \alpha_g + \tau_g = 1$.

Therefore,

$$1 - \alpha_g = \tau_g = e^{-\kappa L} \tag{14.21}$$

For given T_i, T_g, how to obtain q_i, q_g?

Perform energy balance on surface i, net heat transfer rate = radiosity (energy out) − irradiation (energy in)

$$q_i = A_i(J_i - G_i) \tag{14.22}$$
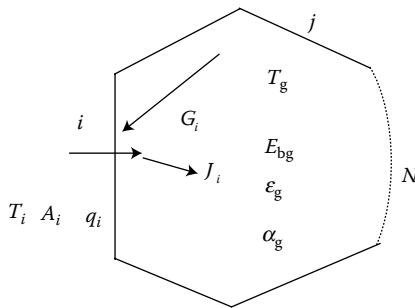
FIGURE 14.6
Radiation heat transfer through gases in an enclosure.

and

$$J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i)G_i \qquad (14.23)$$

Therefore,

$$q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/(A_i\varepsilon_i)} \qquad (14.24)$$

Performing energy balance between surface i and the rest of surfaces j through radiation gases,

$$
\begin{aligned}
q_i &= A_i(J_i - G_i) \\
&= A_i\left[J_i - \sum F_{ij}J_j(1 - \alpha_{ij,g}) - E_{bg}\varepsilon_{i,g}\right] + A_i\varepsilon_{i,g}J_i - A_i\varepsilon_{i,g}J_i \\
&= A_i\varepsilon_{i,g}(J_i - E_{bg}) + \sum A_i F_{ij}(1 - \alpha_{ij,g})(J_i - J_j) \qquad (14.25)
\end{aligned}
$$

where

$$
\begin{aligned}
A_i G_i &= \sum A_j F_{ji} J_j (1 - \alpha_g) + A_i \varepsilon_{i,g} E_{bg} \\
&= \sum A_i F_{ij} J_j (1 - \alpha_g) + A_i \varepsilon_{i,g} E_{bg} \qquad (14.26)
\end{aligned}
$$

And from the following relationships:

$$
\begin{aligned}
A_i(J_i - \varepsilon_{i,g}J_i) &= A_i J_i(1 - \varepsilon_{i,g}) \\
&= A_i J_i(1 - \alpha_{i,g}) \\
&= \sum A_i F_{ij}(1 - \alpha_{i,g})J_i
\end{aligned}
$$

Therefore,

$$q_i = \frac{J_i - E_{bg}}{\left(\dfrac{1}{A_i\varepsilon_{i,g}}\right)} + \sum_{j=1}^{N} \frac{J_i - J_j}{\left(\dfrac{1}{A_i F_{ij}(1 - \alpha_{ij,g})}\right)} \qquad (14.27)$$

$$\underbrace{\phantom{\frac{1}{A_i\varepsilon_{i,g}}}}_{\substack{\text{resistance due to} \\ \text{gas emissivity}}} \qquad \underbrace{\phantom{\frac{1}{A_i F_{ij}(1-\alpha_{ij,g})}}}_{\substack{\text{resistance due to view factor} \\ \text{and gas absorptivity}}}$$

If gas has no radiation properties, that is, $\varepsilon_g = \alpha_g = 0, \tau_g = 1$, then the above equation returns to the one we have seen before as

$$q_i = \sum_{j=1}^{N} \frac{J_i - J_j}{1/A_i F_{ij}}$$

14.2.1   Matrix Linear Equations

Apply method 2—the matrix method for N surfaces enclosure with participating gases. For case A problem, given temperatures to determine heat fluxes. Let the right side of Equation 14.24 = the right side of Equation 14.27 to form the matrix as before:

$$a_{11}J_1 + a_{12}J_2 + \cdots + a_{1N}J_N = c_1$$
$$a_{21}J_1 + a_{22}J_2 + \cdots + a_{2N}J_N = c_2$$
$$a_{31}J_1 + a_{32}J_2 + \cdots + a_{3N}J_N = c_3$$
$$a_{N1}J_1 + a_{N2}J_2 + \cdots + a_{NN}J_N = c_N$$

Therefore,

$$[A][J] = [C]$$
$$[J] = [A]^{-1}[C]$$

In addition, combining Equations 14.23 and 14.26, we obtain J_i = emission from surface i + reflection from surface j

$$= \varepsilon_i E_{bi} + (1 - \varepsilon_i) \left[\sum_{j=1}^{N} F_{ij} J_j (1 - \alpha_g) + \varepsilon_{i,g} E_{bg} \right] \tag{14.28}$$

Similarly, Equation 14.28 can be used to form the matrix $[A][J] = [C]$ as follows:

$$J_1 = \varepsilon_1 E_{b1} + (1 - \varepsilon_1)[F_{11}J_1(1 - \alpha_g) + \varepsilon_{1,g}E_{bg} + F_{12}J_2(1 - \alpha_g) + \varepsilon_{1,g}E_{bg} + \cdots]$$
$$J_2 = \varepsilon_2 E_{b2} + (1 - \varepsilon_2)[F_{21}J_1(1 - \alpha_g) + \varepsilon_{2,g}E_{bg} + F_{22}J_2(1 - \alpha_g) + \varepsilon_{2,g}E_{bg} + \cdots]$$

$$\vdots$$

$$\vdots$$

$$J_N = \varepsilon_N E_{bN} + (1 - \varepsilon_N)[F_{N1}J_1(1 - \alpha_g) + \varepsilon_{N,g}E_{bg}$$
$$+ F_{N2}J_2(1 - \alpha_g) + \varepsilon_{N,g}E_{bg} + \cdots]$$

Once matrix $[J]$ has been solved, then surface heat transfer rate can be determined from Equation 14.24 as

$$\Rightarrow q_i = \frac{E_{bi} - J_i}{(1 - \varepsilon_i)/(\varepsilon_i A_i)}$$

If we consider energy balance between gases and enclosure surfaces i, the heat transfer rate (energy releases) from gases to the enclosure is

$$q_g = \sum_{i=1}^{N} A_i \varepsilon_{i,g} (E_{bg} - J_i) = \sum_{i=1}^{N} \frac{E_{bg} - J_i}{1/A_i \varepsilon_{i,g}} \tag{14.29}$$

However, we still need Equations 14.24 and 14.27 or Equation 14.28 to solve J_i using the matrix method. The aforementioned gas radiation problems can also be solved by method 1—the electric network analogy method.

14.2.2 Electric Network Analogy

Special case 1: Figure 14.7a shows several combustion furnaces that can be modeled as radiation between two surface enclosures containing hot radiation gases, if $T_g > T_1 > T_2$: By using Equations 14.24, 14.27, and 14.29, Figure
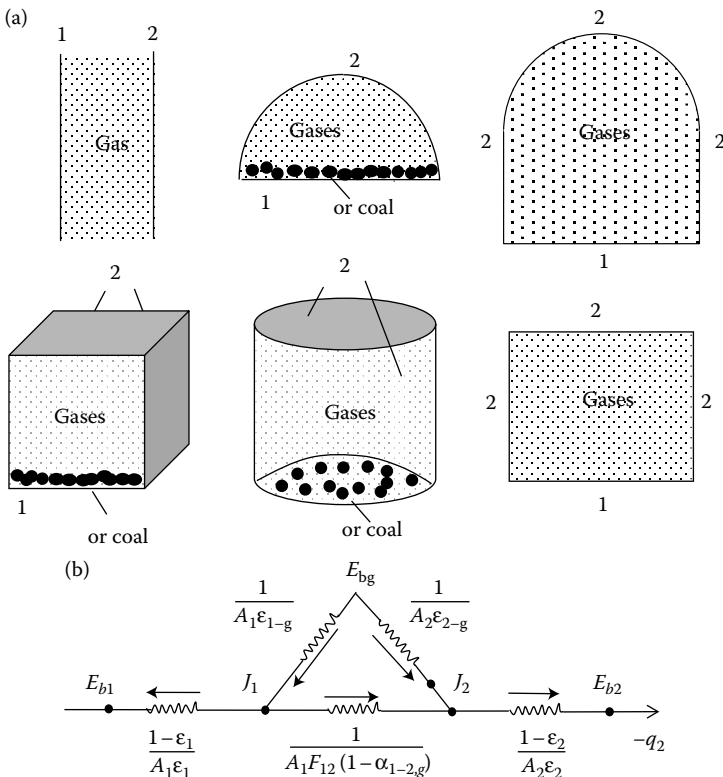
FIGURE 14.7
(a) Radiation between hot gases and two-surface enclosures; (b) Electric network for radiation from hot gas to two-surface enclosure.

14.7b shows the associated electric network from hot gas to surfaces 1 and 2. Hot gases release energy to surfaces 1 and 2 through their resistances (with gas emissivity less than unity); each surface has its own resistance (with surface emissivity less than unity). There is a reduced view factor between two surfaces because gas cannot be completely seen through between two surfaces (due to the gas absorption effect). If gas absorptivity is zero, the view factor is the same as the one with nonparticipating gases (such as air). Energy balance on surfaces 1 and 2 must be performed in order to solve for radiosities J_1 and J_2, respectively. Then, energy releases from hot gases, and heat transfer to surfaces 1 and 2 can be determined.

Special case 2: Figure 14.8 shows that several combustion furnaces can be modeled as heat transfer between hot gases and a single gray surface enclosure (assume an enclosure at a uniform temperature). The simple electric network can be used to solve this type of problem.

From Equations 14.24 and 14.28, solve for q_1 as

$$J_1 = \varepsilon_1 E_{b1} + (1 - \varepsilon_1)[J_1(1 - \alpha_g) + \varepsilon_g E_{bg}] \tag{14.30}$$

$$q_1 = (E_{b1} - J_1)A_1\varepsilon_1/(1 - \varepsilon_1) \tag{14.31}$$

Substituting J_1 into q_1, we obtain

$$q_1 = \frac{A_1\varepsilon_1\alpha_g\sigma T_s^4}{1 - (1 - \varepsilon_1)(1 - \alpha_g)} - \frac{A_1\varepsilon_1\varepsilon_g\sigma T_g^4}{1 - (1 - \varepsilon_1)(1 - \alpha_g)}$$

$$= \frac{A_1\varepsilon_1}{1 - (1 - \varepsilon_1)(1 - \alpha_g)}(\alpha_g\sigma T_s^4 - \varepsilon_g\sigma T_g^4) \tag{14.32}$$

Special case 3—Net radiation heat transfer between nongray gases and a single black enclosure: To further simplify the problem, assume that the whole furnace
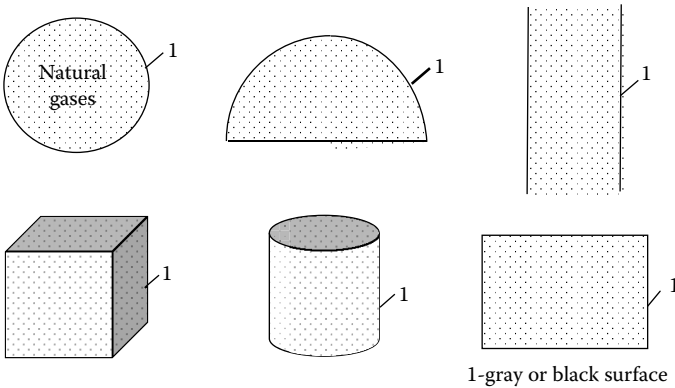
FIGURE 14.8
Radiation between hot gases and single-surface enclosure.
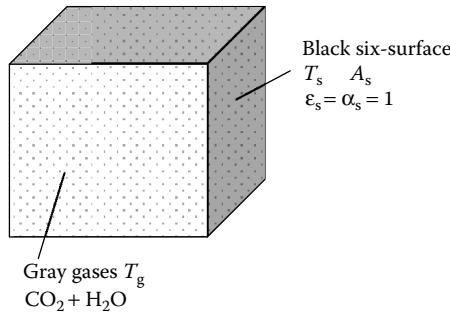
FIGURE 14.9
Radiation between hot gases and single black enclosure.

surface is a blackbody at a uniform temperature as shown in Figure 14.9. Net heat transfer rate = hot gas emission and absorbed by the black surface-black surface emission and absorbed by gases [1].

$$q_{net} = \underbrace{(\varepsilon_g A_s \sigma T_g^4) \cdot \alpha_s}_{\substack{\text{emission} \\ \text{from the gas}}} - \underbrace{(\varepsilon_s A_s \sigma T_s^4) \cdot \alpha_g}_{\substack{\text{emission} \\ \text{from the surface}}} = A_s \sigma (\varepsilon_g T_g^4 - \alpha_g T_s^4) \tag{14.33}$$

For gray gas,

$$\varepsilon_g = \alpha_g \quad (\text{Otherwise, } \varepsilon_g \neq \alpha_g)$$

and for nongray gas

$$\varepsilon_g = \varepsilon_c + \varepsilon_w - \Delta\varepsilon$$

$$\alpha_g = \alpha_c + \alpha_w - \Delta\alpha$$

$$\Delta\alpha = \Delta\varepsilon$$

For water vapor,

$$\alpha_w = C_w \left(\frac{T_g}{T_s}\right)^{0.45} \cdot \varepsilon_w \left(T_s, P_w L \frac{T_s}{T_g}\right) \tag{14.34}$$

For carbon dioxide,

$$\alpha_c = C_c \left(\frac{T_g}{T_s}\right)^{0.65} \cdot \varepsilon_c \left(T_s, P_c L \frac{T_s}{T_g}\right) \tag{14.35}$$

Note that the problem will be more complicated if we consider radiation exchange between nongray gases and a single gray enclosure, or radiation exchange between nongray gases and an N-surfaces enclosure, where $N = 1, 2, 3, \ldots, N$.
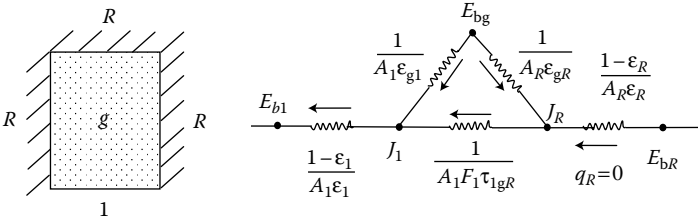
FIGURE 14.10
A gray enclosure and a refractory surface filled with a gray gas.

Special case 4—Gray enclosure filled with a gray gas: Figure 14.10 shows a furnace consisting of a hot or a cold gray surface (1), a refractory surface R, and a gray gas, g, where each element is assumed to be at a uniform temperature $-T_1$, T_R, and T_g; determine the radiation heat transfer between the surface (1) and gas as

$$q_{1g} = \frac{\sigma(T_1^4 - T_g^4)}{(1-\varepsilon_1)/A_1\varepsilon_1 + 1/\{A_1\varepsilon_{g1} + 1/[1/(A_R\varepsilon_R) + 1/(A_1F_{1R}\tau_{1gR})]\}} \quad (14.36)$$

Special case 5—Two gray surfaces with a gray gas: Figure 14.11 shows a furnace consisting of a hot gray surface (1), cold gray surface (2), a refactory surface R, and a gray gas g; determine the radiation heat transfer [1].

Real furnace applications—The zone method: Figure 14.12 shows the concept of the zone method for real-furnace applications proposed by Hottle (MIT) [5]. In a real furnace, combustion gases as well as furnace surface temperatures are nonuniform. The problem can be solved by dividing gases and surfaces into a number of gas zones and surface zones, respectively. Energy balance can be performed on each subsurface (each zone) and between each subsurface and the rest of subsurfaces (zones) through gas zones. View factors need to be calculated between subsurfaces too. The solution procedures are quite complicated.
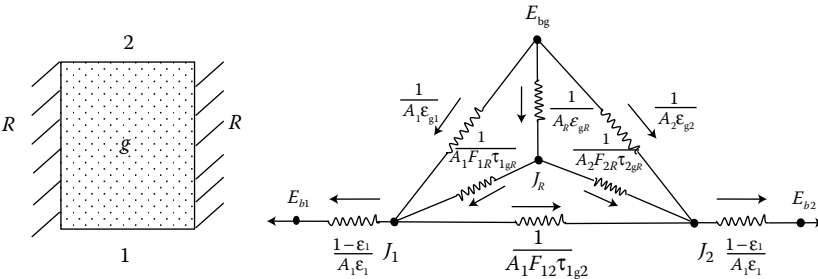
FIGURE 14.11
An enclosure of a gray hot surface, a gray cold surface, and a refractory surface.
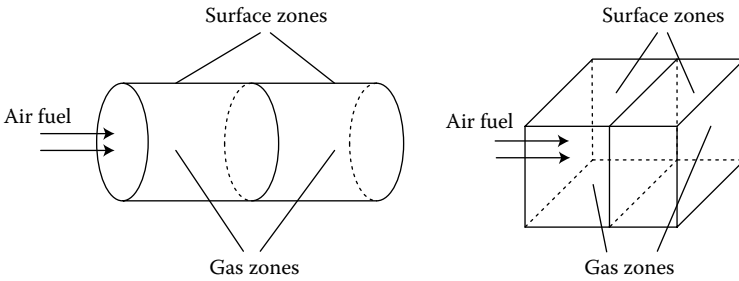
FIGURE 14.12
Concept of zone method for real furnace heat transfer problem.

Zone for gases $T_{g,i}$ = Number of gas zones, a uniform temperature in each zone

Zone for surfaces T_i = Number of surface zones, a uniform temperature in each zone

The problem will be even more complicated if convection effects are considered, which is up to 20% of heat transfer rate for a circulation well-mixed furnace. In reality, soot formation and radiation can further make the problem harder to model.

14.3 Radiation Transfer through Gases with Nonuniform Temperature

14.3.1 Cryogenic Thermal Insulation

In some applications, such as cryogenic thermal insulation, radiation heat is transferred from surface 1 to surface 2 through participating gases with varying temperature as shown in Figure 14.13. For a simple 1-D problem,
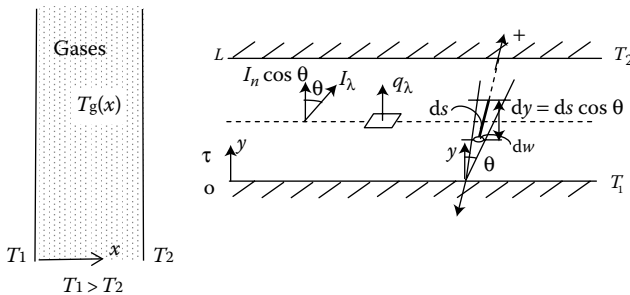
FIGURE 14.13
Radiation heat transfer through gases with varying temperature.

gray gas temperature changes from gray diffuse surface 1 to surface 2 [5,6].

- Nonuniform gas temperature: Cryogenic thermal insulation.
- For simple case: 1-D gray gas, gray and diffuse surfaces.

14.3.2 Radiation Transport Equation in the Participating Medium

$$\frac{\partial I}{\partial \phi} = 0$$

$$\frac{\partial I}{\partial \theta} \neq 0$$

$$dI_\lambda^+(\theta) = -k_\lambda I_\lambda^+ \, ds - r_\lambda I_\lambda^+ \, ds + k_\lambda \frac{e_{b\lambda}(y)}{\pi} \, ds + \frac{r_\lambda G_\lambda(y)}{4\pi} \, ds \qquad (14.37)$$

where k_λ is the absorption coefficient, r_λ the scattering coefficient, $e_{b\lambda}$ the emission power, and $G_\lambda(y)$ the scattering into the area ds from surrounding.
   Also $dI_\lambda^-(\theta) = \ldots$.
   The net heat flux:

1. $$q_\lambda = \int I_\lambda(\tau_\lambda, \theta) \cos\theta \, d\omega$$

    where $I_\lambda = I_\lambda^+ - I_\lambda^-$, τ_λ—Number of mean free path,

    $$\tau_\lambda = \int_0^y (1/\lambda_p) \, dy \Rightarrow \tau_L = (L/\lambda_p) = L\beta_\lambda$$

    With λ_p the mean free path, β_λ volumetric extinction coefficient $\beta_\lambda = k_\lambda + r_\lambda$.

2. $dq/dy = 0$, that is, $q = $ const. if only consider radiation, no conduction, no convection.

   Boundary conditions:
   Radiosity at surface 1,

$$I^+(0) = \frac{J_1}{\pi} = \frac{\varepsilon_1 \sigma T_1^4 + (1 - \varepsilon_1)G_1}{\pi}$$

$$I^-(L) = \frac{J_2}{\pi} = \frac{\varepsilon_2 \sigma T_2^4 + (1 - \varepsilon_2)G_2}{\pi}$$

From the above (1), (2), and BCs, a solution can be achieved by an exponential or numerical method [6].
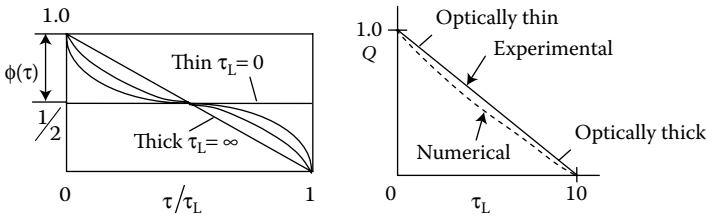
FIGURE 14.14
Nondimensional temperature and heat flux profiles.

For 1-D, gray gases, gray and diffuse surface, radiation only, the heat flux

$$q = \frac{\sigma \left(T_1^4 - T_2^4\right) Q}{1 + Q \left((1/\varepsilon_1) + (1/\varepsilon_2) - 2\right)} \tag{14.38}$$

$$Q \equiv \frac{q}{J_1 - J_2} \equiv \frac{1}{1 + (3/4)\tau_L} \tag{14.39}$$

$$\phi(\tau) \equiv \frac{\sigma T^4(\tau) - J_2}{J_1 - J_2} \tag{14.40}$$

where Q is the nondimensional heat flux, $\phi(\tau)$ is the nondimensional temperature profile, as sketched in Figure 14.14.

Temperature profile:

$$\phi(\tau) = 1 - \frac{1}{2}Q - \frac{3}{4}Q\tau \tag{14.41}$$

Physical significances:

Special case (a)—Optical thin medium: $\tau_L = L\beta_\lambda \ll 1 \approx 0$, or system dimension $\ll$ mean free path, then, $Q \to 1$ and the physical heat flux is

$$\Rightarrow q = \frac{\sigma \left(T_1^4 - T_2^4\right)}{(1/\varepsilon_1) + (1/\varepsilon_2) - 1} \tag{14.42}$$

This equals to surface radiation problem.

Special case (b)—Optical thick medium: β is large or $\tau_L \to \infty$, then $Q = (4/3)(1/\tau_L)$ is small,

$$q = \sigma \left(T_1^4 - T_2^4\right) \frac{4}{3} \frac{1}{\tau_L} \tag{14.43}$$

If considering black surfaces, $\varepsilon_1 = \varepsilon_2 = 1$, from Equations 14.39 through 14.41, the temperature profile between two surfaces with participating medium can be obtained and sketched in Figure 14.15,

a. $\phi = 1/2 = (T^4 - T_2^4)/(T_1^4 - T_2^4)$ for the optical thin medium.
b. $\phi = 1 - (\tau/\tau_L) = (T^4 - T_2^4)/(T_1^4 - T_2^4)$ for the optical thick medium.
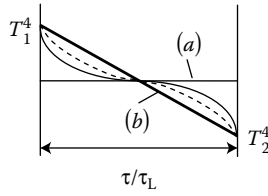
FIGURE 14.15
Temperature profile between two black surfaces through participating gas.

If considering conduction and radiation,

$$q = q_c + q_r = k\frac{T_1 - T_2}{L} + \frac{\sigma\left(T_1^4 - T_2^4\right)Q}{1 + Q((1/\varepsilon_1) + (1/\varepsilon_2) - 2)} \tag{14.44}$$

If considering convection and radiation,

$$q = q_{conv} + q_r = \bar{h}\Delta\bar{T} + \frac{\sigma\left(T_1^4 - T_2^4\right)Q}{1 + Q((1/\varepsilon_1) + (1/\varepsilon_2) - 2)} \tag{14.45}$$

Remarks

In the undergraduate-level heat transfer, from charts, students are expected to know how to read the emissivity and absorptivity values of water vapor and carbon dioxide in a furnace at given size, temperature, and pressure, and then apply these properties to calculate radiation transfer between these gases and the furnace wall, assuming the blackbody furnace wall at a uniform temperature.

In the intermediate-level heat transfer, this chapter focuses on how to derive volumetric absorption; geometric mean beam length; radiation transfer between gray gases at a uniform temperature and an N-surfaces furnace with each surface at different gray diffuse uniform temperature conditions. Students are expected to know how to analytically solve gas radiation problems by using the matrix linear equations method for an N-surfaces furnace with various surface thermal BCs. By using electric network analogy, this chapter has also provided several relevant engineering applications such as radiation transfer between gas at a high uniform temperature and one-surface furnace assuming the gray diffuse surface at a low uniform temperature; or between gas at a high uniform temperature and two-surfaces furnace assuming gray diffuse surfaces with each surface at different low uniform temperatures.

This chapter does not go into much detail on real-furnace applications with varying gas temperature and surface temperature by using the zoning method. We only deal with the 1-D varying gas temperature, steady-state, and gray diffuse surface problem. However, in real-life applications, there are many gas radiation problems involving flow convection; 2-D or 3-D varying

gas temperature; cylindrical or spherical furnace geometry; nongray gases; nongray nondiffuse surfaces; gases with scattering particulates and soot formation; and sphere fillet or fiber porous medium. These topics belong to advanced radiation transfer.

PROBLEMS

14.1. A hemispherical furnace is shown in Figure 14.7. If the furnace contains $CO_2 + N_2$ gases at 1 atm pressure and temperature T_g, determine the total radiation heat transfer from gases to surfaces 1 and 2 (assume $T_g > T_1 > T_2$, and make other necessary assumptions).

  a. Based on the analogy of electric resistance network, draw a radiation heat transfer network from gases to surfaces 1 and 2.

  b. Determine the total radiation from gases to surfaces 1 and 2. The final solutions should be the function of given temperatures, surface area, and radiation properties.

14.2. A hemispherical furnace is shown in Figure 14.7.

  a. If the furnace contains N_2 gas at 5 atm pressure, determine the net radiation heat transfer from surface 1 to surface 2 (assume $T_1 > T_2$, and make other necessary assumptions).

  b. If the furnace contains $CO_2 + N_2$ gases at 5 atm pressure and temperature T_g, determine total radiation heat transfer from gases to surfaces 1 and 2 (assume $T_g > T_1 > T_2$, and make other necessary assumptions).

  c. Reconsider (b), if surface 1 now is a reradiating surface, determine the total radiation heat transfer to the surface 2 of the furnace. In this new condition, comment on whether the radiation transfer to surface 2 will be higher, the same, or lower than that of (b) (make necessary assumptions).

14.3. A long hemicylindrical furnace is shown.

  a. Determine the net radiation heat transfer from surface 1 to surface 2, q_{12}.

  b. If $T_2 = T_2(\theta)$, describe how to determine q_{12}.

  c. Consider combustion gray gas with a uniform temperature T_g and emissivity ε_g inside the furnace, and determine the total radiation heat transfer from gas to surfaces 1 and 2, q_g. Assume T_1, T_2 constant.

14.4. A hemispherical furnace, with a reradiating floor and a water-cooled ceiling, contains CO_2 and N_2 gases at 1 atm pressure and 1000°C. Take $\varepsilon_1 = 0.8, \varepsilon_2 = 0.7, D = 1\,\mathrm{m}$, and $T_2 = 500°C$. Determine the radiant heat transfer to the ceiling of the furnace. Assume gray gases.

  Given: $\sigma = 5.67 \times 10^{-8}(\mathrm{w/m^2K^4})$.

  Volume of a sphere $= (4/3)\pi((1/2)D)^2$

  Surface of a sphere $= 4\pi((1/2)D)^2$

14.5. A hemispherical furnace, with a reradiating floor and a water-cooled ceiling, contains $2CO_2$ and $8N_2$ gases at 1 atm pressure

and 1000°C. Take $\varepsilon_1 = 0.8, \varepsilon_2 = 0.7, D = 1\,\text{m}$, and $T_2 = 500$°C. Determine the radiant heat transfer to the ceiling of the furnace.

14.6. A cryogenic storage chamber has double walls for the purpose of insulation against heat loss. The gap between the walls is filled with a gas whose properties are

Thermal conductivity: $K(T) = 2 \times 10^{-7} \times T$°$K$ $(KW/(m - $°$C))$

Volumetric radiation extinction coefficient: $\beta = 10^{-6}\,(\text{m}^2/\text{m}^3)$

a. Determine the rate of heat loss if one wall is at 500°K and the other wall is at 100°K. Take $\varepsilon_1 = \varepsilon_2 = 0.1, L = 0.2\,\text{m}$.
b. If $\beta = 100\,(\text{m}^2/\text{m}^3)$, what would be the result in (a)?

14.7. A cryogenic storage chamber has double walls for the purpose of insulation against heat loss. The gap between the walls is filled with a gas whose properties are

Thermal conductivity: $K(T) = 1 \times 10^{-7} \times T$°$K(KW/(m - $°$C))$

Volumetric radiation extinction coefficient: $\beta = 10^{-6}(\text{m}^2/\text{m}^3)$

The walls are made of a polished metal, with an emissivity of 0.2. The gap between the walls is 0.5 m.

a. Determine the rate of heat loss if one wall is at 300°K and the other wall is at 100°K.
b. If $\beta = 100\,(\text{m}^2/\text{m}^3)$, what would be the result in (a)?

14.8. A cryogenic storage chamber has double walls for the purpose of insulation against heat loss. The gap between the walls is filled with a gas whose properties are

Thermal conductivity: $K(T) = 3 \times 10^{-7} \times T$°$K(KW/(m - $°$C))$

Volumetric radiation extinction coefficient: $\beta = 10^{-6}\,(\text{m}^2/\text{m}^3)$

The walls are made of a polished metal, with an emissivity of 0.1. The gap between the walls is 0.3 m.

a. Determine the rate of heat loss if one wall is at 400°K and the other wall is at 100°K.
b. If $\beta = 100\,(\text{m}^2/\text{m}^3)$, what would be the result in (a)?

14.9. A gas turbine combustion chamber may be approximated as a long tube of 0.4 m diameter. The combustion gas is at a pressure and temperature of 1 atm and 1000°C, respectively, while the chamber surface temperature is 500°C. If the combustion gas contains CO_2 and water vapor, each with a mole fraction of 0.15, what is the net radiative heat flux between the gas and the chamber surface, which may be approximated as a blackbody?

14.10. Consider a hemispherical furnace, with a reradiating floor and a water-cooled ceiling, contains $2CO_2 + 8N_2$ gases at 1 atm pressure and 1200°C. Take $\varepsilon_1 = 0.9, \varepsilon_2 = 0.6, D = 1.5\,\text{m}$, and $T_2 = 350$°C. Determine the radiant heat transfer to the ceiling of the furnace.

14.11. Consider a hemispherical furnace radiation heat transfer problem. The furnace floor (surface 1) has area A_1 and emissivity ε_1 at temperature T_1, whereas the furnace enclosure (surface 2) has area A_2 and emissivity ε_2 at temperature T_2. If the furnace contains $CO_2 + N_2$ gases at 1 atm pressure and temperature T_g, determine

the total radiation heat transfer from gases to surfaces 1 and 2 (assume $T_g > T_1 > T_2$, and make other necessary assumptions).

a. Based on the analogy of electric resistance network, draw a radiation heat transfer network from gases to surfaces 1 and 2.

b. Determine the total radiation from gases to surfaces 1 and 2. The final solutions should be the function of given temperatures, surface area, and radiation properties.

14.12. Consider a hemispherical furnace. The hemispherical furnace wall has surface area A_1, emissivity ε_1, at temperature T_1, whereas the furnace floor has surface area A_2, emissivity ε_2, at temperature T_2. If the furnace contains $CO_2 + N_2$ gases at 10 atm pressure and temperature T_g, determine the total radiation heat transfer from gases to surfaces 1 and 2 (assume $T_g > T_2 > T_1$, and make other necessary assumptions). Assume that the gas emissivity is ε_g.

a. Based on the analogy of electric resistance network, draw a radiation heat transfer network from gases to surfaces 1 and 2.

b. Determine the total radiation from gases to surfaces 1 and 2. The final solutions should be the function of given temperatures, surface area, and radiation properties.

14.13. For a cylindrical furnace (top wall 1, bottom wall 2, side wall 3) with hot gray gases, solve the following problems by using the matrix method.

a. Given T_1, T_2, T_3, T_g, determine q_1, q_2, q_3, q_g.

b. Given q_1, q_2, q_3, q_g, determine T_1, T_2, T_3, T_g.

c. Given T_1, T_2, q_3, T_g, determine q_1, q_2, T_3, q_g.

d. Given q_1, q_2, T_3, T_g, determine T_1, T_2, q_3, q_g.

e. Given T_1, T_2, T_3, q_g, determine q_1, q_2, q_3, T_g.

14.14. For a cubic furnace (top wall 1, bottom wall 2, four side wall 3) with hot gray gases, solve the following problems by using the matrix method if the side wall is a reradiating surface.

a. Given T_1, T_2, T_R, T_g, determine q_1, q_2, q_R, q_g.

b. Given q_1, q_2, q_R, q_g, determine T_1, T_2, T_R, T_g.

c. Given T_1, T_2, q_R, T_g, determine q_1, q_2, T_R, q_g.

d. Given q_1, q_2, T_R, T_g, determine T_1, T_2, q_R, q_g.

e. Given T_1, T_2, T_R, q_g, determine q_1, q_2, q_R, T_g.

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Appendix A: Mathematical Relations and Functions

A.1 Useful Formulas

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$d\sin x = \cos x\, dx; \quad d\cos x = -\sin x\, dx$$

$$d\sinh x = \cosh x\, dx; \quad d\cosh x = +\sinh x\, dx$$

$$\int \sin x\, dx = -\cos x + c; \quad \int \cos x\, dx = \sin x + c$$

$$\int \sin^2 x\, dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x + c = -\frac{1}{4}\sin 2x + \frac{1}{2}x + c$$

$$\int \cos^2 x\, dx = \frac{1}{2}\sin x \cos x + \frac{1}{2}x + c = \frac{1}{4}\sin 2x + \frac{1}{2}x + c$$

$$\int \sinh x\, dx = \cosh x + c; \quad \int \cosh x\, dx = \sinh x + c$$

$$\int \sinh^2 x \, dx = \frac{1}{2} \sinh x \cosh x - \frac{1}{2} x + c$$

$$\int \cosh^2 x \, dx = \frac{1}{2} \sinh x \cosh x + \frac{1}{2} x + c$$

A.2   Hyperbolic Functions [1]

x	$\sinh x$	$\cosh x$	$\tanh x$
0.00	0.0000	1.0000	0.00000
0.10	0.1002	1.0050	0.09967
0.20	0.2013	1.0201	0.19738
0.30	0.3045	1.0453	0.29131
0.40	0.4108	1.0811	0.37995
0.50	0.5211	1.1276	0.46212
0.60	0.6367	1.1855	0.53705
0.70	0.7586	1.2552	0.60437
0.80	0.8881	1.3374	0.66404
0.90	1.0265	1.4331	0.71630
1.00	1.1752	1.5431	0.76159
1.10	1.3356	1.6685	0.80050
1.20	1.5095	1.8107	0.83365
1.30	1.6984	1.9709	0.86172
1.40	1.9043	2.1509	0.88535
1.50	2.1293	2.3524	0.90515
1.60	2.3756	2.5775	0.92167
1.70	2.6456	2.8283	0.93541
1.80	2.9422	3.1075	0.94681
1.90	3.2682	3.4177	0.95624
2.00	3.6269	3.7622	0.96403
2.10	4.0219	4.1443	0.97045
2.20	4.4571	4.5679	0.97574
2.30	4.9370	5.0372	0.98010
2.40	5.4662	5.5569	0.98367
2.50	6.0502	6.1323	0.98661
2.60	6.6947	6.7690	0.98903
2.70	7.4063	7.4735	0.99101

(continued)

x	$\sinh x$	$\cosh x$	$\tanh x$
2.80	8.1919	8.2527	0.99263
2.90	9.0596	9.1146	0.99396
3.00	10.018	10.068	0.99505
3.50	16.543	16.573	0.99818
4.00	27.290	27.308	0.99933
4.50	45.003	45.014	0.99975
5.00	74.203	74.210	0.99991
6.00	201.71	201.72	0.99999
7.00	548.32	548.32	1.00000
8.00	1490.5	1490.5	1.00000
9.00	4051.5	4051.5	1.00000
10.00	11013	11013	1.00000

A.3 Bessel Functions

A.3.1 Bessel Functions and Properties [2]

Behaviors of Bessel functions for small arguments:

$$J_0(x) = 1 - \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} - \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{(x/2)^3}{1!2!} + \frac{(x/2)^5}{2!3!} - \cdots$$

$$\vdots$$

$$J_v(x) = \frac{(x/2)^v}{\Gamma(v+1)} \left\{ 1 - \frac{(x/2)^2}{1!(v+1)} + \frac{(x/2)^4}{2!(v+1)(v+2)} - \cdots \right\}$$

$$I_0(x) = 1 + \frac{(x/2)^2}{(1!)^2} + \frac{(x/2)^4}{(2!)^2} + \cdots$$

$$I_1(x) = \frac{x}{2} - \frac{(x/2)^3}{1!2!} + \frac{(x/2)^5}{2!3!} - \cdots$$

$$\vdots$$

$$I_v(x) = \frac{(x/2)^v}{\Gamma(v+1)} \left\{ 1 + \frac{(x/2)^2}{1!(v+1)} + \frac{(x/2)^4}{2!(v+1)(v+2)} + \cdots \right\}$$

Behaviors of Bessel functions for large arguments:

$$J_v(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$$

$$Y_v(x) \approx \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right)$$

$$I_v(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4v^2 - 1}{8x}\right)$$

$$K_v(x) \approx \sqrt{\frac{\pi}{2x}} e^{-x}\left(1 + \frac{4v^2 - 1}{8x}\right)$$

Properties of Bessel functions:

$$\frac{d}{dx}[J_0(mx)] = -mJ_1(mx), \quad \frac{d}{dx}[Y_0(mx)] = -mY_1(mx)$$

$$\frac{d}{dx}[I_0(mx)] = mI_1(mx), \quad \frac{d}{dx}[K_0(mx)] = -mK_1(mx)$$

A.3.2 Bessel Functions of the First Kind [1]

x	$J_0(x)$	$J_1(x)$
0.0	1.0000	0.0000
0.1	0.9975	0.0499
0.2	0.9900	0.0995
0.3	0.9776	0.1483
0.4	0.9604	0.1960
0.5	0.9385	0.2423
0.6	0.9120	0.2867
0.7	0.8812	0.3290
0.8	0.8463	0.3688
0.9	0.8075	0.4059
1.0	0.7652	0.4400
1.1	0.7196	0.4709
1.2	0.6711	0.4983
1.3	0.6201	0.5220
1.4	0.5669	0.5419
1.5	0.5118	0.5579
1.6	0.4554	0.5699
1.7	0.3980	0.5778

(continued)

x	$J_0(x)$	$J_1(x)$
1.8	0.3400	0.5815
1.9	0.2818	0.5812
2.0	0.2239	0.5767
2.1	0.1666	0.5683
2.2	0.1104	0.5560
2.3	0.0555	0.5399
2.4	0.0025	0.5202

A.3.3 Modified Bessel Functions of the First and Second Kinds [1]

x	$e^{-x}I_0(x)$	$e^{-x}I_1(x)$	$e^x K_0(x)$	$e^x K_1(x)$
0.0	1.0000	0.0000	∞	∞
0.2	0.8269	0.0823	2.1407	5.8334
0.4	0.6974	0.1368	1.6627	3.2587
0.6	0.5993	0.1722	1.4167	2.3739
0.8	0.5241	0.1945	1.2582	1.9179
1.0	0.4657	0.2079	1.1445	1.6361
1.2	0.4198	0.2152	1.0575	1.4429
1.4	0.3831	0.2185	0.9881	1.3010
1.6	0.3533	0.2190	0.9309	1.1919
1.8	0.3289	0.2177	0.8828	1.1048
2.0	0.3085	0.2153	0.8416	1.0335
2.2	0.2913	0.2121	0.8056	0.9738
2.4	0.2766	0.2085	0.7740	0.9229
2.6	0.2639	0.2046	0.7459	0.8790
2.8	0.2528	0.2007	0.7206	0.8405
3.0	0.2430	0.1968	0.6978	0.8066
3.2	0.2343	0.1930	0.6770	0.7763
3.4	0.2264	0.1892	0.6579	0.7491
3.6	0.2193	0.1856	0.6404	0.7245
3.8	0.2129	0.1821	0.6243	0.7021
4.0	0.2070	0.1787	0.6093	0.6816
4.2	0.2016	0.1755	0.5953	0.6627
4.4	0.1966	0.1724	0.5823	0.6453
4.6	0.1919	0.1695	0.5701	0.6292
4.8	0.1876	0.1667	0.5586	0.6142
5.0	0.1835	0.1640	0.5478	0.6003
5.2	0.1797	0.1614	0.5376	0.5872
5.4	0.1762	0.1589	0.5279	0.5749

continued

(continued)

x	$e^{-x}I_0(x)$	$e^{-x}I_1(x)$	$e^x K_0(x)$	$e^x K_1(x)$
5.6	0.1728	0.1565	0.5188	0.5633
5.8	0.1696	0.1542	0.5101	0.5525
6.0	0.1666	0.1520	0.5019	0.5422
6.4	0.1611	0.1479	0.4865	0.5232
6.8	0.1561	0.1441	0.4724	0.5060
7.2	0.1515	0.1405	0.4595	0.4905
7.6	0.1473	0.1372	0.4476	0.4762
8.0	0.1434	0.1341	0.4366	0.4631
8.4	0.1398	0.1312	0.4264	0.4511
8.8	0.1365	0.1285	0.4168	0.4399
9.2	0.1334	0.1260	0.4079	0.4295
9.6	0.1305	0.1235	0.3995	0.4198
10.0	0.1278	0.1213	0.3916	0.4108

A.4 Gaussian Error Function [1]

η	*erf* η	η	*erf* η	η	*erf* η
0.00	0.00000	0.36	0.38933	1.04	0.85865
0.02	0.02256	0.38	0.40901	1.08	0.87333
0.04	0.04511	0.40	0.42839	1.12	0.88679
0.06	0.06762	0.44	0.46622	1.16	0.89910
0.08	0.09008	0.48	0.50275	1.20	0.91031
0.10	0.11246	0.52	0.53790	1.30	0.93401
0.12	0.13476	0.56	0.57162	1.40	0.95228
0.14	0.15695	0.60	0.60386	1.50	0.96611
0.16	0.17901	0.64	0.63459	1.60	0.97635
0.18	0.20094	0.68	0.66378	1.70	0.98379
0.20	0.22270	0.72	0.69143	1.80	0.98909
0.22	0.24430	0.76	0.71754	1.90	0.99279
0.24	0.26570	0.80	0.74210	2.00	0.99532
0.26	0.28690	0.84	0.76514	2.20	0.99814
0.28	0.30788	0.88	0.78669	2.40	0.99931
0.30	0.32863	0.92	0.80677	2.60	0.99976
0.32	0.34913	0.96	0.82542	2.80	0.99992
0.34	0.36936	1.00	0.84270	3.00	0.99998

The Gaussian error function is defined as

$$erf\,\eta = \frac{2}{\sqrt{\pi}} \int\limits_0^{\eta} e^{-u^2}\,du$$

The complementary error function is defined as

$$erfc\,\eta \equiv 1 - erf\,\eta$$

$$\eta = \frac{x}{\sqrt{4\alpha t}}$$

References

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Index